On the Dynamics of Floating Structures

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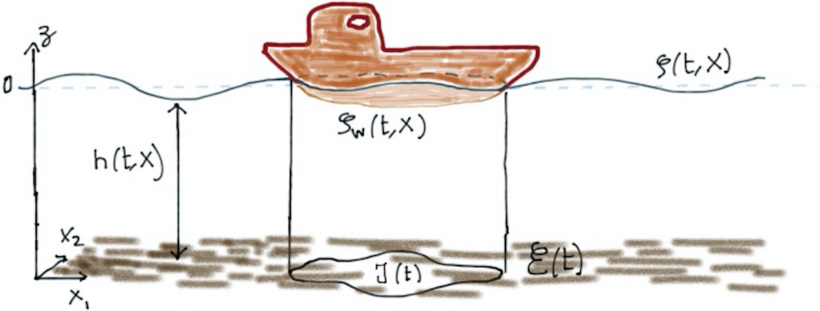
Theoretical and Computational Aspects of Nonlinear Surface Waves

Notations



Floating device: ship or wave energy convertor

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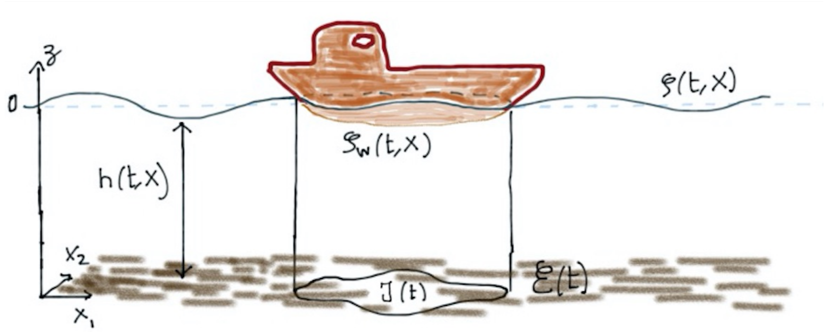

Floating device: ship or wave energy convertor

Notation

If f is defined on $\mathbb{R}^d$, we write

$$f_{\mathrm{e}} = f_{|_{\mathcal{E}}} \quad \text{and} \quad f_{\mathrm{i}} = f_{|_{\mathcal{I}}}$$

Basic equations

In the fluid domain Ω_t

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\begin{aligned}
\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_{X,z}\mathbf{U} &= -\frac{1}{\rho}\nabla_{X,z}P - g\mathbf{e}_z \\
\mathrm{div}\ \mathbf{U} &= 0, \\
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At the surface

$$\boxed{\forall X \in \mathcal{E}(t),} \qquad \underline{P}_{\mathrm{e}}(t,X) := P(t,X,\zeta_{\mathrm{e}}(t,X)) = P_{\mathrm{atm}},$$

$$\forall X \in \mathbb{R}^d, \qquad \partial_t \zeta - \underline{U} \cdot N = 0 \qquad \text{with } N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix},$$

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At the bottom

$$U_b \cdot N_b = 0 \qquad \text{with } N_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}.$$

Interior equations and coupling

Constraint in the interior domain

The surface of the fluid coincides with the wetted portion of the body

$$\zeta_{\mathrm{i}} = \zeta_{\mathrm{w}}$$

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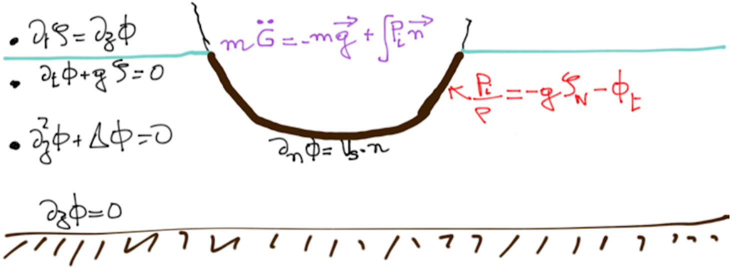
Coupling conditions on $\Gamma(t) := \partial\mathcal{I}(t) = \partial\mathcal{E}(t)$

Continuity of the surface elevation and of the surface pressure

$$\zeta_{\rm e}(t,\cdot) = \zeta_{\rm i}(t,\cdot) \quad \text{and} \quad \underline{P}_{\rm e}(t,\cdot) = \underline{P}_{\rm i}(t,\cdot) \quad \text{on} \quad \Gamma(t)$$

History of the problem

- Fritz John, CPAM, 1949 and 1950: The floating body problem



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  - Variations of the wetted zone neglected
  - Choice of variables not adapted to boundary conditions

