Equations of loci in tables of commuting Jordan types

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March 2016

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- The commutator of P:
  $\mathcal{C}_P = \{A \in \mathcal{M}at_n(\mathsf{k}) \,|\, [A, J_P] = 0\}$.
- The nilpotent commutator of P:
  $\mathcal{N}_P = \{A \in \mathcal{C}_P \,|\, A^n = 0\}$.

Proposition.

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Describe $Q(P)$ in terms of P.

Dominance order

For $R = [r_1, r_2, \cdots] \vdash n$ and $Q = [q_1, q_2, \cdots] \vdash n$,

$$R \preccurlyeq Q \quad \text{iff } \mathcal{O}_R \subseteq \overline{\mathcal{O}}_Q$$

$$\text{iff } \sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} q_i, \text{ for all } k \geq 1.$$

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• More Generally (Panyushev): Let G be a connected simple algebraic group and $\mathfrak{g}$ =Lie G. For a nilpotent $e \in \mathfrak{g}$, let $\mathcal{O} = G.e$ and define $Q(\mathcal{O})$ as the largest nilpotent orbit meeting the centraliser of e.

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$$J_{[7]}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$J_{[3,2,2]} = \left[\begin{array}{ccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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(Basili '03) Number of parts of $Q(P) = r(P)$ where $r(P)$ is $\min\{r \mid P = [P_1, \cdots, P_r]$ s.t. each P_i is almost rectangular$\}$.

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Example.

If $P = [5, 4, 3, 3, 2, 1]$ then $[5, 4 \quad 3, 3, 2 \quad 1]$ or $[5 \quad 4, 3, 3 \quad 2, 1]$ etc. So $Q(P)$ has 3 parts.

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Theorem.(Basili-Iarrobino '08)

$Q(P) = P$ iff parts of P differ pairwise by at least 2.

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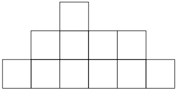

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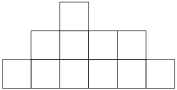

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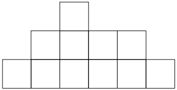

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Let char $\mathsf{k} = 0$ or $> n$. If $A \in \mathcal{N}_P$ is generic then $\mathcal{A}$ is Gorenstein.

$\Rightarrow$[Macaulay] $H(\mathcal{A}) = (1, 2, \cdots, d, h_d, h_{d+1}, \cdots, h_k)$ such that $d \geq h_d \geq h_{d+1} \geq \cdots \geq h_k = 1$ and $h_{i-1} - h_i \leq 1$, for all i.

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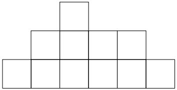

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Corollary.

For all P, parts of $Q(P)$ differ pairwise by at least 2, and therefore $Q(Q(P)) = Q(P)$. So the map $\mathcal{Q}$ is idempotent.

Poset $\mathcal{D}_P$

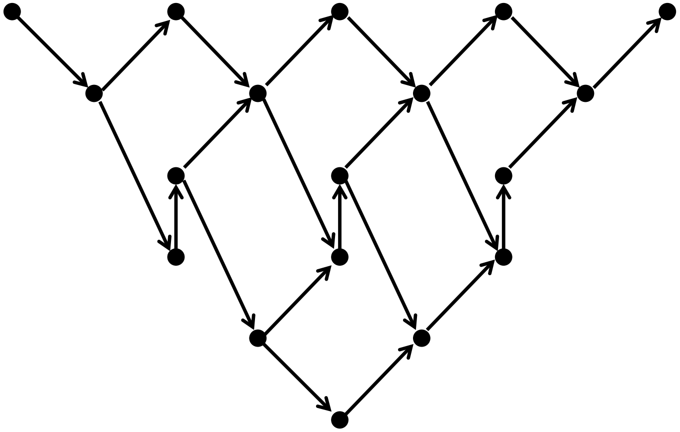
Let V be an n-dimensional k vectors space and $X \in \mathrm{End}(V)$ be nilpotent of Jordan type $P \vdash n$. We define the poset $\mathcal{D}_P$ as follows: ([Basili-Iarrobino-K, 10] inspired by P. Oblak's work)

$$\mathcal{D}_P = \text{the basis of } V \text{ in which } X \text{ is given by } J_P;$$

$$v' < v \text{ in } \mathcal{D}_P \iff \exists\, A \in \mathcal{U}_P \text{ such that } Av\,|_{v'} \neq 0.$$

