Existence and qualitative theory for stratified solitary water waves

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We are interested in studying two-dimensional traveling water waves with heterogeneous density or stratification.

Stratification occurs naturally in many situations due to salinity or temperature gradients. This permits novel types of wavelike motion, and is believed to have significant effects on ocean circulation.

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Shifting to a moving reference frame, we assume that the wave inhabits a steady fluid region $\Omega \subset \mathbb{R}^2$:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}.$$

Here the axes are fixed so that the wave propagates in the positive x-direction with speed $c > 0$, and the mean ocean depth is d.

The unknowns in the system are

- velocity field $(u, v) : \Omega \to \mathbb{R}^2$,
- free surface profile $\eta : \mathbb{R} \to \mathbb{R}$,
- pressure $P : \Omega \to \mathbb{R}$, and
- density $\varrho : \Omega \to \mathbb{R}_+$.

Governing equations

$$\begin{cases} v = (u-c)\eta_x \\ P = P_{\mathrm{atm}} \end{cases}$$

$y = \eta(x)$

$$\begin{cases} (u-c)\varrho_x + v\varrho_y = 0 \\ \varrho(u-c)u_x + \varrho v u_y = -P_x \\ \varrho(u-c)v_x + \varrho v v_y = -P_y - g\varrho \\ u_x + v_y = 0 \end{cases}$$

$y = -d$

$v = 0$

We assume the absence of horizontal stagnation:

$$u - c < 0 \qquad \text{in } \overline{\Omega}.$$

One implication of this is that the integral curves of the relative velocity field $(u - c, v)$, called the streamlines, extend from $x = -\infty$ to $x = +\infty$.

A solitary wave is a solution of the above system that is spatially localized:

$$
\begin{aligned}
(u - c, v) &\longrightarrow (-Fu^*, 0) \\
\varrho &\longrightarrow \varrho^* \\
\eta &\longrightarrow 0
\end{aligned}
$$

as $|x| \to \infty$, uniformly in y, where

- ϱ^* is a given upstream density profile,
- $u^* > 0$ is a given (scaled) asymptotic relative velocity, and
- $F > 0$ is the Froude number.

The continuity equation

$$(u - c)\varrho_x + v\varrho_y = 0 \qquad \text{in } \Omega$$

implies that ϱ is constant on the streamlines. We therefore introduce a streamline density function $\rho : [-1, 0] \to \mathbb{R}_+$ that prescribes the value of ϱ on each streamline. For physical reasons, we assume that ρ is decreasing (stable stratification).

It will turn out that there is a critical Froude number, F_{cr}, that plays an important role in determining the structure of solutions; we say that a solution with $F > F_{\mathrm{cr}}$ is supercritical.

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Finally, let us introduce some terminology for describing the qualitative features of these waves. A traveling wave is called

- laminar if all of its streamlines are parallel to the bed,
- wave of elevation if it is a solitary wave where the height of each streamline above the bed attains its minimum value only at infinity,
- symmetric provided that u and η are even in x while v is odd.
- A symmetric wave of elevation is monotone if the height of every streamline (except the bed) is strictly decreasing on either side of the crest line $\{x = 0\}$.

Theorem (Existence).

Let a Hölder exponent $\alpha \in (0, 1/2]$; streamline density function $\rho \in C^{2+\alpha}([-1, 0], \mathbb{R}_+)$; and positive asymptotic relative velocity $u^* \in C^{2+\alpha}([-d, 0], \mathbb{R}_+)$ be given.

There exists a continuous curve

$$\mathcal{C} = \{(u(s), v(s), \eta(s), F(s)) : s \in (0, \infty)\}$$

of solitary waves with the regularity

$$(u(s), v(s), \eta(s)) \in C^{2+\alpha}(\overline{\Omega(s)}) \times C^{2+\alpha}(\overline{\Omega(s)}) \times C^{3+\alpha}(\mathbb{R}),$$

where $\Omega(s)$ denotes the fluid domain corresponding to $\eta(s)$.