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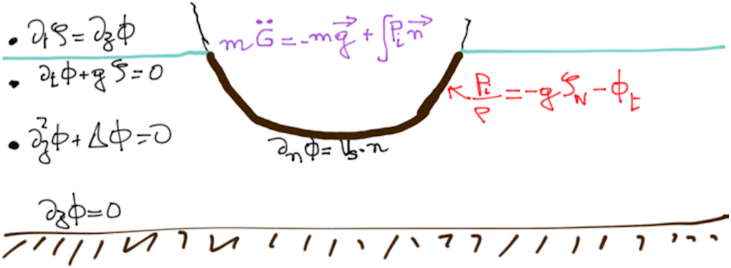

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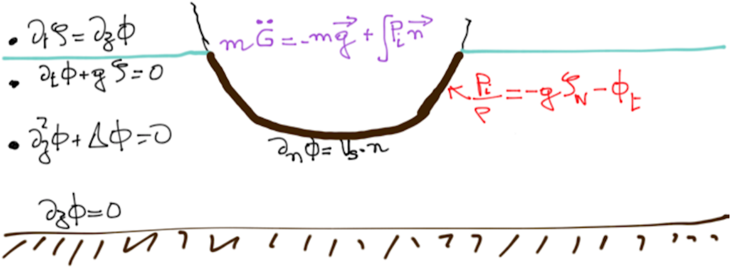

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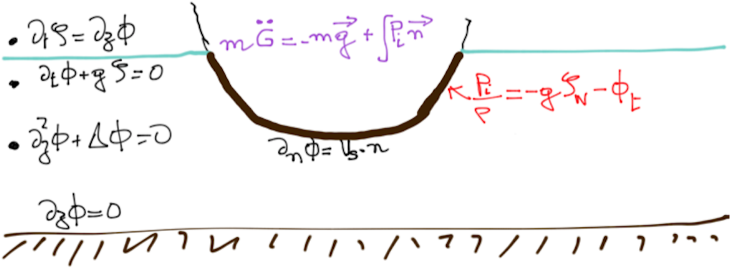

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- Numerical computations: linear models (e.g. Wamit) or CFD
- Recent works on totally immersed solids in a fixed domain $\mathbb{R}^{d+1}$ or Ω: importance of the added-mass effect
- Similar issues in other fluid-interactions problems [CausinGerbeauNobile05]

The free surface Euler equations in (ζ, Q) variables

- Goal: dimension reduction
  - Zakharov: $\psi = \Phi_{|_{z=\zeta}}$
  - Here: vertical integration

$$Q(t,X) = \int_{-h_0}^{\zeta(t,X)} V(t,X,z).$$

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- Conservation of mass

$$\int_{-h_0}^{\zeta} \big(\nabla \cdot V + \partial_z w\big) = 0 \quad \rightsquigarrow \quad \boxed{\partial_t \zeta + \nabla \cdot Q = 0}$$

- Momentum equation
  - Pressure from vertical component of the Euler equation

$$\int_z^{\zeta} \left(\partial_t w + \mathbf{U} \cdot \nabla_{X,z} w + g + \frac{1}{\rho} \partial_z P \right) = 0$$

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$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + gh\nabla\zeta + \nabla \cdot \Big(\int_{-h_0}^{\zeta} V \otimes V \Big) + h\mathbf{a}_{\rm NH} = 0, \end{cases}$$

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- ⇝ They are of compressible type

Acceleration of the free surface

$$\begin{aligned} \partial_t^2 \zeta &= -\nabla \cdot \partial_t Q \\ &= \underbrace{\nabla \cdot \big[gh\nabla\zeta + \dots \big]}_{:= \mathbf{a}_{\rm FS}(\zeta, Q)} \end{aligned}$$

Adaptation in the presence of a floating body

The equations on ζ and Q

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + gh\nabla\zeta + \nabla \cdot \Big(\int_{-H_0}^{-H_0+h} V \otimes V \Big) + \mathbf{a}_{\rm NH} = -\frac{h}{\rho}\nabla \underline{P}, \end{cases}$$

with $\underline{P}_{\rm e} = P_{\rm atm}$ on $\mathcal{E}(t)$ and $\underline{P}_{\rm i}$ unknown on $\mathcal{I}(t)$.

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with $\underline{P}_{\rm e} = P_{\rm atm}$ on $\mathcal{E}(t)$ and $\underline{P}_{\rm i}$ unknown on $\mathcal{I}(t)$.

