Mixing in Noisy Nonlinear Oscillators

Application to Low-Frequency Climate Variability

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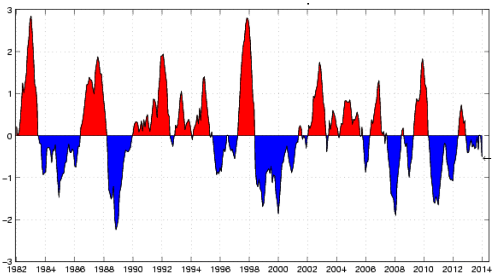
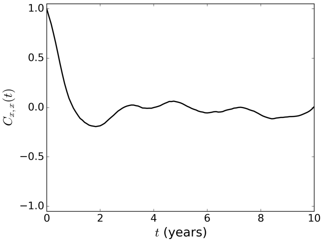
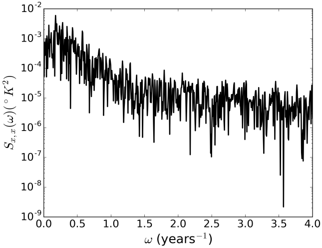
Motivation



(a) ???



(b) Sample correlation

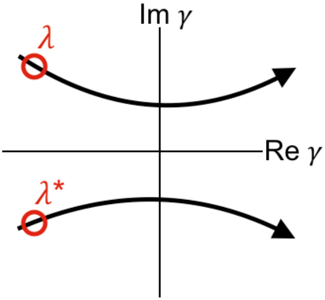


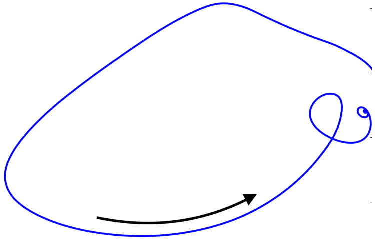
(c) Periodogram

Deterministic Hopf bifurcation

Normal Form [1] of Hopf bifurcation:

$$\begin{cases} dr &= (\delta r - r^3)dt \\ d\theta &= (\gamma - \beta r^2)dt \end{cases}$$





$\delta << \delta_c \rightarrow$ fast relaxation to fixed point

[1] Guckenheimer and Holmes [1983]

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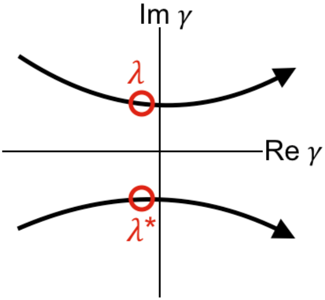

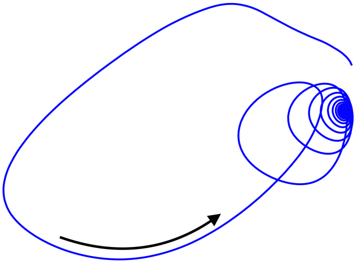

$\delta \approx \delta_c \rightarrow$ *Slowing Down*

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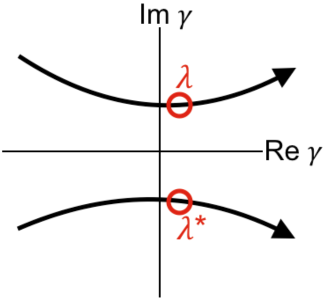

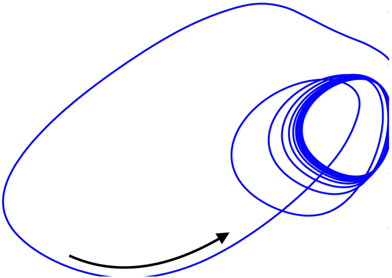

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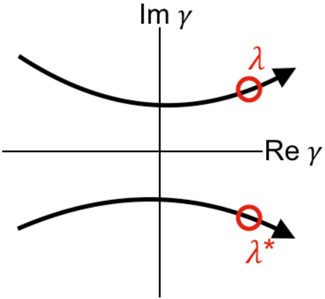
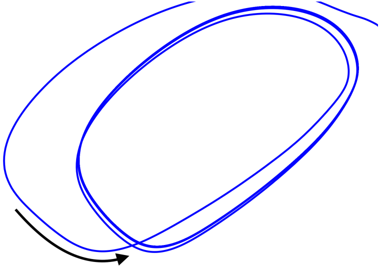




