Neutral genetic patterns for expanding populations

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Outline

Research Questions

1. How do nonlinear dynamics drive genetic patterns of population spread?
   - Connect ecological concepts with mathematical tools.

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    - Allee effects.
    - Overcompensation.

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   - Connect ecological concepts with mathematical tools.
2. How does growth/dispersal alter the genetic diversity of an expanding population?
   - Thin-tailed versus fat-tailed dispersal kernels.
   - Allee effects.
   - Overcompensation.
3. Applications to a biological system?
   - Range expansion of mountain pine beetle

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- **Adaptive genetic diversity**
  - Helps organisms cope with current environmental variability.
  - A diverse array of genotypes are especially important in disease resistance.
  - Diversity within populations reduces potentially negative effects of breeding among close relatives.

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  - A diverse array of genotypes are especially important in disease resistance.
  - Diversity within populations reduces potentially negative effects of breeding among close relatives.
- **Neutral genetic diversity**
  - Gene variants do not have any direct effect on fitness.
  - Useful for investigating processes as gene flow, migration, or dispersal.

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Ecology concepts

- *Founder effect*: The establishment of a new population by a few original founders which carry only a fraction of the total genetic variation.[2]

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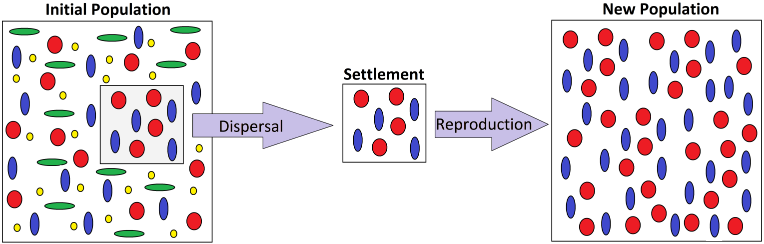

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What are inside dynamics?

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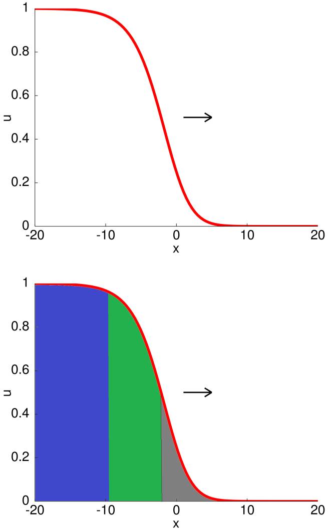
- The term "inside dynamics" refers to studying the underlying structure of the population.
- From a mathematical standpoint we classify the inside dynamics of traveling wave solutions as pulled or pushed fronts.



Previous work

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INSIDE DYNAMICS OF SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS

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Allee effect promotes diversity in traveling waves of colonization

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Inside dynamics of pulled and pushed fronts

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Inside dynamics of delayed travelling waves

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Fundamental Concepts

Definition 1 (Traveling wave solution)

An integrodifference equation is said to have a **traveling wave solution** provided that there exists a function, $U(x - ct)$, that satisfies

$$U(x - c) = \int_{-\infty}^{\infty} k(x - y)g(U(y))U(y)\,dy$$

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Definition 2 (Asymptotic spreading speed)

The rightward **asymptotic spreading speed**, c^*, satisfies the following properties: for any positive ε,

$$\lim_{t\to\infty} \sup_{x\geq t(c^*+\varepsilon)} u_t(x) = 0 \quad \text{and} \quad \lim_{t\to\infty} \sup_{x\leq t(c^*-\varepsilon)} [K - u_t(x)] = 0$$

Pulled and pushed fronts

- **Pulled front:** A traveling wave solution where the speed of propagation is determined by the growth rate at the leading edge of the front.

[3] Weinberger, H. F. “Long-time behavior of a class of biological models." *SIAM journal on mathematical analysis* 13.3 (1982): 353-396.

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  - Initial conditions with fast decay spread at c^*.
    - Convergence to c^*?