$\mathcal{U}_P$ is a maximal nilpotent subalgebra of $\mathcal{C}_P$.
$\forall\, N \in \mathcal{N}_P$, $\exists\, C \in \mathcal{C}_P$ s.t. $CNC^{-1} \in \mathcal{U}_P$.

Let P=[5,4,3,3,2,1].



Properties of $\mathcal{D}_P$

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The set of vertices that belong to an almost rectangular subpartition of P and the first and last vertices of every row above them make a (maximum) chain in $\mathcal{D}_P$. Such a chain is called a simple U-chain.

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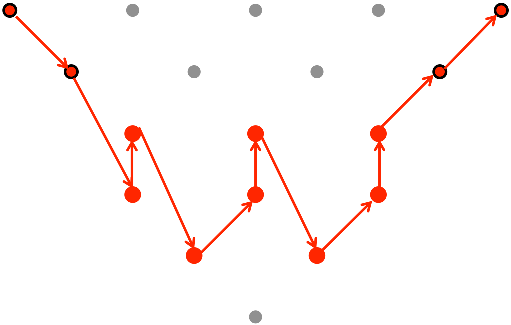

Theorem.
([Oblak, '08]) Let $P = (\cdots, p^{n_p}, \cdots)$ with $n_p \geq 0$. Then the biggest part of $Q(P)$ is $i(P)$ defined as

$$\max\{an_a + (a+1)n_{a+1} + 2\sum_{p>a+1} n_p \,|\, a \geq 1\}.$$

Theorem.
([K, '14]) Let $P = (p_s^{n_s}, \cdots p_1^{n_1})$ with $n_i > 0$. Then the smallest part of $Q(P)$, is $\mu(P)$ defined as follows.
If $p_{i+1} = p_i + 1$ for $1 \leq i \leq s$ (P is a "spread"), then

$$\mu(P) = \min\{p_1\, n_{2i-1} + (p_1+1)n_{2j} \,|\, 1 \leq i \leq j \leq r(P)\}.$$

For an arbitrary P, write $P = (P_\ell, \cdots, P_1)$ such that each P_k is a spread. Then $\mu(P) = \min\{\mu(\overline{P}_k) \,|\, 1 \leq k \leq r(P)\}$, where $\overline{P}_k$ is obtained from P_k subtracting $2\sum_{i=1}^{k-1} r(P_i)$ from each part.

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Example

Let $P = [5, 4, 3, 3, 2, 1]$. Then $i(P) = 12$ and $\mu(P) = 1$. So $Q(P) = [12, 5, 1]$.

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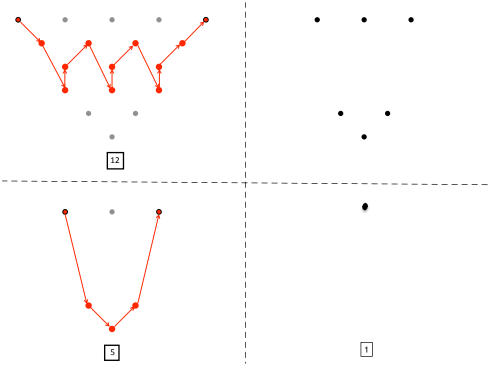

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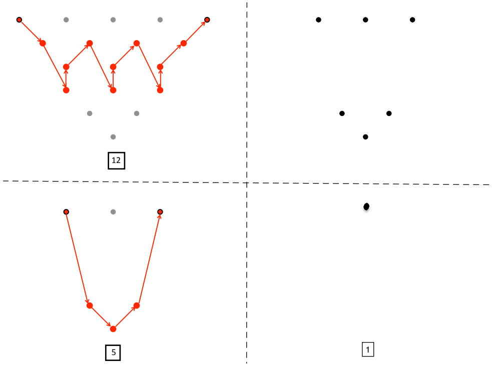

[K '13] The partition from Oblak's conjecture is well-defined.