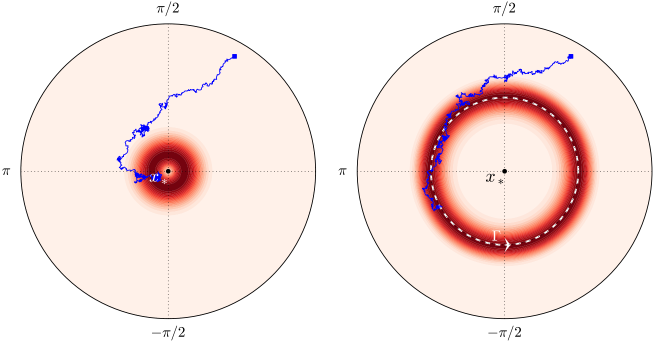
$\delta >> \delta_c \rightarrow$ Fast relaxation to periodic orbit

[1] Guckenheimer and Holmes [1983]

Stochastic Hopf equation

Perturbation of Hopf by white noise on x, y:

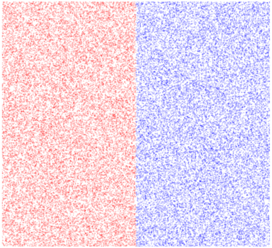
$$\begin{cases} dr &= (\delta r - r^3 + \frac{\epsilon^2}{2r})dt + \epsilon dW_r \\ d\theta &= (\gamma - \beta r^2)dt + \frac{\epsilon}{r} dW_\theta \end{cases}$$



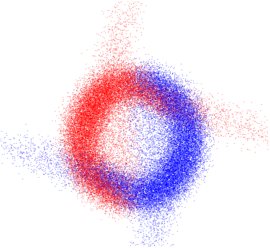
(a) $\delta < 0$ (b) $\delta > 0$

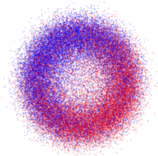
Ensemble mixing

(a) $t = 0$

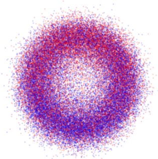


(b) $t = 1$





(c) $t = 5$



(d) $t = 10$

Markov semigroup

Process $X_t(\omega, x)$ governed by Îto SDE:

$$dX_t = F(X_t)dt + \Sigma(X)dW_t, \quad X_0 = x$$

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Process $X_t(\omega, x)$ governed by Îto SDE:

$$dX_t = F(X_t)dt + \Sigma(X)dW_t, \quad X_0 = x$$

Conditional expectations $u(x,t) = \mathbb{E}[f(X_t)|X_0 = x]$ governed by *Backward Kolmogorov Equation*

$$\partial_t u = \underbrace{F \cdot \nabla u + \frac{1}{2}\Sigma\Sigma^t : \nabla\nabla u,}_{\mathcal{G}u} \quad u(x,0) = f(x)$$

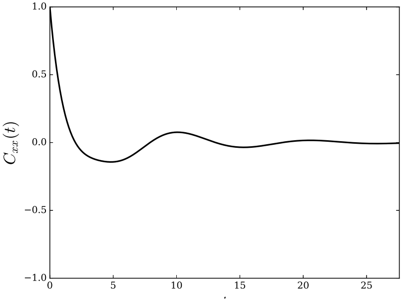
generating *Markov semigroup*

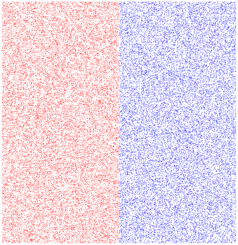
$$P_t u = e^{t\mathcal{G}} u, \quad u \in L^2_\mu(X)$$

Mixing and correlations

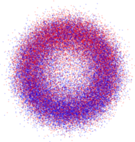
Strong mixing [2] implies:

$$C_{f,g}(t) = \int f \ P_t g \ d\mu \underset{t\to\infty}{\to} 0$$





$\xrightarrow{\mathcal{L}_{t\to\infty}}$



[2] Da Prato and Zabczyk [1996]

Spectral decomposition on $\sigma(\mathcal{G})$

Decomposition of correlation function:

$$C_{f,g}(t) = \sum_{k\geq 1} e^{\lambda_k t} w_k$$

with $w_k = \int f \ \psi_k \ d\mu \ \int \psi_k^* \ g \ d\mu$

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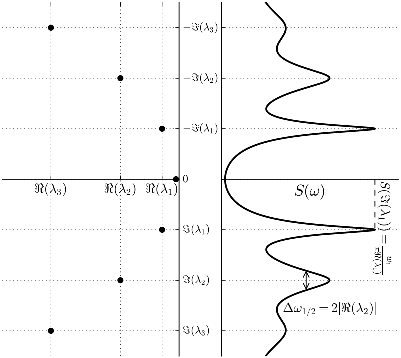
Decomposition of power spectrum:

$$S_{f,g}(\omega) = \sum_{k\geq 1} \frac{w_k}{\pi} \frac{-\Re(\lambda_k)}{(\omega - \Im(\lambda_k))^2 + \Re(\lambda_k)^2}$$