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  - There exist traveling wave solutions for $c \geq c^*$.[3]
  - Initial conditions with fast decay spread at c^*.
    - Convergence to c^*?
- **Pushed front:** A traveling wave solution where the speed of propagation is determined by the population growth at intermediates densities, i.e., behind the front.
  - Strong Allee effect, formula for c^* unknown.

[3] Weinberger, H. F. "Long-time behavior of a class of biological models." *SIAM journal on mathematical analysis* 13.3 (1982): 353-396.

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System of integrodifference equations

- To analyze the inside dynamics we separate the population into separate neutral fractions $v_t^i(x)$.

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- Assuming individuals in each fraction grow and disperse in the same manner and only differ by position and label. Then,

$$v_{t+1}^i(x) = \int_{-\infty}^{\infty} \underbrace{k(x-y)}_{\text{dispersal kernel}} \; \underbrace{g(u_t(y))}_{\text{per capita growth}} \; \underbrace{v_t^i(y)}_{\text{neutral fraction}} \; dy. \tag{1}$$

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- Linear in v^i $\rightarrow$ sum is the equation for the entire population density $\rightarrow$ existence of traveling wave solutions.

Outline

Dispersal and growth functions

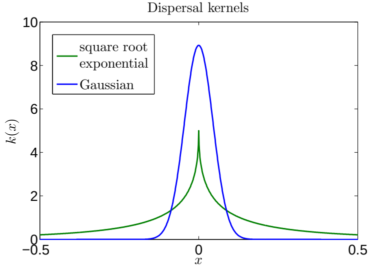

Figure: Plots of the dispersal kernels and growth functions used in the numerical simulations.

Outline

Founder effect: Beverton-Holt

$$\underbrace{k(x-y)}_{\text{dispersal kernel}} = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}}}_{\text{Gaussian}} \quad \text{and} \quad \underbrace{g(u_t(y))}_{\text{per capita growth}} = \underbrace{\frac{R}{1+\frac{R-1}{K}u_t(y)}}_{\text{Beverton-Holt}}$$

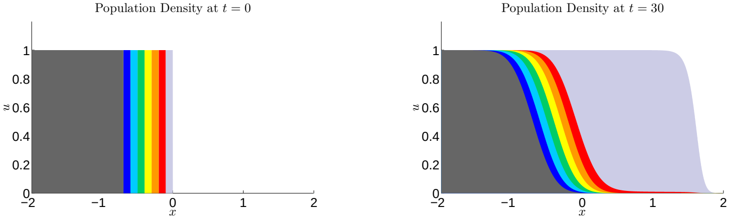

Figure: Numerical simulation using the parameter values $R = 2.5$, $K = 1$, and $\sigma^2 = 0.002$.

Founder effect: Ricker

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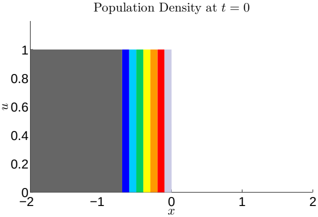

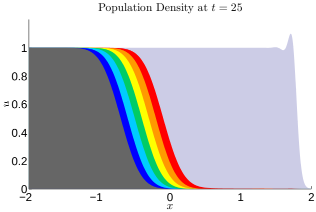

Figure: Numerical simulation using the parameter values $R = 4$, $K = 1$, and $\sigma^2 = 0.002$.

Thin-tailed kernel with per capita growth maximal at zero

Theorem 3

Let k is thin-tailed and g is maximal at zero. If the initial density $v_0^i(x)$ of neutral fraction i converges to 0 faster than the traveling wave solution U as $x \to \infty$, then, for any $A \in \mathbb{R}$, the density of neutral fraction i, $v_t^i(x)$, converges to 0 uniformly as $t \to \infty$ in the moving half-line $[A + ct, \infty)$.

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- Pulled front.
- Theorem applies to functions with overcompensation.
- The neutral fraction at the leading edge dominates as time progresses.
- All other fractions approach zero at the front of the invasion wave.
- If $v_0^i(x)$ has compact support then $v_t^i(x) \to 0$ uniformly as $t \to \infty$ in the moving frame.

