Distinguished models of intermediate Jacobians

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Arithmetic Aspects of Explicit Moduli Problems

The quest for the phantom

Mazur's Question

$X/\mathbb{Q}$ a smooth projective threefold, $h^{3,0} = h^{0,3} = 0$.
Is there an abelian variety $A/\mathbb{Q}$:

$$H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1)) \cong H^1(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)?$$

Such an A is called a phantom.

Joint work with Sebastian Casalaina-Martin (Boulder) and Charles Vial (Bielefeld).

Weights

$Y/\mathbb{Q}$ smooth, projective.

- $H^r(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ pure of weight r:

$$\left|\mathrm{Fr}_p \,|H^r(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\right| = \sqrt{p^r}.$$

- $H^r(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(j))$ is pure of weight $r - 2j$.

Hodge numbers

$Y/\mathbb{C}$ smooth, projective.

- $H^r(Y(\mathbb{C}),\mathbb{Q})$ has Hodge structure of weight r:

$$\begin{aligned} H^r(Y(\mathbb{C}),\mathbb{Q}) \otimes \mathbb{C} &= \oplus_{p+q=r} H^{p,q}(Y) \\ H^{p,q}(Y) &= H^q(Y(\mathbb{C}),\Omega^p_Y) \\ h^{p,q}(Y) &= \dim H^{p,q}(Y) \end{aligned}$$

- Ex: $\dim Y = 3$

$$\begin{array}{ccccccc} & & & h^{00} & & & \\ & & h^{10} & & h^{01} & & \\ & h^{20} & & h^{11} & & h^{02} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{31} & & h^{22} & & h^{13} & \\ & & h^{32} & & h^{23} & & \\ & & & h^{33} & & & \end{array}$$

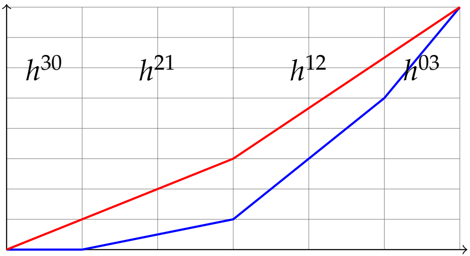
Newton over Hodge

$X/\mathbb{Z}_p$ smooth, projective, good reduction.

- $\mathrm{NP}(X, r)$ Newton polygon of Fr on $H^r_{\mathrm{dR}}(X_{\mathbb{Q}_p}) \cong H^r_{\mathrm{cris}}(X_p)$.
- $\mathrm{HP}(X, r)$ r^{th} Hodge polygon, vertices $(\sum_{0 \le j \le k} h^{r-j,j}, \sum_{0 \le j \le k} j h^{r-j})$.

Theorem (Mazur)

$\mathrm{NP}(X, r)$ *lies on or above* $\mathrm{HP}(X, r)$.



Divisibility

Corollary

If $h^{30}(X) = 0$, then each eigenvalue of Frobenius on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$ is an algebraic integer of size $\sqrt{p}$.

Proof.

- $\mathrm{NP}(X,3)$ over $\mathrm{HP}(X,3)$ implies all slopes of Fr_p on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ are ≥ 1.
- $\implies$ each eigenvalue α of Fr_p on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ divisible by p
- $\implies$ each eigenvalue α/p of Fr_p on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$ is algebraic integer of size $\sqrt{p}$.

$\square$

$H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$ could come from an abelian variety

Jacobians

Jacobians as Phantoms

If X/K smooth projective, then Pic^0_X is a phantom in degree 1.

From Kummer sequence

$$1 \longrightarrow \boldsymbol{\mu}_N \longrightarrow \mathcal{O}_X^\times \xrightarrow{[N]} \mathcal{O}_X^\times \longrightarrow 1$$

get

$$0 \longrightarrow H^1(X_{\overline{K}}, \boldsymbol{\mu}_N) \longrightarrow H^1(X_{\overline{K}}, \mathcal{O}_X^\times) \longrightarrow H^1(X_{\overline{K}}, \mathcal{O}_X^\times)$$

so

$$H^1(X_{\overline{K}}, \mathbb{Z}/N(1)) \cong \ker\left(H^1(X_{\overline{K}}, \mathcal{O}_X^\times) \to H^1(X_{\overline{K}}, \mathcal{O}_X^\times)\right) \cong \mathrm{Pic}^0_X[N](\overline{K}).$$

