Surfaces, orbifolds, and dominance

Shira Viel

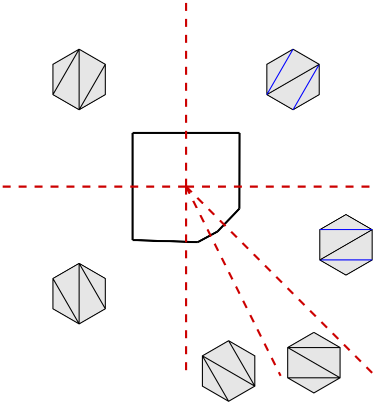
North Carolina State University

Algebraic CombinatoriXX 2

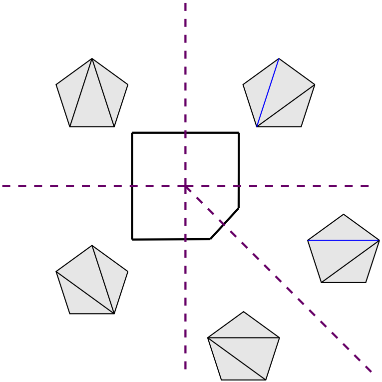
May 16, 2017

Motivation: the cyclohedron and associahedron

The **normal fan** to the n-**cyclohedron** (aka Type B n-associahedron) **refines** the normal fan to the n-**associahedron** (of Type A):



Centrally-symmetric triangulations
of the $(2n+2)$-gon



Triangulations
of the $(n+3)$-gon

Exchange matrices and matrix dominance

An **exchange matrix** is a **skew-symmetrizable** integer matrix.

(
- Fundamental combinatorial datum specifying a **cluster algebra**
- **Finite-type** exchange matrices classified by (finite) **Dynkin diagrams**

)

Definition

Given $n \times n$ exchange matrices $B = [b_{ij}]$ and $B' = [b'_{ij}]$, we say that B **dominates** B' if for each i and j,

- the entries b_{ij} and b'_{ij} weakly agree in sign, and
- $|b_{ij}| \geq |b'_{ij}|$.

Example

$$B = \begin{bmatrix} 0 & 1 \\ \text{-2} & 0 \end{bmatrix} \text{ dominates } B' = \begin{bmatrix} 0 & 1 \\ \text{-1} & 0 \end{bmatrix}.$$

$\big(B$ skew-symmetrizable since $BD = \left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$ for $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] \in \operatorname{diag}(\mathbb{Z}_+)\big)$

Dominance phenomena

Suppose B, B' are exchange matrices such that B dominates B'. Reading has shown that in many cases,

1. There exists an injective ring homomorphism from the cluster algebra $\mathcal{A}_\bullet(B')$ into $\mathcal{A}_\bullet(B)$ (which preserves g-**vectors**),
2. The identity map from $\mathbb{R}^B$ to $\mathbb{R}^{B'}$ is **mutation-linear**,
3. The **scattering fan** $\mathcal{D}_B$ refines the scattering fan $\mathcal{D}_{B'}$,
4. The **mutation fan** $\mathcal{F}_B$ refines the mutation fan $\mathcal{F}_{B'}$.

The mutation fan

Broadly, the **mutation fan** for an $n \times n$ exchange matrix is a complete fan in $\mathbb{R}^n$ which encodes the combinatorics of mutation.

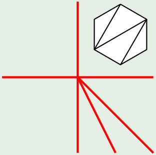
(
- generalization of **g-vector fan**: for finite-type cluster algebras, they coincide
- can be used to construct bases/**universal coefficients**

)

Suppose B, B' are exchange matrices such that B dominates B'. In many cases, the mutation fan $\mathcal{F}_B$ refines the mutation fan $\mathcal{F}_{B'}$.

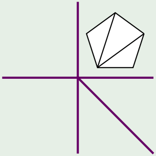
Example

$\mathcal{F}_{\left[\begin{smallmatrix} 0 & 1 \\ -2 & 0 \end{smallmatrix}\right]} =$



refines

$\mathcal{F}_{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} =$



Theorem

$\mathcal{F}_B$ refines $\mathcal{F}'_B$ when B' is obtained from B by **orbifold-resection**.

Surface model ingredients

Marked surface: 2-dim'l compact oriented surface S, possibly with boundary, with designated marked points M.

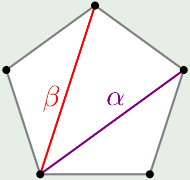
Arcs: special class of curves in S that connect marked points. Considered up to isotopy relative to M.

