# Optimal Designs for Individual Prediction in Multiple Group Random Coefficient Regression Models

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# Contents

# Two groups random coefficient model

| | | | | |
|---|---|---|---|---|
| Group 1 (G1) | - | treatment | - | $n_1$ individuals |
| Group 2 (G2) | - | control | - | $n_2$ individuals |

$$\text{G1}: \quad Y_{1ik} = \alpha_i + \mu_i + \varepsilon_{1ik}, \quad i = i_1, .., i_{n_1}, \quad k = 1, .., K$$

$$\text{G2}: \quad Y_{2ik} = \mu_i + \varepsilon_{2ik}, \quad i = i_{n_1+1}, .., i_N, \quad k = 1, .., K$$

$$N = n_1 + n_2$$

- $K$ number of observations per individual
- $\varepsilon_{jik}$, $j = 1, 2$, observational errors
  - $\mathrm{E}(\varepsilon_{jik}) = 0$
  - $\mathrm{Var}(\varepsilon_{jik}) = \sigma^2$

# Two groups random coefficient model

$$\text{G1}: \quad Y_{1ik} = \alpha_i + \mu_i + \varepsilon_{1ik}, \quad i = i_1, .., i_{n_1}, \quad k = 1, .., K$$

$$\text{G2}: \quad Y_{2ik} = \mu_i + \varepsilon_{2ik}, \quad i = i_{n_1+1}, .., i_N, \quad k = 1, .., K$$

- $\boldsymbol{\theta}_i := (\alpha_i, \mu_i)^\top$ *individual random parameters*
  - $\mathrm{E}(\boldsymbol{\theta}_i) = (\alpha_0, \mu_0)^\top =: \boldsymbol{\theta}_0$ *unknown*
  - $\mathrm{Cov}(\boldsymbol{\theta}_i) = \sigma^2 \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$; $v \,\&\, u$ *known*; $v,\ u > 0$
- *All* $\boldsymbol{\theta}_i$ *and all* $\varepsilon_{ji'k}$ *uncorrelated*

*Search for optimal group sizes for* *a) estimation of* $\alpha_0$
*b) prediction of* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)^\top$

# Best estimation

Best linear unbiased estimator for population parameters $\boldsymbol{\theta}_0$ :

$$\hat{\boldsymbol{\theta}}_0 = (\hat{\alpha}_0, \hat{\mu}_0)^\top$$

$$\hat{\alpha}_0 = \bar{Y}_1 - \bar{Y}_2 \qquad \& \qquad \hat{\mu}_0 = \bar{Y}_2$$

$$\bar{Y}_j = \frac{1}{n_j}\sum_{i=1}^{n_j}\frac{1}{K}\sum_{i=1}^{K} Y_{jik}, \quad j = 1, 2$$

Variance of estimator $\hat{\alpha}_0$ :

$$\mathrm{var}\left(\hat{\alpha}_0\right) = \frac{\sigma^2}{K}\left(\frac{K(u+v)+1}{n_1} + \frac{Ku+1}{n_2}\right)$$

# Best prediction

Best linear unbiased predictor for individual parameters $\boldsymbol{\theta}_i$ :

$$\hat{\boldsymbol{\theta}}_i = (\hat{\alpha}_i, \hat{\mu}_i)^\top$$

$$\hat{\alpha}_i = \begin{cases} \frac{Kv}{K(v+u)+1}\,(\bar{Y}_{1i} - \bar{Y}_2) + \frac{Ku+1}{K(v+u)+1}\,(\bar{Y}_1 - \bar{Y}_2), & \text{ind. "}i\text{" in G1} \\ \bar{Y}_1 - \bar{Y}_2, & \text{ind. "}i\text{" in G2} \end{cases}$$

$$\bar{Y}_{ji} = \frac{1}{K}\sum_{k=1}^{K} Y_{jik}, \quad j = 1, 2$$

$$\hat{\mu}_i = \begin{cases} \frac{Ku}{K(v+u)+1}\,(\bar{Y}_{1i} - \bar{Y}_1) + \bar{Y}_2, & \text{ind. "}i\text{" in G1} \\ \frac{Ku}{Ku+1}\,\bar{Y}_{2i} + \frac{1}{Ku+1}\,\bar{Y}_2, & \text{ind. "}i\text{" in G2} \end{cases}$$

# MSE matrix

MSE matrix of predictor $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, ..., \hat{\alpha}_N)^\top$ :

$$\mathrm{Cov}\,(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

$$\mathbf{A}_{11} = \sigma^2(Ku+1)\left(\left(\frac{N}{Kn_1n_2} - \frac{v}{K(u+v)+1}\right)\mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top + \frac{v}{K(u+v)+1}\,\mathbf{I}_{n_1}\right)$$

$$\mathbf{A}_{12} = \sigma^2(Ku+1)\frac{N}{Kn_1n_2}\mathbf{1}_{n_1}\mathbf{1}_{n_2}^\top = \mathbf{A}_{21}^\top$$

$$\mathbf{A}_{22} = \sigma^2\left(\left(\frac{K(u+v)+1}{Kn_1} + \frac{Ku+1}{Kn_2}\right)\mathbf{1}_{n_2}\mathbf{1}_{n_2}^\top + v\,\mathbf{I}_{n_2}\right)$$