(i) (Stagnation limit) Following $\mathcal{C}$, we encounter waves that are arbitrarily close to having points of (horizontal) stagnation:

$$\liminf_{s \to \infty} \inf_{\Omega(s)} |c - u(s)| = 0.$$

(ii) (Critical laminar flow) The left endpoint of $\mathcal{C}$ is a critical laminar flow,

$$\lim_{s \to 0} (u(s) - c, v(s), \eta(s), F(s)) = (-F_{\mathrm{cr}} u^*, 0, 0, F_{\mathrm{cr}}).$$

(iii) (Symmetry and monotonicity) Every solution in $\mathcal{C}$ is a wave of elevation that is symmetric, monotone, and supercritical.

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Let us point out a few features of this theorem.

- u^* above is allowed to be an arbitrary smooth laminar profile.
- This is the first construction of rotational solitary waves, even without stratification, where the stagnation limit is known to hold for arbitrary u^*.
- To the best of our knowledge, this is the first large-amplitude existence theory for stratified waves in any regime (channel/free surface, periodic/solitary, one-phase/multiple-phase) where stagnation is guaranteed.

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Outline of argument

We begin by non-dimensionalizing and then changing coordinate so that Ω is mapped to a fixed domain. Specifically, we use the Dubreil-Jacotin transformation:

$$(x, y) \mapsto (x, -\psi) =: (q, p),$$

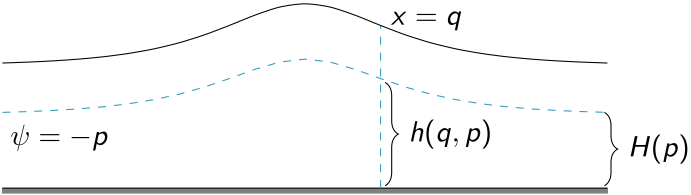
where ψ is the pseudo relative stream function defined uniquely by

$$\nabla^\perp \psi = \sqrt{\varrho}(u - c, v), \qquad \psi|_{y=\eta} = 0.$$

Our non-dimensionalization ensures that $\psi = 1$ on the bed, and hence the image of Ω is the infinite strip

$$R := \mathbb{R} \times (-1, 0),$$

where $\{p = 0\}$ corresponds to the free surface and $\{p = -1\}$ the bed.



We also reformulate the problem in terms of a new unknown $h = h(q, p) \in C^{3+\alpha}(\overline{R})$, that describes the height above the bed of the point with $x = q$ and lying on the streamline $\{-\psi = p\}$.

The asymptotic structure is determined by $H = H(p)$, the height of the streamlines at infinity, and $\rho = \rho(p)$.

The governing equation then becomes the following quasilinear PDE with completely nonlinear boundary conditions:

$$\begin{cases} \left(-\dfrac{1+h_q^2}{2h_p^2}+\dfrac{1}{2H_p^2}\right)_p+\left(\dfrac{h_q}{h_p}\right)_q-\dfrac{1}{F^2}\rho_p(h-H)=0 & \text{in } R, \\ \dfrac{1+h_q^2}{2h_p^2}-\dfrac{1}{2H_p^2}+\dfrac{1}{F^2}\rho(h-1)=0 & \text{on } p=0, \\ h=0 & \text{on } p=-1. \end{cases}$$

The effect of stratification is the presence of a zeroth order term that has the "bad sign" in the sense that it violates the hypotheses of the maximum principle. This is a major technical annoyance.

We will frequently rewrite this in terms of

$$w = w(q,p) := h(q,p) - H(p),$$

which measures the deviation of the streamlines from their asymptotic height.

