Variational existence and stability theory for hydroelastic solitary waves

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Joint with Mark Groves and Benedikt Hewer, Saarland University

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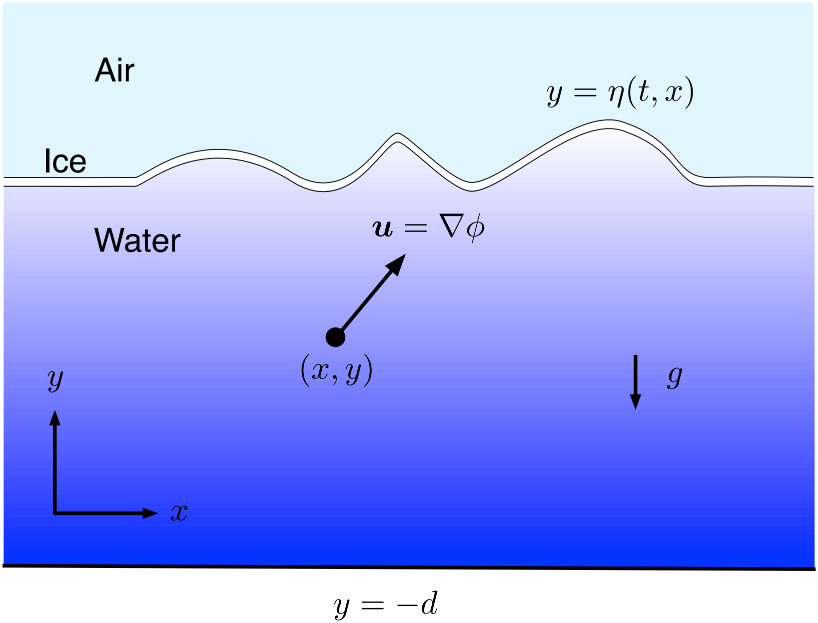

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$y = \eta(t, x)$
Ice
Water
$\boldsymbol{u} = \nabla\phi$
(x, y)
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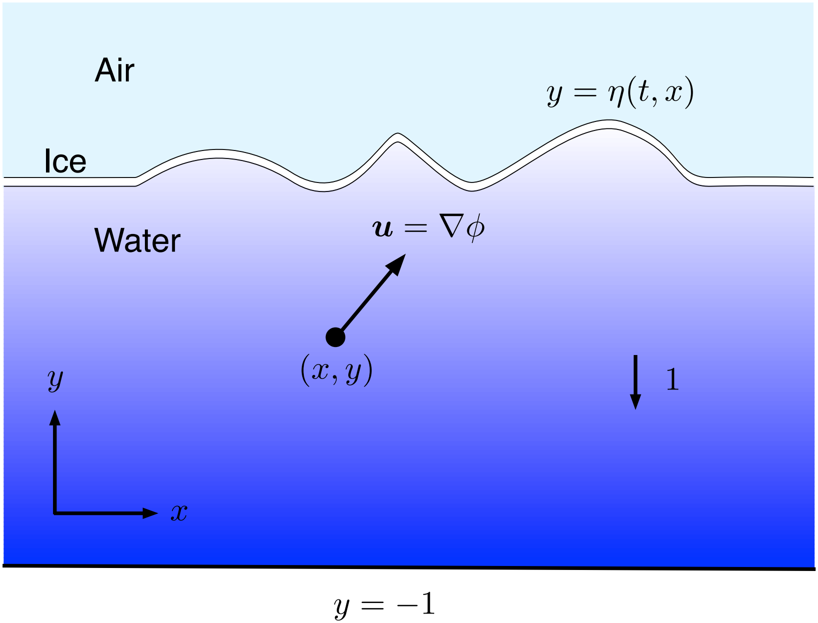
g
x
$y = -d$



Air
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Equations of motion

(Plotnikov & Toland '11, Blyth, Parau & Vanden-Broeck '11)

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\begin{aligned}
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\eta_t + \phi_x\eta_x - \phi_y &= 0 && y = \eta \\
\phi_t + \frac{1}{2}|\nabla\phi|^2 + \eta + \gamma H(\eta) &= 0 && y = \eta
\end{aligned}
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H(\eta) = \kappa_{ss} + \frac{1}{2}\kappa^3, \qquad \gamma = \frac{\mathcal{D}}{\rho g d^4}
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$$H(\eta) = \frac{1}{(1+\eta_x^2)^{\frac{1}{2}}}\left[\frac{1}{(1+\eta_x^2)^{\frac{1}{2}}}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}\right)_x\right]_x + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}\right)^3$$

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Travelling (solitary) wave: $\eta = \eta(x-ct)$, $\phi = \phi(x-ct, y)$

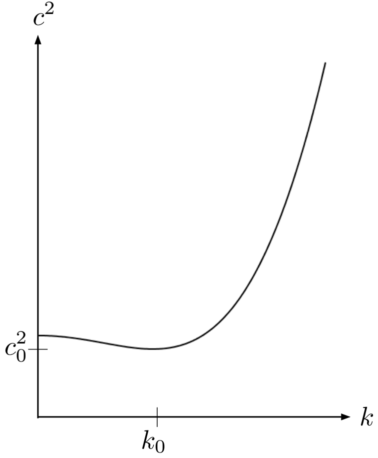
Other models

- Euler-Bernoulli: $H(\eta) = \eta_{xxxx}$
- Kirchhoff-Love: $H(\eta) = \partial_x^2 \kappa$

Heuristics

$$\eta(x-ct) \sim e^{ik(x-ct)}, \qquad c^2 = (1+\gamma k^4)\frac{\tanh k}{k}$$

Dispersion relation:



For $c = c_0 - \varepsilon^2$, one can derive the NLS equation

$$v_{xx} - v \pm |v|^2 v = 0$$

where

$$\eta(x) = \varepsilon v(\varepsilon x) e^{ik_0 x} + \text{c.c.} + O(\varepsilon^2)$$

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Focusing $(+)$ if $\gamma > \gamma_0 \approx 3.37 \times 10^{-10}$

Typical values:
$\gamma \approx 10^{-5}$ (McMurdo sound)
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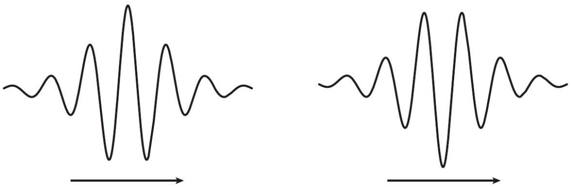
In the focusing case NLS has solitary waves



Stability theory

Hamiltonian formulation

$$\mathcal{H}(\eta, \xi) = \frac{1}{2} \iint_{\Omega} |\nabla \phi|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} \eta^2 \, dx + \frac{\gamma}{2} \int_{\mathbb{R}} \kappa^2 \, ds$$

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$$\mathcal{H}(\eta, \xi) = \frac{1}{2} \int_{\mathbb{R}} \left\{ \xi G(\eta) \xi + \eta^2 + \gamma \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\} dx$$

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Here $\xi = \phi|_{y=\eta}$ and $G(\eta)$ is the *Dirichlet-Neumann operator*

$$G(\eta)\xi = \sqrt{1+\eta_x^2}\,\partial_n\phi,$$

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Travelling waves are critical points of $\mathcal{H} - c\mathcal{I}$

Main Theorem

Theorem

Let $B_R(0) = \{\eta \in H^2(\mathbb{R})\colon \|\eta\|_{H^2} < R\}$, $R > 0$ *given.*

Assume that $\gamma > \gamma_0$ *and* $0 < \mu \ll 1$.

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- *The set* D_μ *of minimisers of* $\mathcal{H}(\eta, \xi)$ *subject to the constraint* $\mathcal{I}(\eta, \xi) = 2\mu$ *in the set* $B_R(0) \times H_\star^{1/2}(\mathbb{R})$ *is nonempty.*

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Function space for ξ:

$$H_\star^{1/2}(\mathbb{R}) = \{u \in H^s_{\mathsf{loc}}(\mathbb{R})\colon u' \in H^{-1/2}(\mathbb{R})\}/\mathbb{R}$$
$$\|u\|_{H_\star^{1/2}} = \|u'\|_{H^{-1/2}}$$

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- Better existence and stability results than for capillary-gravity waves
- So far no global existence results under these conditions
- Local well-posedness: Ambrose & Siegel (to appear in PRSE)

Proof of the theorem

Step 1. Reduction from phase space to configuration space

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- Fix η and minimise $\mathcal{H}(\eta, \xi)$ subject to $\mathcal{I}(\eta, \xi) = 2\mu$.
  Unique minimiser:

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$$\xi_\eta = c_\eta G(\eta)^{-1} \eta_x$$

- Minimise

$$\mathcal{J}(\eta) = \mathcal{H}(\eta, \xi_\eta) = \frac{\mu^2}{\mathcal{L}(\eta)} + \mathcal{K}(\eta)$$

  where

$$\mathcal{L}(\eta) = \frac{1}{2} \int_{\mathbb{R}} \eta_x G(\eta)^{-1} \eta_x \, dx$$

$$\mathcal{K}(\eta) = \int_{\mathbb{R}} \left\{ \frac{1}{2}\eta^2 + \gamma \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\} dx$$

Step 1.5. Periodic problem with large period

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- For $0 < \mu \ll 1$, $\exists$ test function

$$\begin{aligned}\eta_\star^\mu(x) = {} & \mu \operatorname{sech}(\mu x)\cos(k_0 x) \\ & - A\mu^2 \operatorname{sech}^2(\mu x)\cos(2k_0 x) - B\mu^2 \operatorname{sech}^2(\mu x),\end{aligned}$$

such that

$$\mathcal{J}(\eta_\star^\mu) < 2\mu$$

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To rule out dichotomy, we show that

$$I(\mu) = \inf\{\mathcal{J}(\eta) \colon \eta \in B_R(0)\}, \quad 0 < \mu \ll 1,$$

is strictly subhomogeneous:

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Implies strict subadditivity:

$$I(\mu_1 + \mu_2) < I(\mu_1) + I(\mu_2)$$

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Solutions:

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- The other problem is the interesting part

Idea:

For the test function we find that

$$\mathcal{J}(\eta_\star^\mu) = 2c_0\mu + I_{\mathsf{NLS}}\mu^3 + O(\mu^4),$$

where $I_{\mathsf{NLS}} < 0$ is the ground state energy for NLS

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Problem: Are near minimisers similar to $\eta_\star^\mu$?

Properties of minimising sequences

Any minimising sequence satisfies

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We use these properties to show that η_n has a form similar to $\eta_\star^\mu$

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- $\hat{\eta}_{n,2}$ *has support near* $0, \pm 2k_0$,
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- $$\|\eta_{n,3}\|_{H^2}^2 \le C\mu^5$$

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Also yields convergence of v_n to solution of NLS using compactness of minimising sequences for the NLS variational problem

Reference

M. D. Groves, B. Hewer & E. Wahlén, Variational existence theory for hydroelastic solitary waves, C. R. Math. Acad. Sci. Paris **354** (2016)

Thank you for staying awake!