Hook formulas for skew shapes

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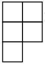
Standard Young Tableaux

Irreducible representations of S_n:

Specht modules $\mathbb{S}_\lambda$, for all $\lambda \vdash n$.

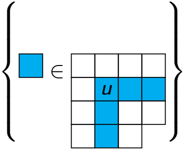
Basis for $\mathbb{S}_\lambda$: **Standard Young Tableaux** of shape λ:

$\lambda = (2,2,1)$: :

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$$

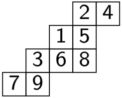
Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h_u} = \frac{5!}{4*3*2*1*1}$$

Hook length of box $u = (i,j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#$ 

Counting skew SYTs

Outer shape λ, inner shape μ, e.g. for $\lambda = (5,4,4,2), \mu = (3,2,1)$



Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

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			2	4
		1	5	
	3	6	8	
7	9			

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Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu}$$

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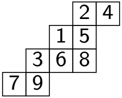

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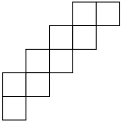
No product formula, e.g. $\lambda/\mu = \delta_{n+2}/\delta_n$:



$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

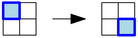
Hook-Length formula for skew shapes

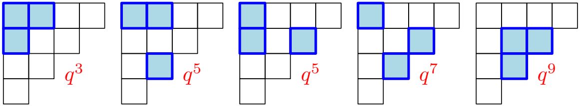
Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda]\setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ.

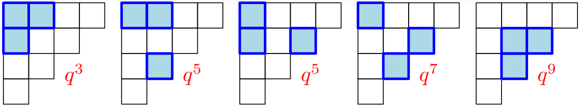
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1, q, q^2, \dots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)}$$

$$+ \frac{q^7}{(1-q)^2(1-q^3)^3(1-q^5)^2} + \frac{q^9}{(1-q)^2(1-q^3)^2(1-q^5)^2(1-q^7)}$$

Theorem (Morales-Pak-P)

$$\sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

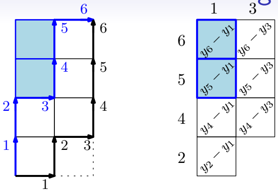
Theorem (Morales-Pak-P)

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu)$ is the set of pleasant diagrams.

Other recent proof by [M. Konvalinka]

Algebraic proof for SSYTs:



$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]:
Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then the Schubert class X_w for w at point v is:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \left(y_{v(d+j)} - y_{v(d-i+1)} \right).$$

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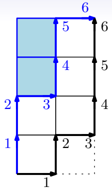

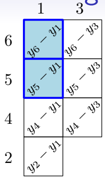

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Factorial Schur functions:

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\left(y_{v(1)}, \dots, y_{v(d)} | y_1, \dots, y_{n-1}\right).$$

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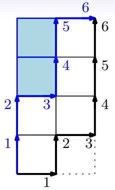

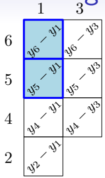

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Evaluation at $y = 1, q, q^2, \ldots$, $v(d+1-i) = \lambda_i + d + 1 - i$, $x_i \to y_{v(i)} = q^{\lambda_i+d+1-i}$
$\to$ Jacobi-Trudi

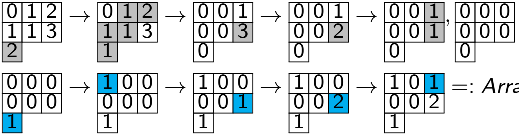
$$s_\mu^{(d)}(q^{v(1)}, \ldots | 1, q, \ldots) = \frac{\det[\prod_{r=1}^{\mu_j+d-j}(q^{\lambda_i+d+1-i} - q^r)]_{i,j=1}^{d}}{\prod_{i<j}(q^{\lambda+d+1-i} - q^{\lambda_j+d+1-j})} = \ldots.$$

$$\ldots[\mathit{simplifications}]\ldots = \det[h_{\lambda_i - i - \mu_j + j}(1, q, \ldots)] = s_{\lambda/\mu}(1, q, \ldots)$$

Combinatorial proofs:

Hillman-Grassl algorithm/map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$RRP \quad P =$



$=: Array \quad A = \Phi(P)$

$Weight(P) = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 = \sum_{i,j} A_{i,j} hook(i,j) =$
$1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 = weight(A)$

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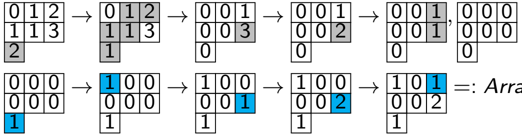
RRP $P =$  $=:$ *Array* $A = \Phi(P)$

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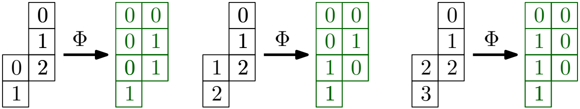
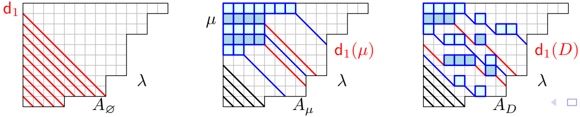


Theorem (Morales-Pak-P)

The restricted Hillman-Grassl map is a bijection from the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .