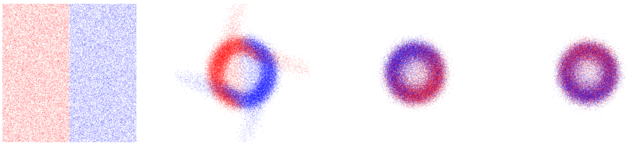
Stochastic Hopf

Hopf SDE:

$$\begin{cases} dr &= (\delta r - r^3 + \frac{\epsilon^2}{2r})dt + \epsilon dW_r \\ d\theta &= (\gamma - \beta r^2)dt + \frac{\epsilon}{r} dW_\theta \end{cases}$$

Backward Kolmogorov:

$$\partial_t u = \underbrace{\left(\delta r - r^3 + \frac{\epsilon^2}{2r}\right)\partial_r u + (\gamma - \beta r^2)\partial_\theta u + \frac{\epsilon^2}{2}\partial_{rr}^2 u + \frac{\epsilon^2}{2r^2}\partial_{\theta\theta}^2 u}_{\mathcal{G}u}$$



Small-noise expansion $\epsilon << 1$

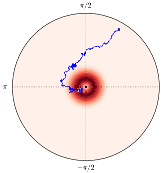
Expansion of the eigenvalues:

$$\lambda_k = \lambda_k^{(0)} + \epsilon \lambda_k^{(1)} + \dots$$

Rescaled deviations from deterministic trajectories:

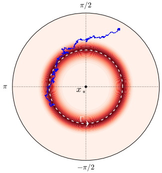
- $\delta < 0$

$$x' = \frac{x}{\epsilon}, \quad y' = \frac{y}{\epsilon}.$$



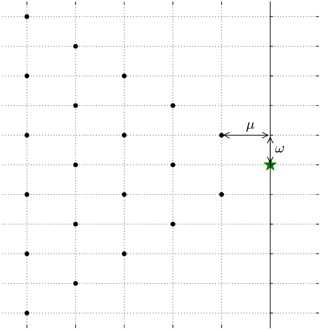
- $\delta > 0$

$$r' = \frac{r - \sqrt{\delta}}{\epsilon}, \qquad \phi' = \phi - \omega t.$$



Spectrum for $\delta < 0$

$$\lambda_{ln} = (l+n)\delta - i(l-n)\gamma + \mathcal{O}(\epsilon^2), \quad l, n \in \mathbb{N}$$



(a) Eigenvalues

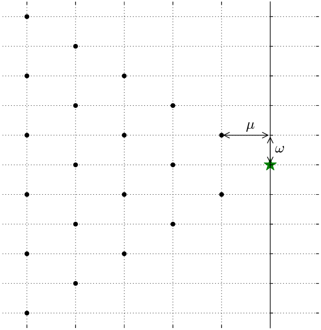
[3] Chen and Liu [2014]

Spectrum for $\delta < 0$

$$\lambda_{ln} = (l+n)\delta - i(l-n)\gamma + \mathcal{O}(\epsilon^2), \quad l, n \in \mathbb{N}$$

$$\psi_{ln}(r,\theta) \approx l! \left(\frac{\epsilon^2}{\delta}\right)^l e^{i(l-n)\theta} r^{n-l} L_l^{n-l}\left(-\frac{\delta r^2}{\epsilon^2}\right), \quad l < n$$

[3]



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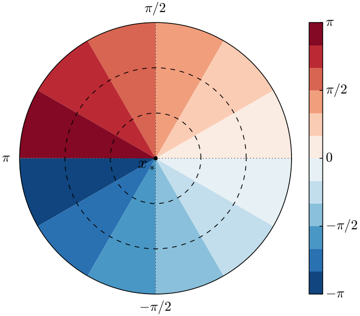
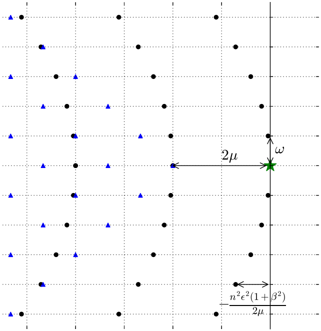


(b) Eigenvector ψ_{01}

[3] Chen and Liu [2014]