We formulate the entire system abstractly as

$$\mathscr{F}(w,F) = 0,$$

where $\mathscr{F} : X \times \mathbb{R} \to Y$ and

$$X := \left\{ w \in C^{3+\alpha}_{\mathrm{b,e}}(\overline{R}) \cap C^2_0(\overline{R}) : w(\cdot,-1) = 0 \right\},$$

and $Y = Y_1 \times Y_2$ for

$$Y_1 := C^{1+\alpha}_{\mathrm{b,e}}(\overline{R}) \cap C^0_0(\overline{R}), \qquad Y_2 := C^{2+\alpha}_{\mathrm{b,e}}(T) \cap C^1_0(\overline{R}).$$

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Small-amplitude theory

The first step is to construct small-amplitude solutions bifurcating from the critical laminar flow.

Our strategy is to use a spatial dynamics method in the spirit of [Groves and Wahlén, 2008], who studied small-amplitude homogeneous rotational solitary waves.

A few comments:

- It is not immediately obvious that there is a critical laminar flow due to the bad sign of ρ_p.
- Once the spectral theory is established, we show that one can reformulate the problem as a Hamiltonian system that has a 0^2 resonance at F_{cr}.

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Abstract global continuation theory

The result of the above analysis is a local curve of solutions

$$\mathcal{C}_{\mathrm{loc}} = \{(\tilde{w}(s), \tilde{F}(s)) : s \in (0,1)\}.$$

To prove the theorem, we must first continue $\mathcal{C}_{\mathrm{loc}}$ to a globally defined solution curve.

Because the mapping $\mathscr{F}$ is real analytic, a promising tool is the analytic global bifurcation theory of [Dancer, 1973] and [Buffoni and Toland, 2003]. However, there are a number of obstacles:

- The point of bifurcation is on the boundary of the set where $\mathscr{F}_w$ is Fredholm; and
- we do not expect in general that $\mathscr{F}^{-1}(0)$ is locally compact.

Both of these are common features of bifurcation theoretic studies of solitary waves.

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Theorem. (Analytic continuation)

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $\mathcal{I} \subset \mathbb{R}$ an open interval, and let $\mathcal{U} \subset \mathcal{X}$ be an open set with $0 \in \partial\mathcal{U}$.

For an analytic map $\mathcal{G} : \mathcal{U} \times \mathcal{I} \to \mathcal{Y}$, consider the set of solutions $\mathcal{Z} := \mathcal{G}^{-1}(0)$. Assume that:

(i) $\mathcal{G}_x(x, \lambda)$ is Fredholm with index 0 for any $(x, \lambda) \in \mathcal{Z}$,

(ii) there exists a continuous solution curve

$$\mathscr{C}_{\mathrm{loc}} = \{(\tilde{x}(s), \tilde{\lambda}(s)) : s \in (0, 1)\} \subset \mathcal{Z}$$

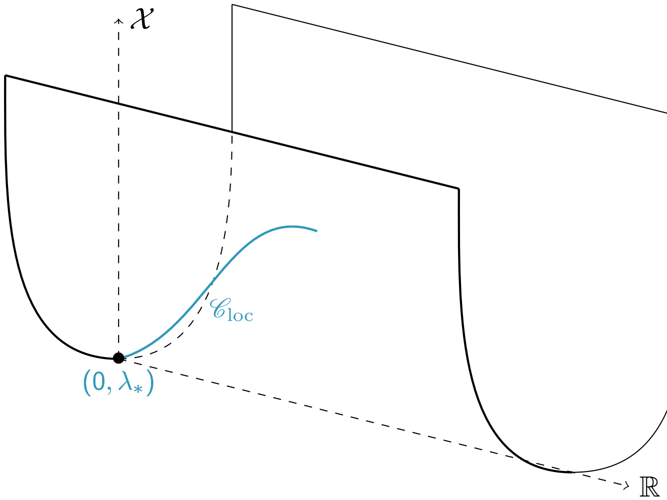
with $\lim_{s \searrow 0} \tilde{x}(s) = 0$, and

