Resolving Stanley's e-Positivity of Claw-Contractible-Free Graphs

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BIRS May 18, 2017

Chromatic symmetric functions

Given G with vertex set V a proper colouring κ of G is

$$\kappa : V \to \{1, 2, 3, \ldots\}$$

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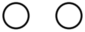
$G =$  has $X_G =$

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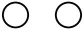
$G =$  has $X_G =$

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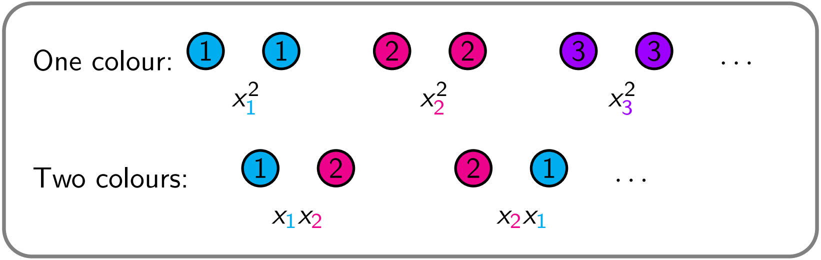
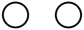


$G =$  has $X_G = x_1^2 + x_2^2 + x_3^2 + \cdots$

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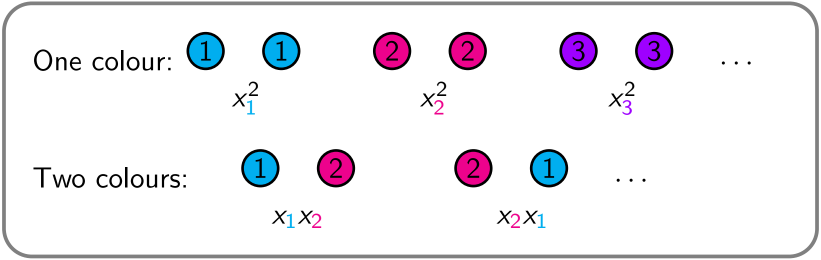
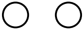


$G =$  has $X_G = x_1^2 + x_2^2 + x_3^2 + \cdots$

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Let Λ be the algebra of symmetric functions

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \subset \mathbb{Q}[[x_1, x_2, \ldots]]$$

$$\Lambda^N = \mathrm{span}_{\mathbb{Q}}\{e_\lambda \mid \lambda \vdash N\}.$$

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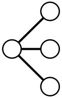 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘

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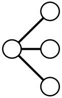 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘

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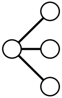 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘

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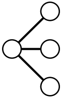 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘

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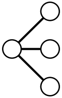 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘

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Stanley 1995:

We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive.

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Contracts to the claw: shrinking edges + identifying vertices + removing multiple edges = claw.

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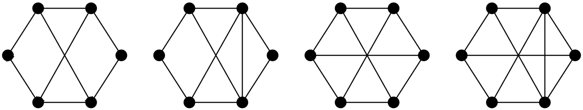

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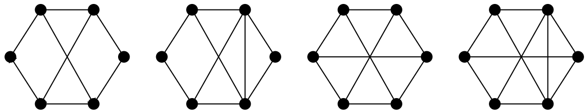

Proposition (Brouwer-Veldman 1987)

G is claw-contractible-free if and only if deleting all sets of 3 non-adjacent vertices gives disconnection.

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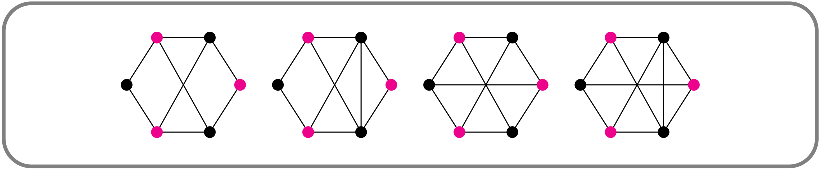

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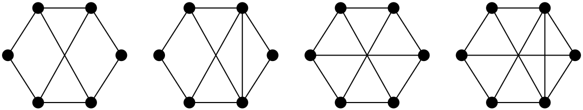
...WITH CHROMATIC SYMMETRIC FUNCTION



$2e_{222}$	$-$	$6e_{33}$	$+$	$26e_{42}$	$+$	$28e_{51}$	$+$	$102e_6$
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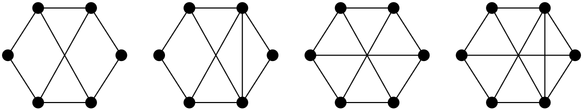
$2e_{321}$	$-$	$6e_{33}$	$+$	$20e_{42}$	$+$	$32e_{51}$	$+$	$228e_6$



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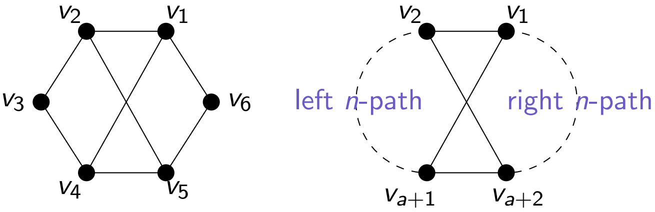
Smallest counterexamples to Stanley's statement.

Infinite family: saltire graphs

The saltire graph $SA_{n,n}$ for $n \geq 3$ is given by



with $SA_{3,3}$ on the left.