But we also know that

$$\zeta = \zeta_{\rm w} \quad \text{on} \quad \mathcal{I}(t).$$

and therefore

$$\nabla \cdot Q = -\partial_t \zeta_{\rm w}$$

Finding the pressure in the interior domain

Formulation as an incompressible problem in the interior region

$$\begin{cases} \nabla \cdot Q = -\partial_t \zeta_{\mathrm{w}}, \\ \partial_t Q + \nabla \cdot \Big(\int_{-h_0}^{\zeta} V \otimes V \Big) + gh\nabla\zeta + h\mathbf{a}_{NH} = -\frac{h}{\rho}\nabla \underline{P}_{\mathrm{i}}. \end{cases}$$

$\rightsquigarrow$ Recall that a_{FS} is the acceleration of the FS without floating body

The wetted pressure is found by solving

$$\begin{cases} -\nabla \cdot (\frac{h}{\rho}\nabla \underline{P}_{\mathrm{i}}) = -\partial_t^2 \zeta_{\mathrm{w}} + a_{\mathrm{FS}} \quad \text{on} \quad \mathcal{I}(t) \\ \underline{P}_{\mathrm{i}}|_{\Gamma(t)} = P_{\mathrm{atm}}. \end{cases}$$

Removing the constraint

If $(\zeta, \overline{V})$ and $\Gamma(t)$ solve

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \int_{-h_0}^{\zeta} V \otimes V + gh\nabla\zeta + h\mathbf{a}_{\rm NH} = -\frac{h}{\rho}\nabla \underline{P}, \end{cases}$$

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with the surface pressure $\underline{P}$ given by

$$\underline{P}_{\rm e} = P_{\rm atm} \quad \text{and} \quad \begin{cases} -\nabla \cdot (\frac{h}{\rho}\nabla P_{\rm i}) = -\partial_t^2 \zeta_{\rm w} + \mathbf{a}_{\rm FS} \quad \text{on} \quad \mathcal{I}(t), \\ P_{\rm i}{}_{|_{\Gamma(t)}} = P_{\rm atm}, \end{cases}$$

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and with the coupling conditions at the contact line

$$\zeta_{\rm e} = \zeta_{\rm i} \quad \text{and} \quad \boxed{Q_{\rm e} = Q_{\rm i}} \quad \text{on} \quad \Gamma(t)$$

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$\rightsquigarrow$ Interior pressure recovered through a *d dimensional* elliptic equation.

Coupling with the solid dynamics

The interior pressure equation

$$-\nabla \cdot (\frac{h}{\rho}\nabla \underline{P}_{\rm i}) = \mathbf{a}_{\rm FS} - \partial_t^2 \zeta_{\rm w} \quad \text{on} \quad \mathcal{I}(t).$$

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- From the continuity of the normal velocity

$$\begin{aligned}\partial_t\zeta_{\rm w} &= \underline{U}_{\rm w}\cdot N_{\rm w}\\ &= (\mathbf{U}_G + \boldsymbol{\omega}\times\mathbf{r}_G)\cdot N_{\rm w}\end{aligned}$$

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$$-\nabla\cdot(\frac{h}{\rho}\nabla \underline{P}_{\rm i}) = \mathbf{a}_{\rm FS} - (\dot{\mathbf{U}}_G + \dot{\boldsymbol{\omega}}\times\mathbf{r}_G)\cdot N_{\rm w} + \mathrm{II}(\underline{U}_{{\rm w},\tau},\underline{U}_{{\rm w},\tau}) + \dots$$

$\rightsquigarrow$ Three different components of the pressure

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- Force exerted on the solid

$$F_{\rm fluid}=\int_{\mathcal{I}(t)}\underline{P}_{\rm i}N_{\rm w}=F^{\rm I}+F^{\rm II}+F^{\rm III}$$

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- Force exerted on the solid

$$F_{\rm fluid} = \int_{\mathcal{I}(t)} \underline{P}_{\rm i} N_{\rm w} = F^{\rm I} + F^{\rm II} + F^{\rm III}$$

- Added mass effect $\boxed{F^{II} = -M_a(t)\dot{\mathbf{U}}_G}$.

The case of a freely floating structure

Newtons's laws – take $\omega = 0$ for simplicity –

$$\mathfrak{m}\dot{\mathbf{U}}_G = -\mathfrak{m}g\mathbf{e}_z + F_{\rm fluid}$$

The resulting force exerted by the fluid is

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$$\boxed{F^{\rm II} = -M_{\rm a}(t)\dot{\mathbf{U}}_G}.$$

- Newton's law then becomes – with $\omega = 0$ –

$$\boxed{\big(\mathfrak{m} + M_{\rm a}(t)\big)\dot{\mathbf{U}}_G = -\mathfrak{m}g\mathbf{e}_z + F^{\rm I} + F^{\rm III}}$$

Evolution of the contact line

NOT kinematic

Simplified models in shallow water

$$\begin{cases} \partial_t h + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \big(\int_{-h_0}^{\zeta} V \otimes V \big) + gh\nabla\zeta + h\mathbf{a}_{\rm NH} = -\frac{h}{\rho}\nabla \underline{P} \end{cases}$$

- Two terms can be simplified in shallow water
  1. The "Reynolds tensor"

$$\int_{-h_0}^{\zeta} V \otimes V \approx h\overline{V} \otimes \overline{V} = \frac{1}{h} Q \otimes Q$$

$\rightsquigarrow$ Valid with very good precision except if a significant vorticity is present and a "turbulence-like" analysis is needed [Castro-L. 2015].