Triangulations: maximal compatible collections T of arcs, always of the same cardinality n.

Signed adjacency matrix: $n \times n$ skew-symmetric integer matrix $B(T)$ encoding adjacencies of arcs in triangulation.

Example

$(S, M), T =$



$$B(T) = \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 0 & 1 \\ \beta & -1 & 0 \end{array}$$

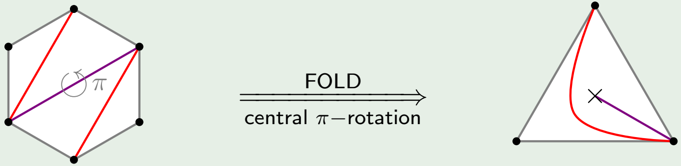
Folding

Orbifold: topological space, locally looks like quotient space of . . .

- (in general) . . . $\mathbb{R}^n$ under linear action of finite group
- (for me) . . . marked surface under symmetry of surface
- (today) . . . marked surface under central π-rotation

Fixed points under action are called **orbifold points**, denoted $\times$.
(Think of the origin of the complex plane under the $z \mapsto z^2$ map)

Example



$$B(T) = \begin{array}{cc} & \begin{array}{ccc} \alpha & \beta_1 & \beta_2 \end{array} \\ \begin{array}{c} \alpha \\ \beta_1 \\ \beta_1 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{array} \qquad B(\bar{T}) = \begin{array}{cc} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \end{array}$$

Extending the model to orbifolds: additional ingredients

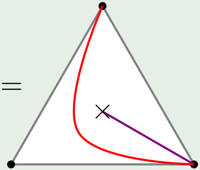
Marked orbifold: 2-dim'l compact oriented surface S, possibly with boundary, with designated marked points M, and orbifold points Q endowed with angle π. $M \cap Q = \emptyset$.

(Pending) arcs: special class of curves in S connecting marked points, or connecting a marked and orbifold point. Considered up to isotopy relative to $M \cup Q$.

Signed adjacency matrix: $n \times n$ skew-symmetrizable integer matrix $B(T)$ encoding adjacencies of arcs in triangulation.

Example

$(S, M, Q), T =$

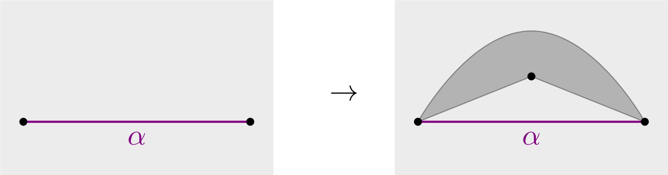


$$B(T) = \begin{array}{c|cc} & \alpha & \beta \\ \alpha & 0 & 1 \\ \beta & -2 & 0 \end{array}$$

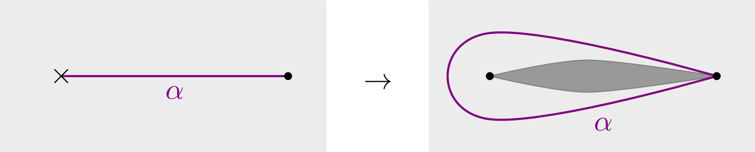
Note: If $Q = \emptyset$, then $(S, M, Q) = (S, M)$ is a marked surface.

Resection and orbifold-resection

Reading defines **resection**, an operation on marked surfaces that induces a dominance relation on signed adjacency matrices:

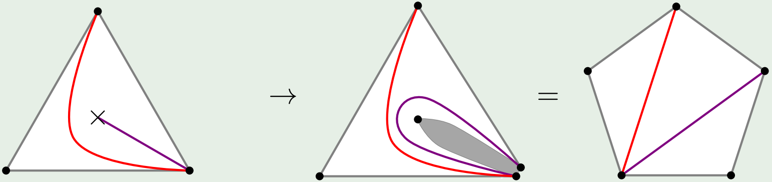


We introduce an analogous **orbifold-resection** operation which also induces dominance on signed adjacency matrices.