# Optimal design

Exact design:

$$\xi := \begin{pmatrix} G1 & G2 \\ w_1 & w_2 \end{pmatrix}$$

$$w_1 = \frac{n_1}{N} \quad \& \quad w_2 = \frac{n_2}{N}$$

Approximate design:

$$\xi := \begin{pmatrix} G1 & G2 \\ w & 1-w \end{pmatrix}; \quad 0 \le w \le 1$$

*Search for optimal weight $w^*$ to minimize*

*a) variance of $\hat{\alpha}_0$*

*b) MSE matrix of $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, ..., \hat{\alpha}_N)^\top$*

# $A$-optimal design for estimation of population parameter

$A$-criterion for estimation of population parameter $\alpha_0$

$$\Phi_{A,\alpha_{\mathbf{0}}} := \mathrm{var}\left(\hat{\alpha}_0\right)$$

$$\text{for} \quad \xi = \begin{pmatrix} \text{G1} & \text{G2} \\ w & 1-w \end{pmatrix}$$

$$\Phi_{A,\alpha_{\mathbf{0}}}(w) = \frac{K(v+u)+1}{w} + \frac{Ku+1}{1-w}$$

Optimal weight:

$$w^*_{A,\alpha_{\mathbf{0}}} = \frac{1}{1+\sqrt{\frac{Ku+1}{K(v+u)+1}}}$$

# $D$-optimal design for estimation of population parameter

$D$-criterion for estimation of population parameter $\alpha_0$

$$\Phi_{D,\alpha_0} := \log\left(\mathrm{var}\left(\hat{\alpha}_0\right)\right)$$

$$\text{for} \quad \xi = \begin{pmatrix} \text{G1} & \text{G2} \\ w & 1-w \end{pmatrix}$$

$$\Phi_{D,\alpha_0}(w) = \log\left(\frac{K(v+u)+1}{w} + \frac{Ku+1}{1-w}\right)$$

Optimal weight:

$$w^*_{D,\alpha_0} = w^*_{A,\alpha_0}$$

# $A$-optimal design for prediction of individual parameters

$A$-criterion for prediction of individual parameters $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)^\top$:

$$\Phi_{A,\alpha} := \mathrm{tr}\left(\mathrm{Cov}\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\right)\right)$$

$$\text{for} \quad \xi = \begin{pmatrix} \text{G1} & \text{G2} \\ w & 1-w \end{pmatrix}$$

$$\Phi_{A,\alpha}(w) = \frac{K(v+u)+1}{w} + \frac{Ku+1}{1-w} - \frac{K^2 v^2 N w}{K(v+u)+1}$$

No formula for $w^*_{A,\alpha}$ $\Rightarrow$ Numerical example

# $D$-optimal design for prediction of individual parameters

$D$-criterion for prediction of individual parameters $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)^\top$:

$$\Phi_{D,\alpha} := \log\left(\det\left(\mathrm{Cov}\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\right)\right)\right)$$

$$\text{for} \quad \xi = \begin{pmatrix} \mathsf{G1} & \mathsf{G2} \\ w & 1-w \end{pmatrix}$$

$$\Phi_{D,\alpha}(w) = N\, w \log\left(\frac{Ku+1}{K(v+u)+1}\right) - \log\left(w(1-w)\right)$$

Optimal weight:

$$w^*_{D,\alpha} = \frac{1}{2} + \frac{1}{Nt} + \sqrt{\frac{1}{4} + \frac{1}{(Nt)^2}}, \qquad t = \log\left(\frac{Ku+1}{K(v+u)+1}\right)$$

