A Murnaghan-Nakayama rule for $qH^*(F\ell_n)$

Carolina Benedetti Velásquez
York University - Fields Institute
with N. Bergeron, L. Colmenarejo, F. Saliola, F. Sottile.

Algebraic Combinatorixx2
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Background

- Hook product
- Murnaghan-Nakayama in $H^*(F\ell_n)$

Quantum results

- quantum hook product
- Murnaghan-Nakayama in $qH^*(F\ell_n)$

SCHUBERT POLYNOMIALS $\mathfrak{S}_w$

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Open problem

Find a combinatorial rule for

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w d_{u,v}^w \mathfrak{S}_w$$

Monk's rule

Let $u \in S_n$ and $k < n$. Then

$$\mathfrak{S}_{(k,k+1)}\mathfrak{S}_u = \sum \mathfrak{S}_v$$

summing over all $v = u(i,j)$ such that $i \leq k < j$ and $\ell(u)+1 = \ell(u(i,j))$.

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The *k-Bruhat order* on S_n is given by $u \lessdot_k u(i,j)$ whenever $i \leq k < j$ and $\ell(u) + 1 = \ell(u(i,j))$.

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- $1246|357 \lessdot_4 1247356$

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- $1246|357 \lessdot_4 1247356 \rightsquigarrow 1246357 \xrightarrow[(4,7)]{(6,7)} 1247356.$

Fact: If $w_1 < \cdots < w_k,\ w_{k+1} < \cdots < w_n$ then $\mathfrak{S}_w = s_\lambda(x_1, \ldots, x_k)$.
For instance, $\mathfrak{S}_{(k,k+1)} = s_\square(x_1, \ldots, x_k)$.

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Theorem (Sottile '96)

Let $u \in S_n$, $k < n$ and $\lambda = (b, 1^{a-1})$ with $a \leq k, b \leq n-k$. Then

$$\mathfrak{S}_u \cdot s_\lambda = \sum_w \mathfrak{S}_w$$

where $u \xrightarrow{\gamma}_k w$ with γ peakless and $shape(\gamma) = \lambda$.

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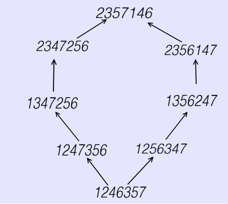
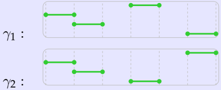
Example

$n = 7, k = 4$
$\lambda = (2, 1^2)$





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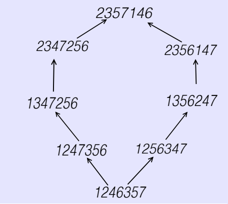
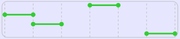


γ_1 :



γ_2 :

then $[\mathfrak{S}_w]\, s_\lambda \mathfrak{S}_u \ = \ 2.$

Murnaghan-Nakayama in $H^*(F\ell_n)$

Theorem 2 (Morrison-Sottile'16)

Let $p_{(r)}(x_1, \ldots, x_k) = s_{(r)} - s_{(r-1,1)} + \cdots + (-1)^{r-1} s_{(1^r)}$. Then

$$p_{(r)} \cdot \mathfrak{S}_u = \sum_w (-1)^{leg(\gamma)} \mathfrak{S}_w$$

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Example: $u = 38254671,\ r = k = 4,\ w = 38572461,\ v = 48372561$

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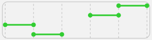

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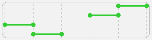
γ' : 

then $[\mathfrak{S}_w]\, p_{(4)} \mathfrak{S}_u = (-1)^1$ but $[\mathfrak{S}_v]\, p_{(4)} \mathfrak{S}_u = 0$

QUANTUM WORLD

Quantum world

Algebraic Geometry

Quantum world

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Problem

Find a combinatorial rule for

$$p^q_{(r)} \mathfrak{S}^q_u = \sum_w d^w_{u,v} \mathfrak{S}^q_w$$

q-Monk's rule (Fomin-Gelfand-Postnikov '97)

Let $u \in S_n$ and $k < n$. Then

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Let $S_n[q] := \{q^\alpha v \ : \ v \in S_n,\ q^\alpha$ monomial in the $q_i's\}$. The *quantum k-Bruhat order* on $S_n[q]$ is given by

- $u \lessdot_k^q u(i,j)$ if $i \leq k < j$ and $\ell(u) + 1 = \ell(u(i,j))$, and
- $u \lessdot_k^q q_i \cdots q_{j-1} u(i,j)$ if $i \leq k < j$ and $\ell(u) + 1 = \ell(u(i,j)) + 2(j-i)$

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Theorem q1 (B-B-C-S-S'17)

Let $u \in S_n, k < n, \lambda = (b, 1^{a-1})$ with $a \leq k, b \leq n-k$ and $\rho = (1, \ldots, n)$. Then

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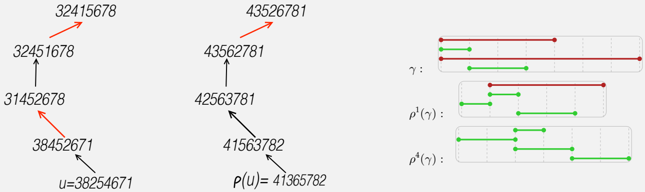
Example: $n = 8, k = 4$ $\lambda = (2, 1^2)$. Then $[\mathfrak{S}^q_w]\, s^q_\lambda \mathfrak{S}^q_u = q_2 q_3 q_4^2 q_5 q_6 q_7$



Murnaghan-Nakayama in $qH^*(F\ell_n)$

Theorem q2 (B-B-C-S-S'17)

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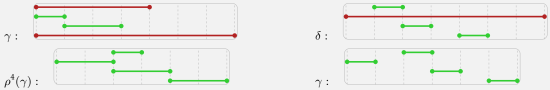
Example: $u = 38254671,\ r = k = 4,\ w = 32415678,\ v = 41362578$



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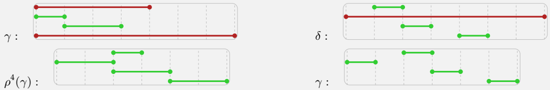

then $[\mathfrak{S}^q_w]\, p^q_4 \mathfrak{S}^q_u = (-1)^2 q_2 q_3 q_4^2 q_5 q_6 q_7$ but $[\mathfrak{S}^q_v]\, p^q_4 \mathfrak{S}^q_u = 0$

THANK YOU!

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