Spectrum for $\delta > 0$

$$\lambda_{ln} = -2l\delta - in(\gamma - \beta\delta) - \frac{n^2\epsilon^2(1+\beta^2)}{2\delta} + o(\epsilon^3)$$

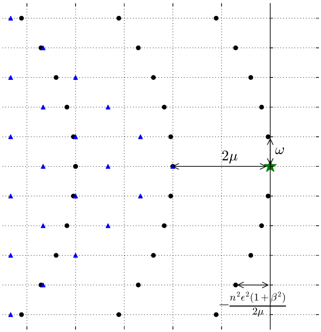


(a) Eigenvalues

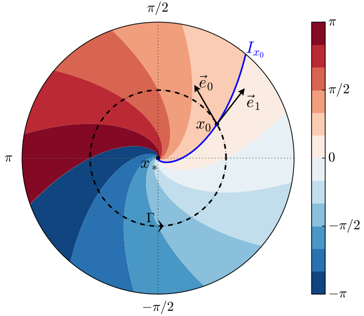
Spectrum for $\delta > 0$

$$\lambda_{ln} = -2l\delta - in(\gamma - \beta\delta) - \frac{n^2\epsilon^2(1+\beta^2)}{2\delta} + o(\epsilon^3)$$

$$\phi_{ln} \approx e^{in\left(\theta - \beta \log \frac{r}{\sqrt{\delta}}\right)} H_l\left(\frac{\sqrt{2\delta}}{\epsilon}\left(r - \sqrt{\delta}\right)\right)$$



(a) Eigenvalues



(b) Eigenvector ψ_{01}

Smoothing noise: geometric perspective

Hopf + only one radial forcing vector field $\epsilon\partial_r$:

$$\begin{cases} dr &= (\delta r - r^3)dt + \epsilon dW_r \\ d\theta &= (\gamma - \beta r^2)dt \end{cases}$$

[4] Hörmander [1968]

Smoothing noise: geometric perspective

Hopf + only one radial forcing vector field $\epsilon\partial_r$:

$$\begin{cases} dr & = (\delta r - r^3)dt + \epsilon dW_r \\ d\theta & = (\gamma - \beta r^2)dt \end{cases}$$

Test Hörmander's Lie bracket condition [4]:

$$\mathrm{span}\ \ \epsilon\partial_r = \mathbb{R}^2\ \ ?$$

If not $\rightarrow \mathcal{G}$ not elliptic

$$\mathrm{span}\ \ \{\epsilon\partial_r, [F, \epsilon\partial_r]\} = \mathbb{R}^2\ \ ?$$

$$\dots$$

If yes $\rightarrow \mathcal{G}$ hypoelliptic

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If yes $\rightarrow \mathcal{G}$ hypoelliptic

$$[F, \epsilon\partial_r] = \sigma(\delta - 3r^2)\partial_r - \sigma\beta r\partial_\theta$$

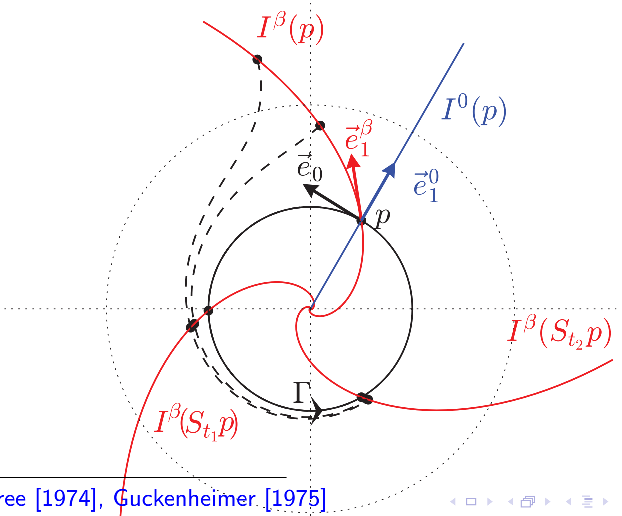
$$\Rightarrow \mathrm{span}\ \ \{\epsilon\partial_r, [F, \epsilon\partial_r]\} = \mathbb{R}^2 \quad \text{iff} \quad \beta \neq 0$$

[4] Hörmander [1968]

Twisting of the isochrons for $\beta \neq 0$

Leaves $I(p)$ of the stable manifold $\to$ generalisation of phase [5]

$$I(p) = \{q : \|p(t) - q(t)\| \underset{t\to\infty}{\to} 0\}, \quad p \in \Gamma$$



[5] Winfree [1974], Guckenheimer [1975]