Outline

Strong Allee effect

$$\underbrace{k(x-y)}_{\text{dispersal kernel}} = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}}}_{\text{Gaussian}} \quad \text{and} \quad \underbrace{g(u_t(y))}_{\text{per captia growth}} = \underbrace{\frac{R(u_t(y))^{\delta-1}}{1+\frac{R-1}{K}(u_t(y))^{\delta}}}_{\text{sigmoid Beverton-Holt}}$$

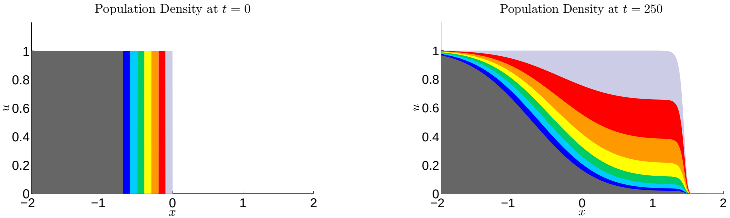

Figure: Numerical simulation using the parameter values $R = 4$, $K = 1$, $\delta = 2$, and $\sigma^2 = 0.002$.

Normal kernel with Allee type growth

Theorem 4

If $k \sim N(\mu, \sigma^2)$ and g has a strong Allee effect, then for any $A \in \mathbb{R}$, the density of neutral fraction i, $v_t^i(x)$, converges to a proportion $p^i[v_0^i]$ of the total population , U, uniformly as $t \to \infty$ in the moving half-line $[A + ct, \infty)$. Moreover, the proportion $p^i[v_0^i]$ can be computed explicitly:

$$p^i[v_0^i] = \frac{\int_{-\infty}^{\infty} v_0^i(x) U(x) e^{\frac{c-\mu}{\sigma^2/2} x} \, dx}{\int_{-\infty}^{\infty} U^2(x) e^{\frac{c-\mu}{\sigma^2/2} x} \, dx}. \quad (2)$$

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- Pushed front.
- All neutral fractions contribute to the spread at the leading edge.
- Result relies heavily on the fact that $k \sim N(\mu, \sigma^2)$.

Outline

Fat-tail dispersal kernel

$$\underbrace{k(x-y)}_{\text{dispersal kernel}} = \underbrace{\frac{1}{4\alpha} e^{-\frac{1}{\alpha}\sqrt{|x-y|}}}_{\text{square root exponential}} \quad \text{and} \quad \underbrace{g(u_t(y))}_{\text{per capita growth}} = \underbrace{\frac{R}{1+\frac{R-1}{K}u_t(y)}}_{\text{Beverton-Holt}}$$

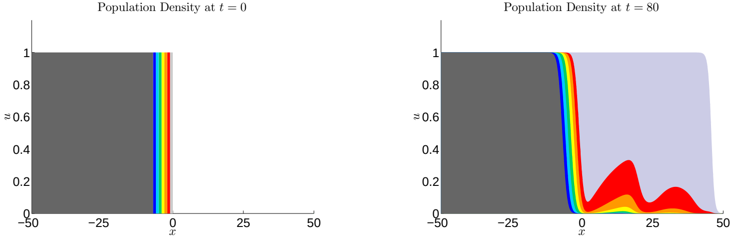

Figure: Numerical simulation using the parameter values $R = 2.5$, $K = 1$, and $\alpha = 0.015$.

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Discussion

Conclusions

- Thin-tailed dispersal kernels per capita growth maximal at zero $\rightarrow$ pulled front solutions.
- Allee effect $\rightarrow$ pushed front solutions.
- Fat-tailed dispersal $\rightarrow$ complicated dynamics?

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Future work

- Obtain analytic results for Model (1) with a fat-tailed dispersal kernel.
- Analyze the contribution of different neutral fractions by providing a measure for the genetic diversity in the population.
- Apply Model (1) to the range expansion of mountain pine beetle across western Canada.

Outline

1. Background
   - Motivation
   - Inside dynamics
   - Mathematical Model
2. Results
   - Dispersal and growth functions
   - Thin-tailed kernel & maximal per capita growth at zero
   - Strong Allee effect
   - Fat-tail dispersal
3. Discussion
   - Conclusions & Future work
   - Acknowledgements

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