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     ⇝ First order approximation (neglect dispersive effects).

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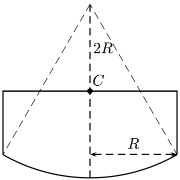
- The same approximations must be done for the interior pressure
- Extension to dispersive models: Boussinesq, Green-Naghdi, etc.

The shallow water equations with an immersed device

$$\begin{cases} \partial_t h + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t (h\overline{V}) + \nabla \cdot (h\overline{V} \otimes \overline{V}) + gh\nabla\zeta = -\frac{h}{\rho}\nabla \underline{P}, \end{cases}$$

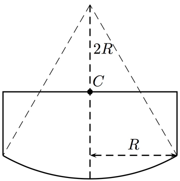
- The same approximations must be done for the interior pressure
- Extension to dispersive models: Boussinesq, Green-Naghdi, etc.
- Extension to numerical scheme: the discretization of the source term must be such that the discrete pressure is the discrete Lagrange multiplier

Numerical simulation



- The solid has vertical walls
- It is allowed to move vertically only
- The hydrodynamic model is the nonlinear shallow water system

Numerical simulation



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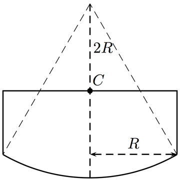
Proposition

- The computations are explicit. The distance δ of the center of mass to its equilibrium satisfies the ODE

$$(m + m_a(\delta))\ddot{\delta} = -2\rho g R\delta \underbrace{+ \rho g R(\zeta_{e,+} + \zeta_{e,-})}_{\text{Damping+coupling}} + NL$$

Numerical simulation



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- The hydrodynamic model is the nonlinear shallow water system

Proposition

- The computations are explicit. The distance δ of the center of mass to its equilibrium satisfies the ODE

$$(m + m_a(\delta))\ddot{\delta} = -2\rho g R\delta \underbrace{- 2\rho g R(h_0 - \tau_0(\frac{R}{2\sqrt{g}}\dot{\delta}_g)^2)}_{\text{Damping+coupling}}$$

- In the return to equilibrium problem one obtains a closed, explicit, ODE

Numerical simulations: Fixed object

Numerical simulations: Fixed object

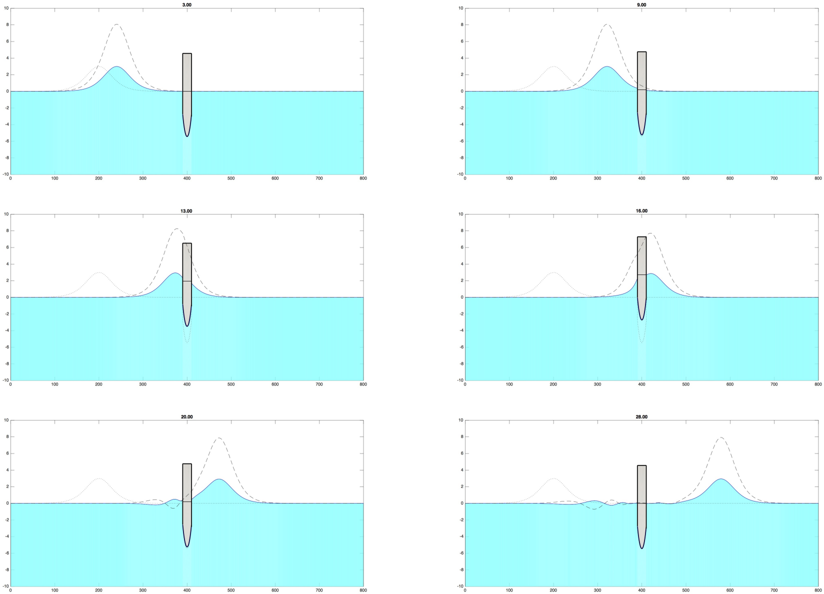
Numerical simulations: Forced motion

Numerical simulations: Floating

$\rightsquigarrow$ Validated with *explicit solution for the solid motion*

Numerical simulations: Floating

Numerical simulations: Floating with dispersive effects



Ongoing