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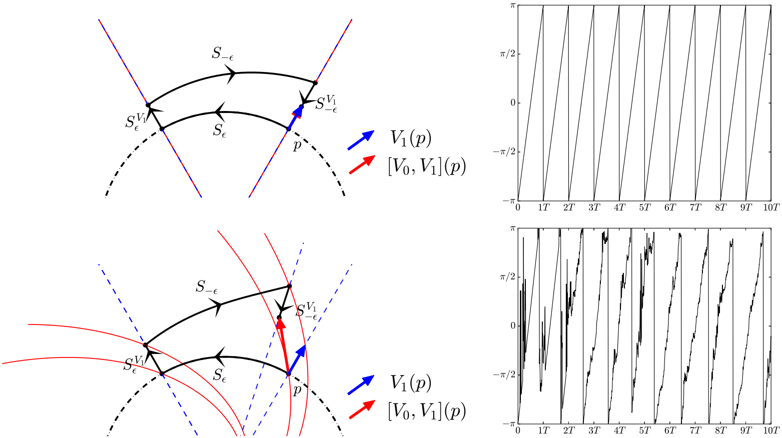
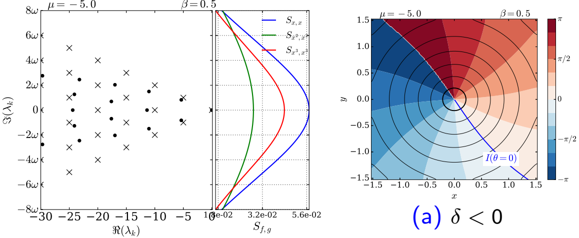
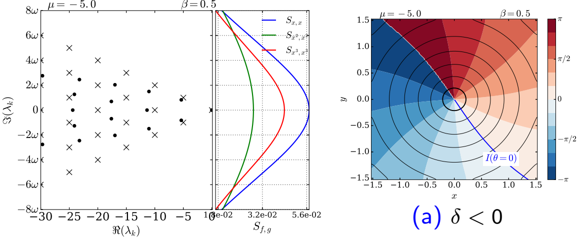




$\mu=-5.0$
$\beta=0.5$
$\Im(\lambda_k)$
$\Re(\lambda_k)$
$S_{f,g}$
$S_{x,x}$
S_{x^2,x^2}
S_{x^3,x^3}
3.2e-02
5.6e-02
$\mu=-5.0$
$\beta=0.5$
$I(\theta=0)$
x
y

(a) $\delta < 0$



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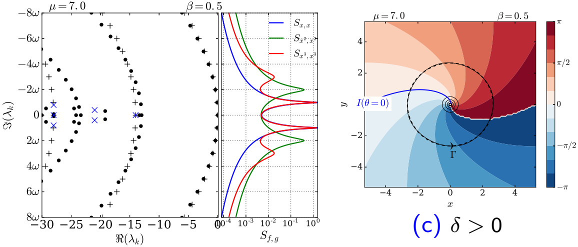
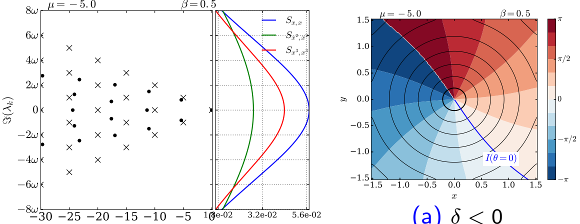


(c) $\delta > 0$



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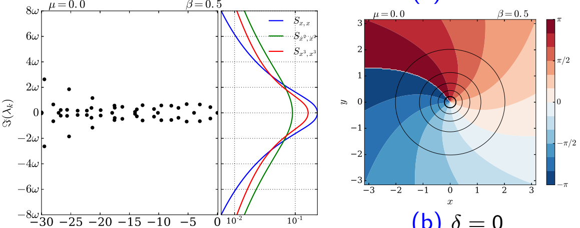
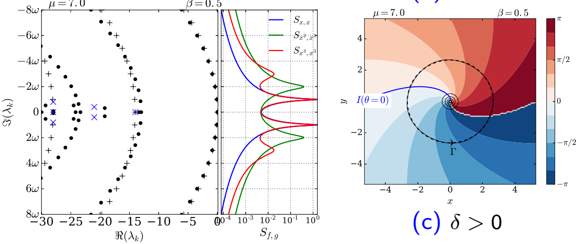


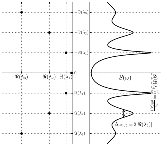
(b) $\delta = 0$

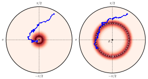


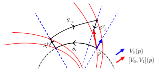
(c) $\delta > 0$

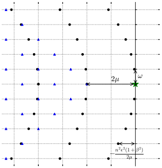
Conclusion

- *Mixing spectrum* relate correlations and power spectrum to the dynamics
- *Stochastic analysis* techniques for bifurcations
- *Geometric* approach to interaction of noise with the dynamics
- Small-noise expansions from *linear stability* of limit set









What about high-dimensional stochastic systems?