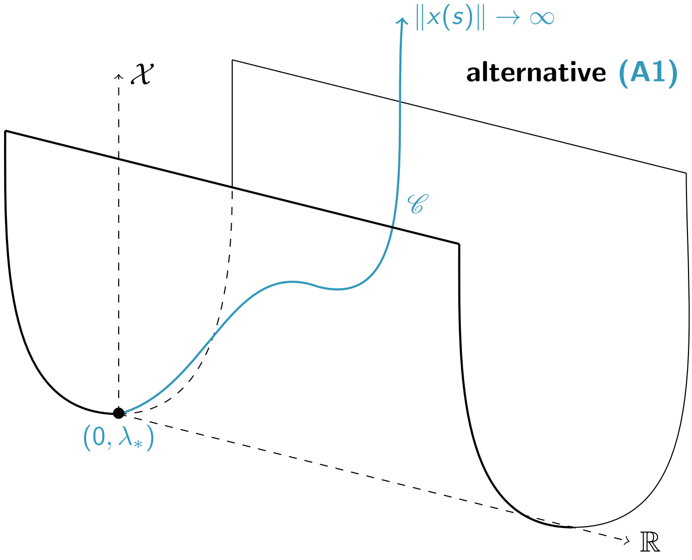
$$\mathcal{G}_x(x, \lambda) : \mathcal{X} \to \mathcal{Y} \text{ is invertible for all } (x, \lambda) \in \mathscr{C}_{\mathrm{loc}}.$$

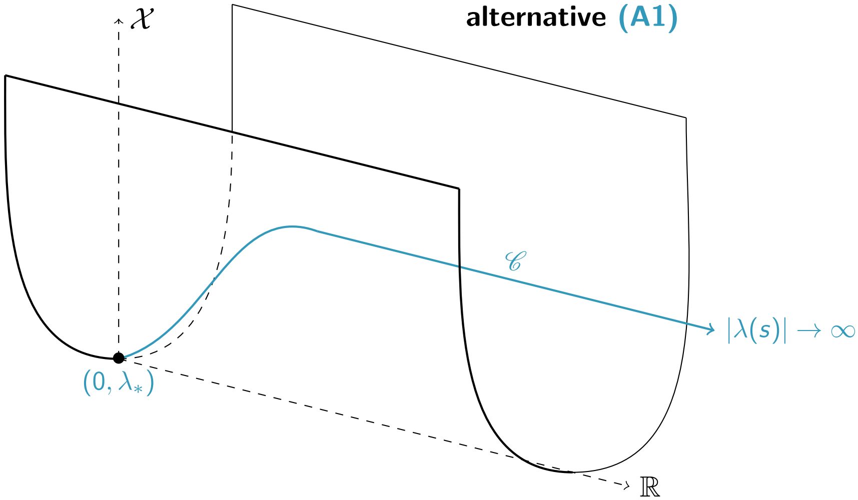
Then there is a continuous path

$$\mathscr{C} = \{(x(s), \lambda(s)) \in \mathcal{U} \times \mathcal{I} : s \in (0, \infty)\} \subset \mathcal{Z}$$

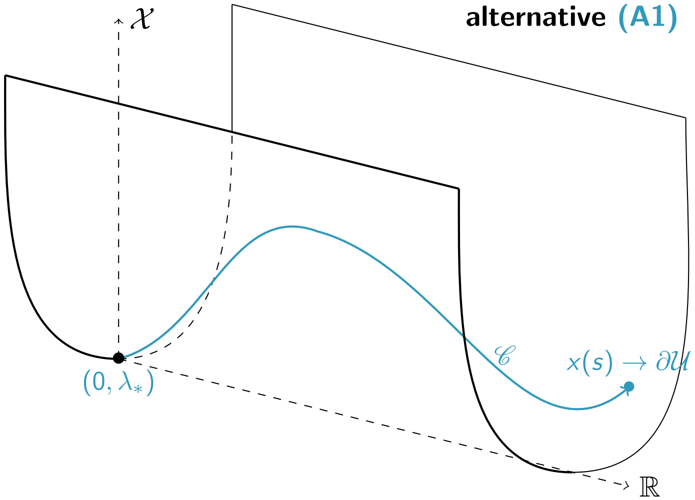
extending $\mathscr{C}_{\mathrm{loc}}$ along which one of the following alternatives must hold:


$\mathcal{X}$
$\mathscr{C}_{\mathrm{loc}}$
$(0, \lambda_*)$
$\mathbb{R}$

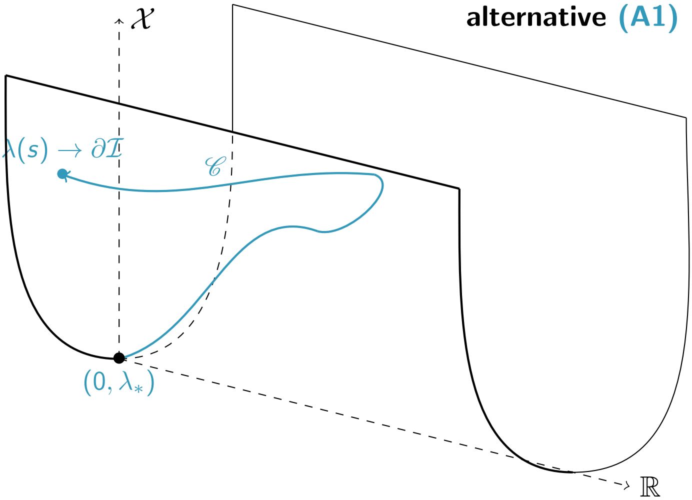

$\|x(s)\| \to \infty$
$\mathcal{X}$
alternative **(A1)**
$\mathscr{C}$
$(0, \lambda_*)$
$\mathbb{R}$



alternative **(A1)**
$\mathcal{X}$
$\mathscr{C}$
$|\lambda(s)| \to \infty$
$(0, \lambda_*)$
$\mathbb{R}$



alternative **(A1)**
$\mathcal{X}$
$\mathscr{C}$
$x(s) \to \partial\mathcal{U}$
$(0, \lambda_*)$
$\mathbb{R}$



alternative **(A1)**
$\mathcal{X}$
$\lambda(s) \to \partial\mathcal{I}$
$\mathscr{C}$
$(0, \lambda_*)$
$\mathbb{R}$

Or,

alternative (A2): There exists a sequence $s_n \to \infty$ such that

$$\sup_n \left[\|x(s_n)\|_{\mathcal{X}} + \frac{1}{\mathrm{dist}\,(x(s_n), \partial\mathcal{U})} + |\lambda(s_n)| + \frac{1}{\mathrm{dist}\,(\lambda(s_n), \partial\mathcal{I})} \right] < \infty$$

but $\{x(s_n)\}$ has no subsequences converging in $\mathcal{X}$.

Alternative (A2) is the price for removing the assumption that $\mathcal{F}^{-1}(0)$ is locally compact. It is essential to be able to either eliminate it, or find a more meaningful interpretation.

Specializing to the case of a nonlinear elliptic PDE, we can characterize precisely the way in which (A2) will occur provided that more qualitative properties of the solutions are known.

Alternative (A2) is the price for removing the assumption that $\mathcal{F}^{-1}(0)$ is locally compact. It is essential to be able to either eliminate it, or find a more meaningful interpretation.

Specializing to the case of a nonlinear elliptic PDE, we can characterize precisely the way in which (A2) will occur provided that more qualitative properties of the solutions are known.

Let $\Omega := \mathbb{R} \times B$ be an infinite cylinder whose base $B \subset \mathbb{R}^{n-1}$ is a bounded $C^{2+\alpha}$ domain. Denote points in Ω as (x, y) where $x \in \mathbb{R}$ and $y \in B$. Partition the components of ∂B as $\partial B = \partial_1 B \cup \partial_2 B$ (either may be empty).