Once-orbifolded triangle $\xrightarrow{o-resect}$ pentagon

Example

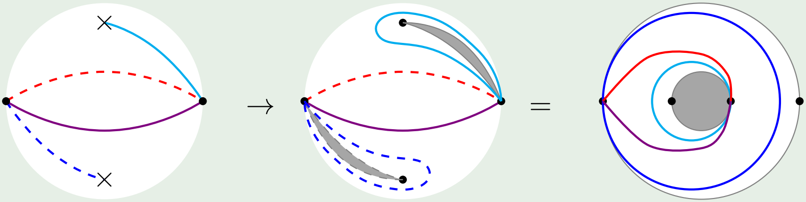


$$B = \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 0 & 1 \\ \beta & -2 & 0 \end{array}$$

$$B(T') = \begin{array}{c|cc} & \alpha' & \beta' \\ \hline \alpha' & 0 & 1 \\ \beta' & -1 & 0 \end{array}$$

2-punctured, 2-orbifolded sphere $\xrightarrow{o-resect}$ annulus

Example



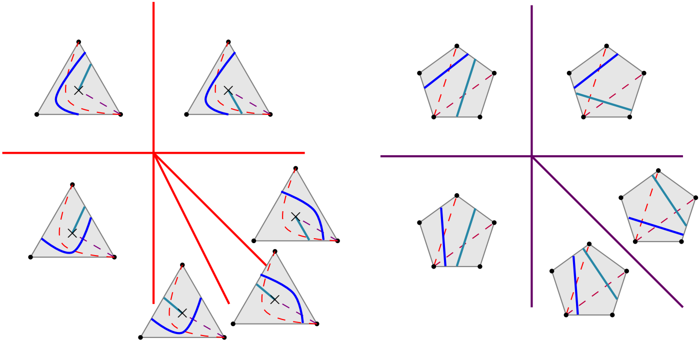
$$B = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & -2 & -2 & 2 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -2 & 2 & 2 & 0 \end{bmatrix} \qquad B' = \begin{array}{c} 1' \\ 2' \\ 3' \\ 4' \end{array} \begin{bmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}$$

Main result: o-resection $\Longrightarrow$ mutation fan refinement

Theorem

Let $\mathcal{O} = (S, M, Q)$, T be a triangulated orbifold and let $\mathcal{O}' = (S', M', Q')$, T' be the triangulated orbifold (or surface) induced by an orbifold-resection of $\mathcal{O}$. Then, the mutation fan $\mathcal{F}_{B(T)}$ refines the mutation fan $\mathcal{F}_{B(T')}$.*

* *(modulo some hypotheses and passing from $\mathbb{R}^n$ to $\mathbb{Q}^n$)*



References

Thank you!

- Anna Felikson, Michael Shapiro, and Pavel Tumarkin, *Cluster algebras and triangulated orbifolds.* Adv. Math. **231** (2012), no. 5, 2953-3002.
- Sergey Fomin, Michael Shapiro and Dylan Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes.* Acta Math. **201** (2008), no. 1, 83–146.
- Nathan Reading, *Universal geometric cluster algebras from surfaces.* Trans. Amer. Math. Soc. **366** (2014), no. 12, 6647–6685.
- Nathan Reading, *The dominance relation on exchange matrices.* In preparation.

Surface model ingredients, part II

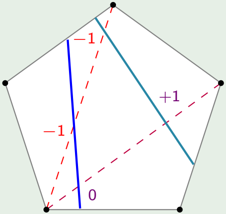
Allowable curves: another special class of curves in S, again considered up to isotopy relative to M.

Rational quasi-laminations: collections L of pairwise-compatible allowable curves with positive rational weights. The underlying set of curves Λ is called the **support** of L.

Shear coordinates: rational vector $\mathbf{b}(T, L)$ encoding the interaction between a quasi-lamination and a triangulation.

Example

$(S, M), T, \Lambda$



$$\mathbf{b}(T, \lambda) = \begin{bmatrix} +1 & -1 \end{bmatrix}$$

$$\mathbf{b}(T, \nu) = \begin{bmatrix} 0 & -1 \end{bmatrix}$$

$\tilde{B}(T, L)$

$$\begin{array}{c c} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \\ \frac{1}{2}\lambda \\ \frac{3}{2}\nu \end{array} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1/2 & -1/2 \\ 0 & -3/2 \end{bmatrix} \end{array}$$

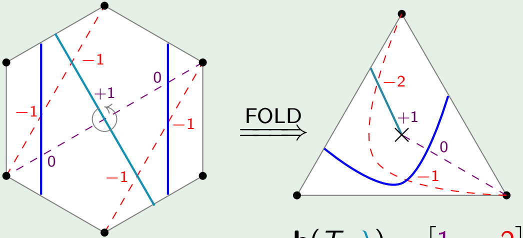
$$\mathbf{b}(T, L) = \tfrac{1}{2}\mathbf{b}(T, \lambda) + \tfrac{3}{2}\mathbf{b}(T, \nu) = \begin{bmatrix} 1/2 & -2 \end{bmatrix}$$

Computing shear coordinates on orbifolds

Given allowable (pending) curve λ in orbifold (S, M, Q) with triangulation T, the **shear coordinate vector** $\mathbf{b}(T, \lambda)$ is indexed by the tagged arcs of T and records intersections of λ with the arcs of T. Each intersection is assigned a value of $\pm 1, 0$ or ± 2.