Reduced Markov operators

- Define reduced state space by $\mathcal{R} : X \to Y, \dim Y < \dim X$

[6] Kallenberg [2002]
[7] Chekroun et al. [2014]

Reduced Markov operators

- Define reduced state space by $\mathcal{R} : X \to Y, \dim Y < \dim X$
- Induced reduced function space $L^2_{\mathfrak{m}}(Y)$, with $\mathfrak{m} = \mathcal{R} * \mu$
- *Conditional measure* [6] $\mu_y : L^2_{\mu}(X) \to L^2_{\mathfrak{m}}(Y), y \in Y$ s.t.:

$$\int f d\mu = \int_Y \int_{\mathcal{R}^{-1}(y)} f d\mu_y d\mathfrak{m}$$

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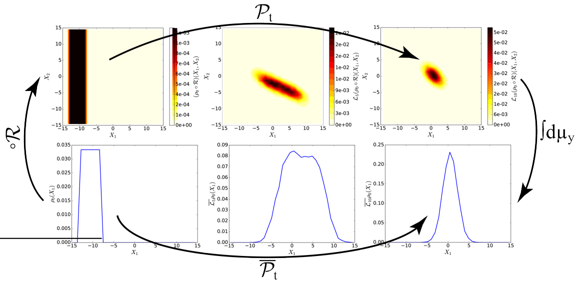
$$\int f d\mu = \int_Y \int_{\mathcal{R}^{-1}(y)} f d\mu_y d\mathfrak{m}$$

- Define *Reduced Markov Operators* $\overline{P}_t : L^2_{\mathfrak{m}}(Y) \to L^2_{\mathfrak{m}}(Y)$ [7]:

$$\overline{P}_t f = \int P_t(f \circ \mathcal{R}) d\mu_y$$



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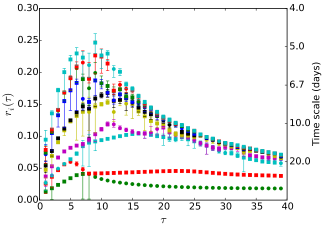
Applicability of reduction

Loss of information in reduction $\rightarrow \overline{P}_{t+s} \neq \overline{P}_t \overline{P}_s$

- Schütte [1999]: early work on almost-invariant sets
- Bittracher et al. [2015]: pseudo-generators for short lags
- Crommelin and Vanden-Eijnden [2011]: multi-scale systems (homogenization)
- Froyland et al. [2014]: multi-scale systems (fiber dynamics)

- Tantet et al. [2015]: leading eigenvalues for long lags.



Estimation from time series

- Galerkin truncation on indicators (bounded domain)

$$(\boldsymbol{P}_\tau)_{ij} = \frac{\left\langle \overline{P}_\tau \mathbb{1}_{B_i}, \mathbb{1}_{B_j} \right\rangle_{\mathfrak{m}}}{\mathfrak{m}(B_j)}.$$

- ML Estimator from a *single long time series* $\{y_s\}_{1\leq s\leq T}$

$$\widehat{(\boldsymbol{P}_\tau)_{ij}} = \frac{\#\{(y_s \in B_j) \wedge (y_{s+\tau} \in B_i)\}}{\#\{y_s \in B_j\}}, \tag{1}$$

- Get eigenvalues $\zeta_k(\tau) \in \mathbb{C}$ and left/right-eigenvectors ψ_k/ψ_k^*
- Convert to "generator" eigenvalues $\lambda_k(\tau)$ s.t. $e^{t\lambda_k(\tau)} = \zeta_k(\tau)$
- Reconstruct correlation functions for any observables

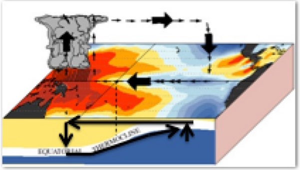
$$C_{f,g}(t) \approx (\boldsymbol{f} D(\boldsymbol{m})\boldsymbol{\Psi}'_\tau)e^{t\boldsymbol{\Lambda}_\tau}(\boldsymbol{\Psi}^*_\tau D(\boldsymbol{m})\boldsymbol{g}') - (\boldsymbol{f}\boldsymbol{m})(\boldsymbol{m}\boldsymbol{g}')$$