Consider a nonlinear elliptic problem

$$\begin{cases} \mathcal{F}(y, u, Du, D^2 u, \lambda) = 0 & \text{in } \Omega, \\ \mathcal{G}(y, u, Du, \lambda) = 0 & \text{on } \mathbb{R} \times \partial_1 B, \\ u = 0 & \text{on } \mathbb{R} \times \partial_2 B, \end{cases}$$

where the parameter $\lambda \in \mathbb{R}^m$, and

$$\mathcal{F} \in C^{\alpha}_{\mathrm{b}}(\overline{B} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n \times n} \times \mathbb{R}^m),$$
$$\mathcal{G} \in C^{1+\alpha}_{\mathrm{b}}(\mathbb{R} \times \partial_1 B \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m).$$

Moreover, assume that $\mathcal{F}$ is uniformly elliptic and $\mathcal{G}$ is uniformly oblique

Lemma (Compactness or front).
Suppose that
$$\{(u_n, \lambda_n)\} \subset C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega}) \times \mathbb{R}$$
is a uniformly bounded sequence of solutions such that, for each $n \geq 1$,
$$u_n \text{ is even in } x \qquad \text{and} \qquad \partial_x u_n \leq 0 \text{ for } x \geq 0,$$
and
$$\lim_{|x|\to\infty} u_n(x, \cdot) = U$$
for some fixed function $U \in C_{\mathrm{b}}^{2+\alpha}(\overline{B})$.

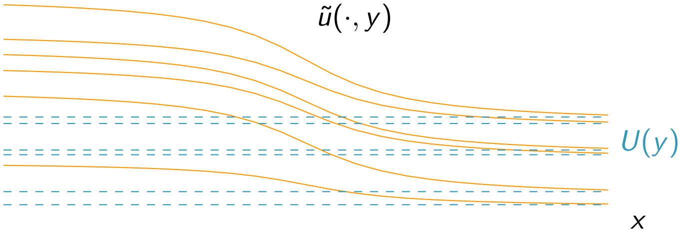
Then, either

(i) (Compactness) $\{u_n\}$ precompact in $C_b^{2+\alpha}(\overline{\Omega})$; or

(ii) (Front) there exists a sequence $x_n \to +\infty$ so that (modulo a subsequence)

$$u_n(\cdot + x_n,\, \cdot) \quad \longrightarrow_{C^2_{\mathrm{loc}}} \qquad \widetilde{u} \in C_b^{2+\alpha}(\overline{\Omega}),$$

where $\widetilde{u}$ is a solution with $\partial_x \widetilde{u} \le 0$ and $\widetilde{u} \not\equiv U$.



Qualitative theory

To apply the "compactness or front" lemma to our system, we must first show that the waves we construct are even and monotonic. This follows from the following more general result.

Theorem (Symmetry)
Let (u, v, η, F) be a supercritical wave of elevation with

$$\|u\|_{C^2(\Omega)},\ \|v\|_{C^2(\Omega)},\ \|\eta\|_{C^3(\mathbb{R})} < \infty,$$

and

$$(u, v) \to (\mathring{u}, 0), \quad (u_y, v_y) \to (\mathring{u}_y, 0) \quad \text{uniformly as } x \to +\infty \text{ (or } -\infty\text{)}.$$

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Then the wave is necessarily even and monotonic.

Next, consider the front-type solution described by the compactness or front lemma. In water waves, we would usually think of this as a hydraulic bore.

Bores have been studied extensively in the applied literature, particularly in channel flows. They have been computed numerically, and rigorous existence results are given by [Amick, 1989], [Makarenko, 1991], and [Tuleuov, 1997].

Typically, the existence/nonexistence of bores hinges on conjugate flow analysis, which is only tractable in a small number of special cases.

All of this is very worrying, since there seems to be no reason that one can exclude the possibility of a bore limit in the global theory.

However. . .