These values can be read off directly, and correspond to those of the preimage $\tilde{\lambda}$ in the "unfolded" $(\tilde{S}, \tilde{M}, \tilde{Q})$, $\tilde{T}$:

Example



$$\mathbf{b}(T, \lambda) = \begin{bmatrix} 1 & -2 \end{bmatrix}$$
$$\mathbf{b}(T, \nu) = \begin{bmatrix} 0 & -1 \end{bmatrix}$$

Main result: o-resection $\Longrightarrow$ mutation fan refinement

Theorem

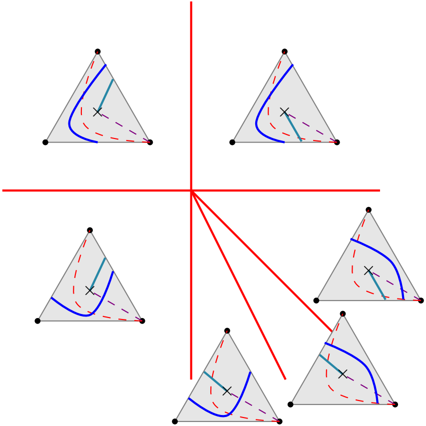
Let $\mathcal{O} = (S, M, Q), T$ be a triangulated orbifold and let $\mathcal{O}' = (S', M', Q'), T'$ be the triangulated orbifold (or surface) induced by an orbifold-resection of $\mathcal{O}$. Then $\mathcal{F}_{B(T)}$ refines $\mathcal{F}_{B(T')}$.*

* *modulo some hypotheses and passing from $\mathbb{R}^n$ to $\mathbb{Q}^n$*

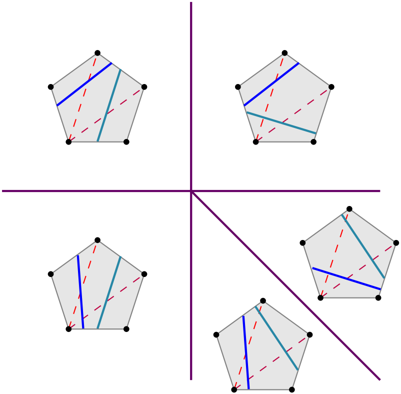
1. Suffices to prove refinement relationship between the **rational quasi-lamination fans** $\mathcal{F}_{\mathbb{Q}}(T)$ and $\mathcal{F}_{\mathbb{Q}}(T')$, whose cones are the rational spans of shear coordinates of collections Λ of pairwise compatible allowable curves in $\mathcal{O}$ and $\mathcal{O}'$, respectively.
2. Let $C = \text{Span}_{\mathbb{Q}_{\geq 0}}\{\mathbf{b}(T, \lambda) : \lambda \in \Lambda\})$ be a cone in $\mathcal{F}_{\mathbb{Q}}(T)$. We show there exists a cone C' in $\mathcal{F}_{\mathbb{Q}}(T')$ such that $C \subseteq C'$.
3. Define a (bijective) map from rational quasi-laminations L in $\mathcal{O}$ to quasi-laminations L' in $\mathcal{O}'$ that preserves shear coordinates and respects support: that is, if $\Lambda_1 = \text{Supp}(L_1) = \text{Supp}(L_2) = \Lambda_2$, then $\Lambda_1' = \text{Supp}(L_1') = \text{Supp}(L_2') = \Lambda_2'$.
4. Set $C' = \text{Span}_{\mathbb{Q}_{\geq 0}}\{\mathbf{b}(T', \lambda') : \lambda' \in \Lambda'\}$. Then $C \subseteq C'$.

Illustration: the cyclohedron and associahedron

The **normal fan** to the n-**cyclohedron** (aka Type B n-associahedron) **refines** the normal fan to the n-**associahedron** (of Type A):



Rational quasi-lamination fan
of the 1-orb'd $(n+1)$-gon



Rational quasi-lamination fan
of the $(n+3)$-gon