Complex Jacobians

$X/\mathbb{C}$ smooth projective

Exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

gives

$$H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^\times) \longrightarrow H^2(X,\mathbb{Z})$$

$$\cong \mathrm{Pic}_X(\mathbb{C})$$

$$\subseteq \mathrm{Pic}^0_X(\mathbb{C})$$

and so

$$\mathrm{Pic}^0_X(\mathbb{C}) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}.$$

Intermediate Jacobians

$$\mathrm{Pic}^0_X(\mathbb{C}) \cong \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$$
$$\cong \mathrm{Fil}^1 H^1(X,\mathbb{C})\backslash H^1(X,\mathbb{C})/H^1(X,\mathbb{Z}).$$

More generally, intermediate Jacobians are

$$J^{2n+1}(X) = \mathrm{Fil}^{n+1}\backslash H^{2n+1}(X,\mathbb{C})/H^{2n+1}(X,\mathbb{Z}).$$

If $H^{2n+1}(X,\mathbb{C})$ has Hodge level one, then

- $H^{2n+1} = H^{n+1,n} \oplus H^{n,n+1}$;
- Complex torus $J^{2n+1}(X)$ is actually an abelian variety.

Complete intersections: Deligne

Theorem (Deligne)

Suppose $X/\mathbb{Q}$ a complete intersection of dimension $2n+1$, and $H^{2n+1}(X,\mathbb{C})$ has Hodge level one. Then $J^{2n+1}(X_{\mathbb{C}})$ descends to an abelian variety $J/\mathbb{Q}$, and J is a phantom for X.

Idea

- Monodromy action on universal $\mathcal{J}^{2n+1}(\mathcal{X})$ over Hilbert scheme is irreducible.
- Descent.

Coniveau

X/K smooth projective.

$$N^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \subseteq \widetilde{N}^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \subseteq H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$$

- $N^r H^i$ from $Y \hookrightarrow X$ of codim r.
- $\widetilde{N}^r H^i$ is maximal $M \subset H^i$; $M(r)$ effective.

Generalized Tate Conjecture

$$N^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell) = \widetilde{N}^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell).$$

Abel–Jacobi

X/K smooth projective.

- $\mathrm{CH}^r(X) = \{\text{codim } r \text{ cycles}\} / \{\text{rat equiv}\}$ Chow group.
- $\mathrm{A}^r(X) \subset \mathrm{CH}^r(X)$ algebraically trivial cycles.

If $X/\mathbb{C}$, have Abel–Jacobi map

$$\mathrm{A}^{n+1}(X) \xrightarrow{\mathrm{AJ}} J^{2n+1}(X)$$

- $J_a^{2n+1}(X) := \mathrm{im}(\mathrm{AJ})$ is an abelian variety.
- $H^1(J_a^{2n+1}) = \mathrm{N}^n\, H^{2n+1}(X)(n)$.

Main result

Theorem (A.–C.-M.–V.)

X/K a smooth projective variety over a subfield of $\mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$. Then there exists an abelian variety J/K such that

$$J_{\mathbb{C}} = J_a^{2n+1}(X_{\mathbb{C}})$$

and the Abel–Jacobi map

$$\mathrm{A}^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J(\mathbb{C})$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant.

Lemma

There exist:

- C/K *a smooth projective geometrically irreducible curve;*
- $\gamma \in \mathrm{CH}^{n+1}(C \times X)$ *a correspondence on* $C \times X$*;*

such that induced map is surjective:

$$H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\gamma_*} \mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)).$$

Strategy

- $\exists f : Y \hookrightarrow X/K$, codim n,

$$f_* H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) = \mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell)(n).$$

- Bertini: $C \hookrightarrow Y$ a curve, $H^1(Y) \hookrightarrow H^1(C)$.
- γ Construct a correspondence via

$$H^1(C) \hookrightarrow H^{2d_Y-1}(Y) \xrightarrow[(L^{d_Y})^{-1}]{\sim} H^1(Y) \longrightarrow H^{2n+1}(X)$$

(Only middle arrow difficult; Lefschetz standard conjecture in degree one.)