Application to Hopf bifurcation in Cane-Zebiak

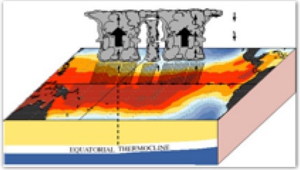
Model configuration:

- fully-coupled Cane-Zebiak model [8]:
  1.5 shallow-water ocean $+$ steady-state atmosphere on β-plane with Galerkin projection (thousands of DOF),
- additive white noise wind-forcing [9],
- $>$ 700 yr of simulation, spin-up removed,
- for different values of the coupling parameter δ around $\delta_c \approx 2.85$.

La Niña



El Niño



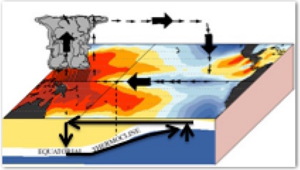
[8] van der Vaart et al. [2000]

[9] Roulston and Neelin [2000]

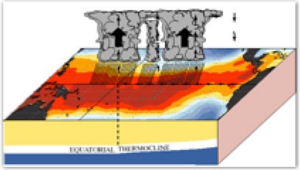
Transition matrix estimation

- reduced space (Easter SST, Western thermocline Depth),
- 100x100 boxes spanning $[-5, 3] \times [-4, 4]$ standard deviations,
- lag τ of 3 month.

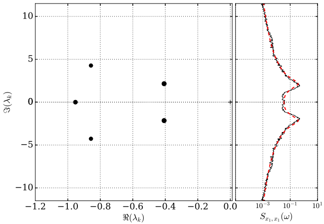
La Niña



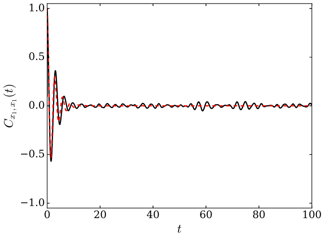
El Niño



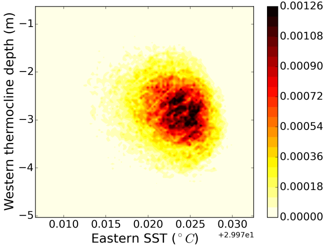
$\delta = 2.5$



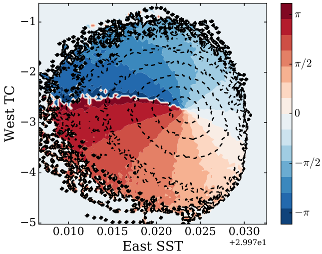
(a) Eigenval. (left), periodogram (black) and reconstruction (right)



(b) Sample correlation (black) and its reconstruction (right)

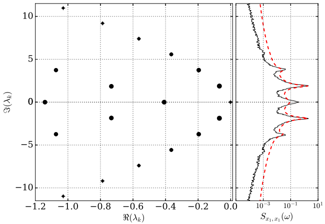


(c) Stationary density

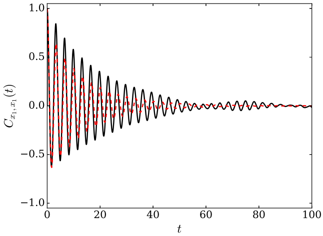


(d) Phase (filled) and amp. (line) of 2nd eigenvector

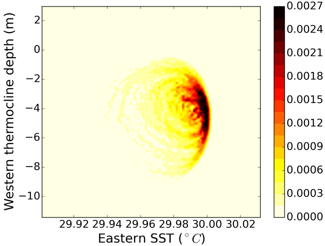
$\delta = 2.8$



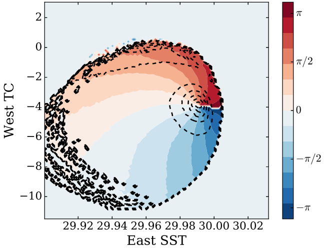
(a) Eigenval. (left), periodogram (black) and reconstruction (right)



(b) Sample correlation (black) and its reconstruction (right)