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[Basili-Iarrobino-K, '10] Let C be a simple U-chain in $\mathcal{D}_P$ and let P' be the partition that corresponds to $\mathcal{D}_P \setminus C$. If C' is a simple U-chain in $\mathcal{D}'_P$ then $C \cup C'$ is the union of two chains in the original poset $\mathcal{D}_P$.

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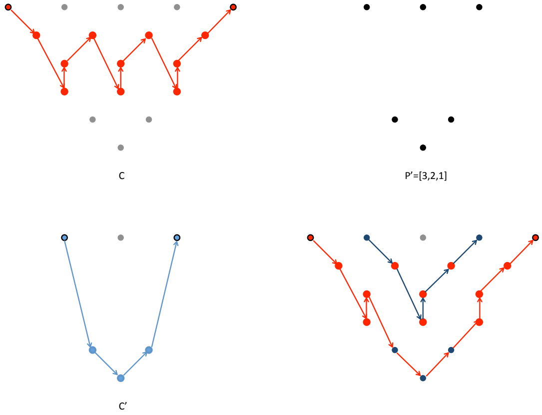

w/ A. Iarrobino, B. Van Steirteghem, and R. Zhao

Let Q be a stable partitions of n. What can we say about

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Fact.

For $Q = [n]$, $\mathcal{Q}^{-1}(Q)$ is the set of all almost rectangular partitions of n.

$$[n], \quad [n]^2, \quad \cdots, \quad [n]^n$$

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Theorem

(Iarrobino, K, Van Steirteghem, and R. Zhao)
Let $Q = (u, u-r)$ for $r \geq 2$ and $u > r$. Then all partitions in $\mathcal{Q}^{-1}(Q)$ can be arranged in a $(r-1) \times (u-r)$ table, $\mathcal{T}(Q)$, such that the partition in row k and column ℓ of the table has $k+\ell$ parts.

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Example.

Let $Q = [8, 3]$.

$(8, 3)$	$(8, [3]^2)$	$(8, [3]^3)$
$(5, [6]^2)$	$([8]^2, [3]^2)$	$([8]^2, [3]^3)$
$(5, [6]^3)$	$([7]^2, [4]^3)$	$([7]^2, [4]^4)$
$(5, [6]^4)$	$(5, [6]^5)$	$(5, [6]^6)$

$\mathcal{T}(Q)$ for $Q = [8,3]$:

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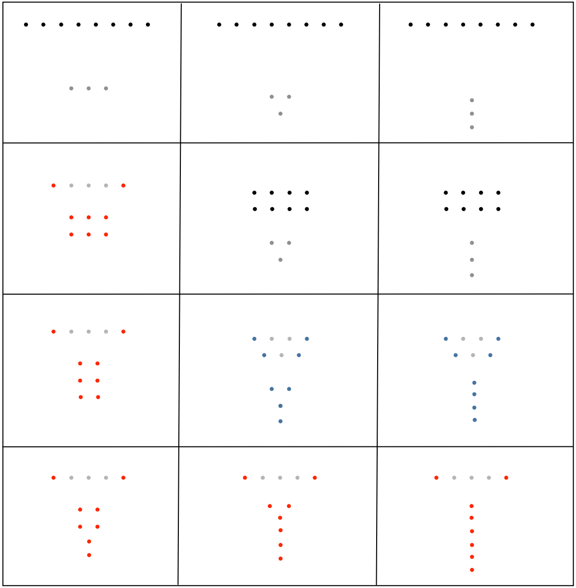