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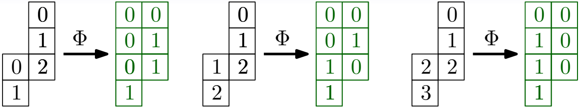

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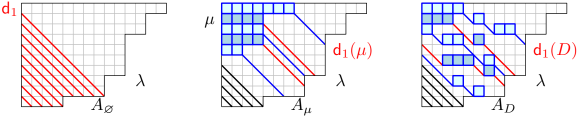
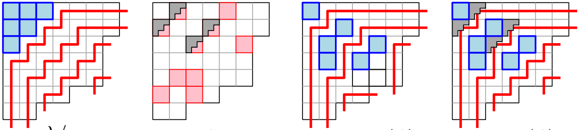


Proof sketch:
Issue: enforce 0s on μ and strict increase down columns on λ/μ.
Show $\Phi^{-1}(A)$ is column strict in λ/μ + support in λ/μ via properties of RSK
(Integer partition on kth diagonal
$(\ldots, P_{2,2+k}, P_{1,1+k}) = \mathit{shape}(RSK(A_k^T))$ is shape of RSK tableau on the corresponding subrectangle of A)
Thus, Φ^{-1} is injective: restricted arrays $\to$ SSYTs of shape λ/μ.
Bijective: use the algebraic identity.

Hillman-Grassl on skew RPPs

Weakly increasing rows:
Skew reverse plane partitions ⇔ arrays/diagrams *"pleasant diagrams"*: $PD(\lambda/\mu)$.
– supersets of $\mathcal{E}(\lambda/\mu)$, identified by the "high peaks".



λ/μ S $D^* = \varrho_1(S)$ $D = \varrho_2(S)$

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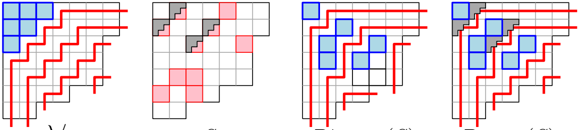
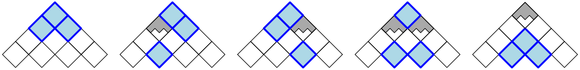


Theorem (MPP)

The HG map is a bijection between skew RPPs of shape λ/μ and arrays with certain nonzero entries (at the "high peaks"):

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$



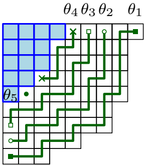
With P-partitions/limit: combinatorial proof of original Naruse Hook-Length Formula for $f^{\lambda/\mu}$..

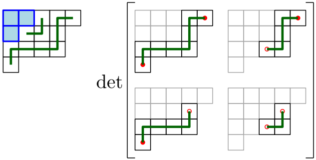
Non-intersecting lattice paths

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \ldots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k}.$$

where $s_\varnothing = 1$ and $s_{\theta_i\#\theta_j} = 0$ if the $\theta_i\#\theta_j$ is undefined.
(Here θ_1 is the border strip following the inner border of λ and θ_i are obtained from the inner border of the remaining partition, until μ is hit, then the border strips are obtained from each connected part etc, and ordered by their corners. The strip $\theta_i\#\theta_j$ is the shape of θ_1 between the diagonals of the endpoints of θ_i and θ_j.)





NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a, b) and (c, d) we have

$$s_\theta(1, q, q^2, \dots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

s  $(1, q, q^2, \dots) = \frac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}$

$\gamma=(3,1),(3,2),(2,2),(2,3),(1,3)$: 

$+ \frac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \frac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \frac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2}$ 

$+ \frac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$ 

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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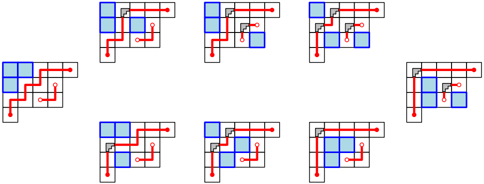
Excited diagrams for λ/μ – NonIntersecting Lattice Paths:



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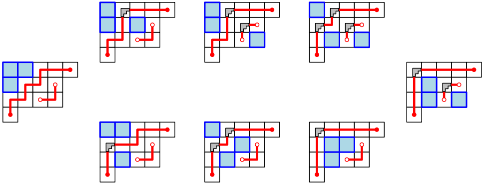
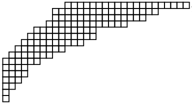


$$s_{\lambda/\mu} \overset{=}{\text{Lascoux-Pragacz}} \det\left[\, s_{\theta_i\#\theta_j} \,\right]_{i,j=1}^{k} \overset{=}{s_{\text{Border Strip}}} \det\left[\sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \right]$$

$$\overset{=}{Lindstrom-Gessel-Viennot} \sum_{NILP:\gamma_1,\ldots} \prod_{u\in\gamma_1\cup\ldots} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \overset{=}{\mathcal{E}(\lambda/\mu)=NILP} \sum_{D\in ED(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdots}}{1-q^{h_u}}$$

Further results and directions

Asymptotics:

$\lambda/\mu =$  , $|\lambda/\mu| = n \to \infty$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

"linear": $\log f^{\lambda^{(n)}/\mu^{(n)}} \sim cn + o(n)$,
"stable": $\sim \frac{1}{2} n \log n + O(n)$,
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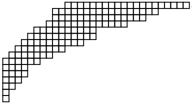

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Multivariate formulas:

Exact product formulas for certain skew shapes (generalizing results by Ch.Krattenthaler et al)
Lozenge tilings with multivariate local weights – determinantal formulas.
Reduced words ....

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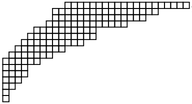
$\lambda/\mu =$  , $|\lambda/\mu| = n \to \infty$

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