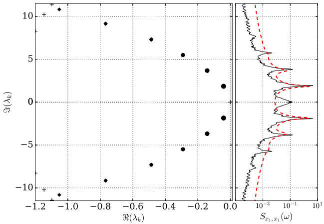


(c) Stationary density

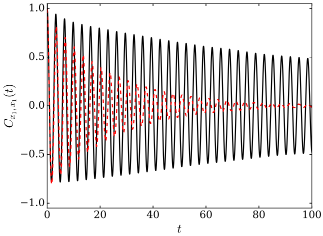


(d) Phase (filled) and amp. (line) of 2nd eigenvector

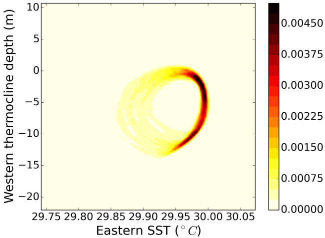
$\delta = 2.9$



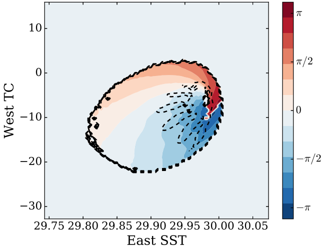
(a) Eigenval. (left), periodogram (black) and reconstruction (right)



(b) Sample correlation (black) and its reconstruction (right)

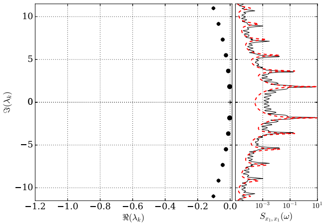


(c) Stationary density

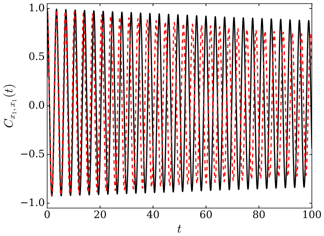


(d) Phase (filled) and amp. (line) of 2nd eigenvector

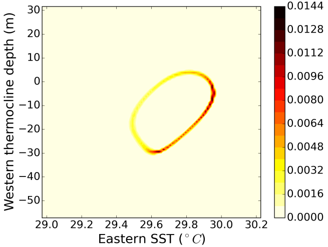
$\delta = 3.5$



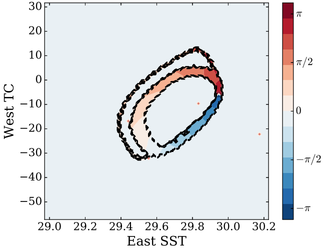
(a) Eigenval. (left), periodogram (black) and reconstruction (right)



(b) Sample correlation (black) and its reconstruction (right)

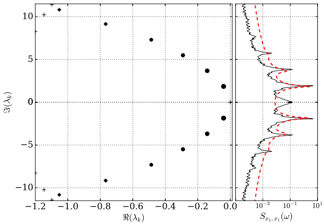


(c) Stationary density

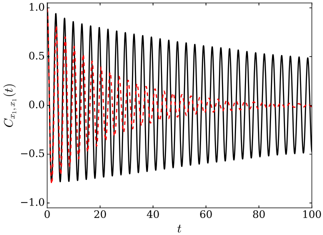


(d) Phase (filled) and amp. (line) of 2nd eigenvector

Dealing with non-Markovianity



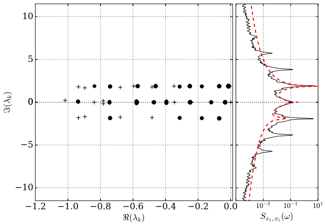
(a) Eigenval. (left), periodogram (black) and reconstruction (right)



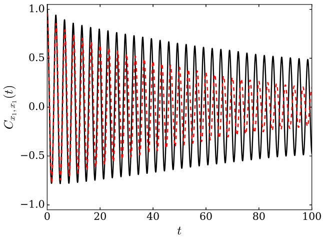
(b) Sample correlation (black) and its reconstruction (right)

Figure: $\tau = 3$ *months*, $\delta = 2.9$

Dealing with non-Markovianity



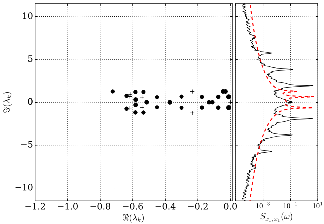
(a) Eigenval. (left), periodogram (black) and reconstruction (right)



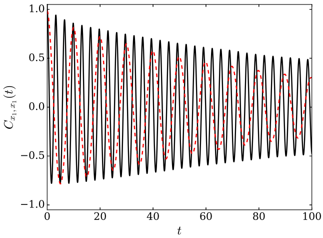
(b) Sample correlation (black) and its reconstruction (right)

Figure: $\tau = 20$ *months*, $\delta = 2.9$

Dealing with non-Markovianity



(a) Eigenval. (left), periodogram (black) and reconstruction (right)



(b) Sample correlation (black) and its reconstruction (right)

Figure: $\tau = 30$ *months*, $\delta = 2.9$

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