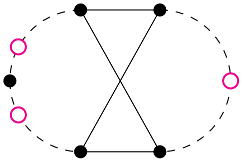
Infinite family: saltire graphs

Theorem (D-Foley-van Willigenburg 2017)

$SA_{n,n}$ for $n \geq 3$ is claw-contractible-free and

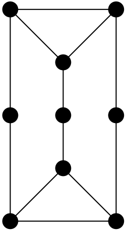
$$[e_{nn}]X_{SA_{n,n}} = -n(n-1)(n-2).$$

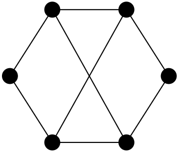
CCF:



And claw-free: triangular tower graphs

Claw-free: does not contain the claw as an induced subgraph.





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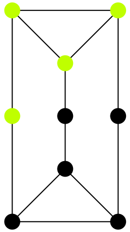

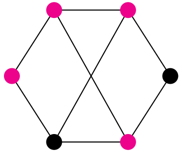

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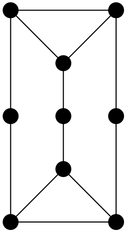

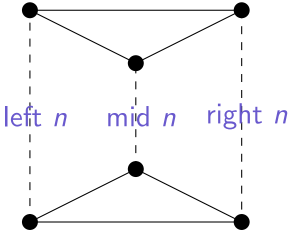

with $TT_{3,3,3}$ on the left.

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$TT_{n,n,n}$ for $n \geq 3$ is claw-contractible-free, claw-free and

$$[e_{nnn}]X_{TT_{n,n,n}} = -n(n-1)^2(n-2).$$

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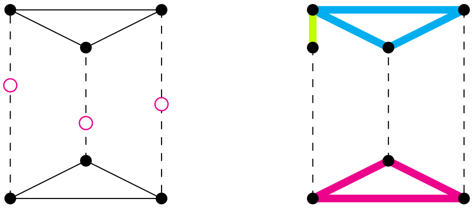
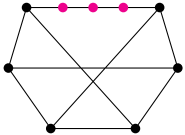


Conjectures

1. Bloated $K_{3,3}$:



with $3n$ vertices has

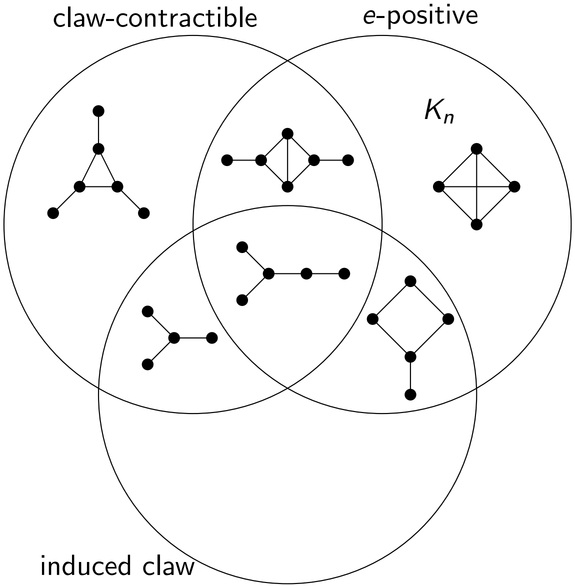
$$-(3 \times 2^n)e_{3^n}.$$

2. No G exists that is connected, claw-contractible-free, claw-free and not e-positive with 10, 11 vertices.

Scarcity

- $N = 6$: 4 of 112 connected graphs ccf and not e-positive.
- $N = 7$: 7 of 853 connected graphs ccf and not e-positive.
- $N = 8$: 27 of 11117 connected graphs ccf and not e-positive.
- Of 293 terms in $TT_{7,7,7}$ only $-$ve at e_{777}.
- Of 564 terms in $TT_{8,8,8}$ only $-$ves at e_{888} and $-1944e_{444444}$.
- Of 1042 terms in $TT_{9,9,9}$ only $-$ves at e_{999} and $-768e_{333333333}$.

A picture speaks 1000 words



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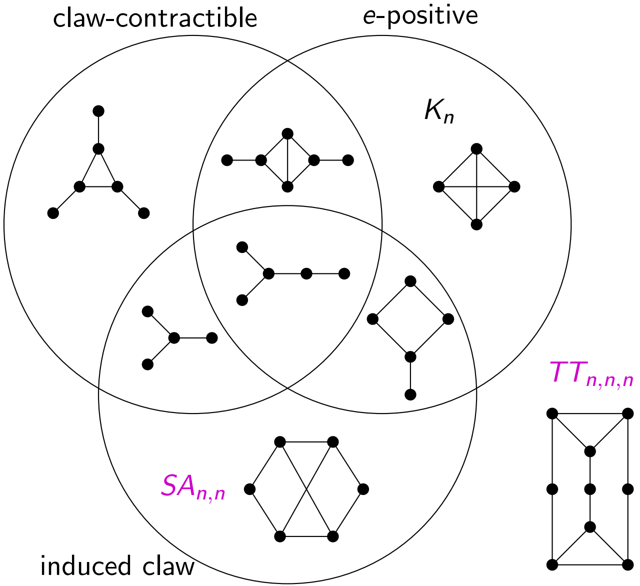

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