Theorem. (Nonexistence of free surface monotone bores)
Suppose that $h \in C_{\mathrm{b}}^2(\overline{R})$ is a bore solution in the sense that

$$h(q, p) \to H_\pm(p) \qquad \text{as } q \to \pm\infty$$

with $\inf_R h_p > 0$. If

$$H_+ \geq H_- = H \text{ on } [-1, 0] \quad \text{or} \quad H_+ \leq H_- = H \text{ on } [-1, 0],$$

then

$$H_+ = H_- = H.$$

A similar result holds with the roles of H_- and H_+ reversed.

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A similar result holds with the roles of H_- and H_+ reversed.

Now, combining the symmetry and‘ “compactness or front” lemma with the above nonexistence of bores theorem, we conclude that alternative (A2) does not occur!

It remains now to study the quantity

$$N(s) := \|w(s)\|_X + \frac{1}{\operatorname{dist}(w(s), \partial U)} + F(s) + \frac{1}{F(s)},$$

and in particular, we must make sense of the statement $N(s) \to \infty$.

Now, combining the symmetry and‘“compactness or front” lemma with the above nonexistence of bores theorem, we conclude that alternative (A2) does not occur!

It remains now to study the quantity

$$N(s) := \|w(s)\|_X + \frac{1}{\operatorname{dist}\left(w(s), \partial U\right)} + F(s) + \frac{1}{F(s)},$$

and in particular, we must make sense of the statement $N(s) \to \infty$.

Theorem (Bounds on F).
The Froude number has the following upper bound

$$F \leq \frac{1}{\pi} \frac{gd}{\min(u^*)^2} \frac{\max \varrho}{\min \varrho} \frac{\sqrt{gd}}{\inf_{\{x=0\}}(c-u)}.$$

Note that we make no additional assumptions on the shear profile u^*, and the above bound implies that $F(s) \to \infty$ forces stagnation (along the crest line).

[Wheeler, 2014] established upper bounds on F that are independent of $\inf_\Omega(c-u)$, but imposed additional requirements on u^*.

In the special case of homogeneous irrotational waves, the argument leading to the above theorem can be modified to recover the estimates of [Starr, 1947], and [Keady and Pritchard, 1974].

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Theorem (Critical flows are laminar)
If $F = F_{\mathrm{cr}}$, then

$$(u, v, \eta) = (c - F_{\mathrm{cr}} u^*, 0, 0).$$

This serves as a type of lower bound for the Froude number; it says that a curve of supercritical waves cannot limit to a subcritical wave without first encountering the critical laminar flow.

However, recall that the analytic continuation theorem states that

$$(w(s), F(s)) \notin \mathscr{C}_{\mathrm{loc}}, \qquad \text{for } s \gg 1.$$

Thus the alternative $F(s) \to 0$ is excluded.

A similar argument also shows that $w(s)$ cannot limit to a laminar flow.

Theorem (Bounds on pressure and velocity).
If $F \geq F_0 > F_{\mathrm{cr}}$, then

$$P - P_{\mathrm{atm}} + MF\psi \geq 0, \qquad (u-c)^2 + v^2 \leq CF^2 \qquad \text{in } \overline{\Omega},$$

where the constants C and M depend only on u^*, ϱ^*, g, d, F_0.

This shows that blow up of the (u, v) is possible only if $F \to \infty$, which we now know also the stagnation limit.

Now. . .

Because (A1) occurs

$$N(s) = \|w(s)\|_X + \frac{1}{\operatorname{dist}(w(s), \partial U)} + F(s) + \frac{1}{F(s)} \to \infty.$$

The above results imply that

$$N(s) \to \infty \qquad \Longleftrightarrow \qquad \|w(s)\|_X + F(s) \to \infty.$$

Finally, via an elliptic regularity argument, we can show that

$$\|w(s)\|_X \to \infty \qquad \Longleftrightarrow \qquad \|(\partial_p w)(s)\|_{C^0(\overline{R})} \to \infty,$$

which translated to the Eulerian variables implies that $|(u(s), v(s)|$ blows up, and hence either stagnation occurs, or $|(u - c, v)|$ blows up.

By the previous results, the latter of these also implies stagnation, and hence the proof is complete.

Thanks for your attention