# $A$-optimal design for prediction of individual parameters

$A$-optimal weight $w^*_{A,\alpha}$

of treatment group G1

for prediction of individual parameters

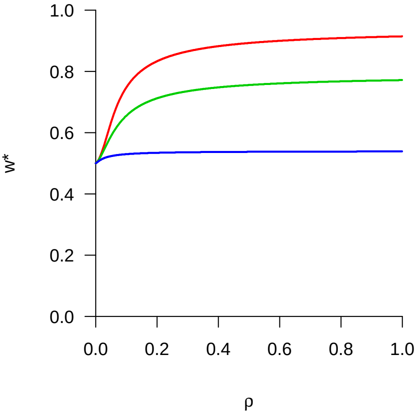
$$q = \frac{v}{u}$$

- $q = 2$
- $q = 0.5$
- $q = 0.1$

$\rho = v/(1+v)$

$N = 100$

$K = 5$



# Efficiency of OD in fixed effects model

Efficiency of $A$-optimal design

in fixed effects model $w_A^* = 0.5$

$$eff = \frac{\Phi_{A,\alpha}(w_{A,\alpha}^*)}{\Phi_{A,\alpha}(w_A^*)}$$

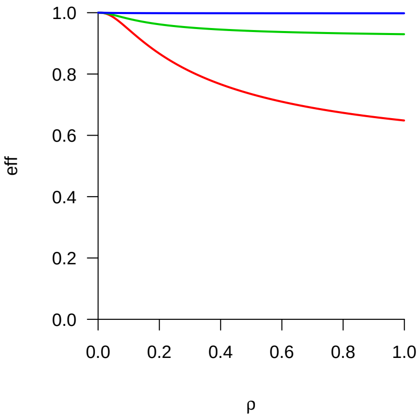
$$q = \frac{v}{u}$$

- $q = 2$
- $q = 0.5$
- $q = 0.1$

$\rho = v/(1+v)$

$N = 100$

$K = 5$



# Efficiency of OD for estimation of population parameters

## Efficiency of $A$-optimal design

for estimation $w^*_{A,\alpha_0}$

$$eff = \frac{\Phi_{A,\alpha}(w^*_{A,\alpha})}{\Phi_{A,\alpha}(w^*_{A,\alpha_0})}$$

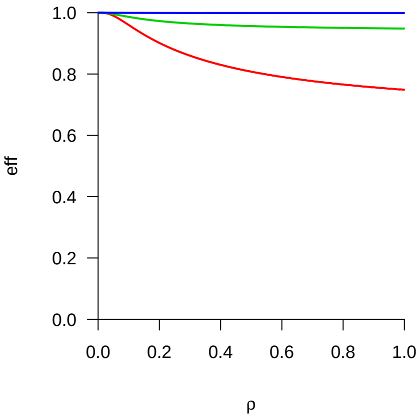
$$q = \frac{v}{u}$$

- $q = 2$
- $q = 0.5$
- $q = 0.1$

$\rho = v/(1+v)$

$N = 100$

$K = 5$



# $D$-optimal design for prediction of individual parameters

$D$-optimal weight $w^*$

of treatment group G1

for prediction of individual parameters

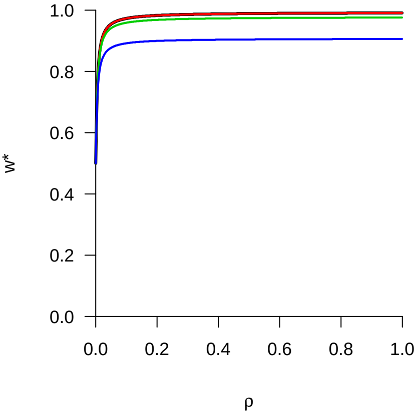
$$q = \frac{v}{u}$$

- $q = 2$
- $q = 0.5$
- $q = 0.1$

$\rho = v/(1+v)$

$N = 100$

$K = 5$



# Efficiency of OD in fixed effects model

Efficiency of $D$-optimal design

in fixed effects model $w_D^* = 0.5$

$$eff = \left( \frac{\exp(\Phi_{D,\alpha}(w^*_{D,\alpha}))}{\exp(\Phi_{D,\alpha}(w^*_D))} \right)^{\frac{1}{N}}$$

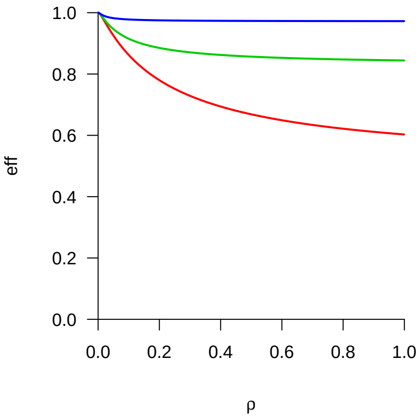
$$q = \frac{v}{u}$$

- $q = 2$
- $q = 0.5$
- $q = 0.1$

$\rho = v/(1+v)$

$N = 100$

$K = 5$



# Efficiency of OD for estimation of population parameters

## Efficiency of $D$-optimal design

for estimation $w^*_{D,\alpha_\mathbf{0}}$

$$eff = \left( \frac{\exp(\Phi_{D,\alpha}(w^*_{D,\alpha}))}{\exp(\Phi_{D,\alpha}(w^*_{D,\alpha_\mathbf{0}}))} \right)^{\frac{1}{N}}$$

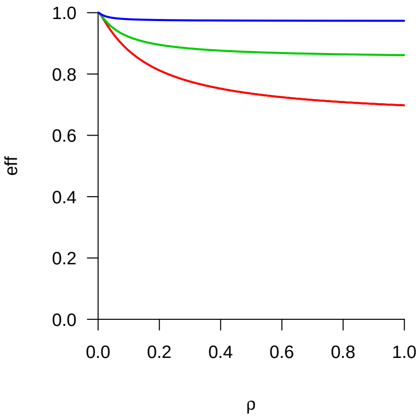
$$q = \frac{v}{u}$$

- $q = 2$
- $q = 0.5$
- $q = 0.1$

$\rho = v/(1+v)$

$N = 100$

$K = 5$



# Directions for future research

- Other design criteria
- More than two groups
- Unknown variance parameters
- OD for prediction of other linear aspects
- Exact designs

# Thank you for your attention!

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