Loci equations for $\mathcal{Q}^{-1}(Q)$
(w/ M. Boij, A. Iarrobino, B. Van Steirteghem, and R. Zhao)
For $Q = [n]$, $A \in \mathcal{U}_Q \Leftrightarrow$

$$A = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_2 \\ 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Matrices in $\mathcal{N}_{[n]}$ with partition $[n]^\ell$ are defined by

$$a_1 = \cdots = a_{\ell-1} = 0.$$

$[n]$	$[n]^2$	$[n]^3$	$\cdots\ \cdots$	$[n]^n$

$-$	$a_1 = 0$	$a_1 = a_2 = 0$	$\cdots\ \cdots$	$a_1 = \cdots = a_{n-1} = 0$

For $Q = (8,3)$

$$A \in \mathcal{N}_{(8,3)} \Leftrightarrow A = \left(\begin{array}{cccccccc|ccc} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & f_1 & f_2 & f_3 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & f_1 & f_2 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & f_1 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 & 0 & 0 \end{array}\right)$$

$\mathcal{Q}^{-1}(8,3)$:

$(8,3)$	$(8,[3]^2)$	$(8,[3]^3)$
$(5,[6]^2)$	$([8]^2,[3]^2)$	$([8]^2,[3]^3)$
$(5,[6]^3)$	$([7]^2,[4]^3)$	$([7]^2,[4]^4)$
$(5,[6]^4)$	$(5,[6]^5)$	$(5,[6]^6)$

	$b_1 = 0$	$b_1 = b_2 = 0$
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$a_1 = a_2 = 0$	$a_1 = a_2 = b_1 = 0$	$a_1 = a_2 = b_1 = q_1 = 0$
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0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & f_1 & f_2 & f_3 \\
0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & f_1 & f_2 \\
0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & f_1 \\
0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & 0 & b_1 & b_2 \\
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$a_1 = a_2 = a_3 = 0$	$a_1 = a_2 = a_3 = q_2 = 0$	$a_1 = a_2 = a_3 = q_1 = Q_2 = 0$

Here $q_1 = \begin{vmatrix} a_3 & f_1 \\ g_1 & b_2 \end{vmatrix}$, $q_2 = \begin{vmatrix} a_4 & f_1 \\ g_1 & b_1 \end{vmatrix}$ and $Q_2 = \begin{vmatrix} a_4 & f_1 \\ g_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_5 & f_2 \\ g_1 & b_1 \end{vmatrix}$.

Definition.

Let $Q = (u, u-r)$, $r \geq 2$. Let $\mathfrak{Z}_{k,\ell}$ denote the locus in $\mathbb{P}(\mathcal{N}_Q)$ defined by functions vanishing on

$$\{A \in \mathcal{N}_Q \mid A \text{ has Jordan type } P_{k,\ell} \in \mathcal{T}(Q)\}.$$

Definition.

Let $Q = (u, u-r)$, $r \geq 2$. Let $\mathfrak{Z}_{k,\ell}$ denote the locus in $\mathbb{P}(\mathcal{N}_Q)$ defined by functions vanishing on

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Conjecture.

The variety $\mathfrak{Z}_{k,\ell}$ is an irreducible complete intersection cut out by $k+\ell-2$ equations in the coordinates of $\mathbb{P}(\mathcal{N}_Q)$. Of these, $\min\{k+\ell-2, r-2\}$ are linear and the rest are quadratic.

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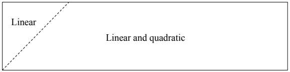

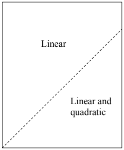

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The m-th quadratic equation in a single B/C hook or A row is the sum of m two by two minors of a $2m \times 2$ matrix that depends on the row or hook, and is a submatrix of the matrix that corresponds to the previous quadratic equation in the same hook or row.

Thank you!