Can take C geometrically irreducible using:

- $\beta : C \to \mathrm{Pic}^0_C$ inducing isomorphism on $H^1(\cdot, \mathbb{Q}_\ell)$;
- Bertini for geometrically irreducible variety Pic^0_C.

We have

$$J^1(C_{\mathbb{C}}) \xrightarrow{\gamma_*} J_a^{2n+1}(X_{\mathbb{C}}).$$

- $J^1(C_{\mathbb{C}}) = (\mathrm{Pic}^0_C)_{\mathbb{C}}$ has a distinguished model over K.
- Use this and γ_* to obtain model for $J_a^{2n+1}(X_{\mathbb{C}})$.

$\mathbb{C}/\overline{K}$

- $\mathbb{C}/\overline{K}$ is a regular extension of fields.
- $\underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}}) := \mathrm{tr}_{\mathbb{C}/\overline{K}}(J_a^{2n+1}(X_{\mathbb{C}}))$ is "largest" abelian variety defined over $\overline{K}$.

Rigidity:

$$\mathrm{Hom}_{\overline{K}}(J(C)_{\overline{K}}, \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})) = \mathrm{Hom}_{\mathbb{C}}(J(C_{\overline{K}})_{\mathbb{C}}, J_a^{2n+1}(X_{\mathbb{C}})).$$

Get surjection

$$J(C_{\overline{K}}) \twoheadrightarrow \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})$$

of abelian varieties over $\overline{K}$.

$\overline{K}/K$

- Need to show

$$J(C_{\overline{K}}) \xrightarrow{\gamma_*} \underset{=a}{J}^{2n+1}(X_{\mathbb{C}})$$

descends to K.

- Suffices to show

$$(\ker \gamma_*)[N](\overline{K})$$

stable under $\mathrm{Gal}(K)$.

Strategy suggested to us by Gabber.

Follow the arrows

$$J(C_{\overline{K}})[N] \longrightarrow \underline{J}_{a}^{2n+1}(X_{\mathbb{C}})[N]$$

Follow the arrows

$$\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{J}_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] & J_a^{2n+1}(X_{\mathbb{C}})[N]
\end{tikzcd}$$

Follow the arrows

$$
\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{\underline{J}}^{2n+1}_{a}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\sim"'] & J^{2n+1}_{a}(X_{\mathbb{C}})[N] \arrow[d, hook] \\
H^1_{\mathrm{an}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] & H^{2n+1}_{\mathrm{an}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1))
\end{tikzcd}
$$

Follow the arrows

$$\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underset{=a}{J}^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\sim"'] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, hook, "\cap"] \\
H^1_{\mathrm{an}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] \arrow[d, "\sim"'] & H^{2n+1}_{\mathrm{an}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1)) \arrow[d, "\sim"] \\
H^1_{\text{ét}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] & H^{2n+1}_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1))
\end{tikzcd}$$

Follow the arrows

$$
\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{J}_{=a}^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\sim"'] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, hook] \\
H^1_{\mathrm{an}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] \arrow[d, "\sim"'] & H^{2n+1}_{\mathrm{an}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1)) \arrow[d, "\sim"] \\
H^1_{\mathrm{\acute{e}t}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] \arrow[d] & H^{2n+1}_{\mathrm{\acute{e}t}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1)) \arrow[d, "\sim"] \\
H^1(C_{\overline{K}}, \mathbb{Z}/N(1)) \arrow[r, "\gamma_*"'] & H^{2n+1}(X_{\overline{K}}, \mathbb{Z}/N(n+1))
\end{tikzcd}
$$

Models

- Since $\ker(J^1(C)_{\overline{K}} \to \underline{J}^{2n+1}_a(X_{\mathbb{C}}))$ stable under $\mathrm{Gal}(K)$, we have a model J/K for $J^{2n+1}_a(X_{\mathbb{C}})$.
- How do we know this is the right model?

Recall the Abel–Jacobi map

$$A^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J_a^{2n+1}(X_{\mathbb{C}}).$$

Lemma

The model J/K of $J_a^{2n+1}(X_{\mathbb{C}})$ makes AJ $\mathrm{Gal}(K)$*-equivariant on torsion.*

Recall the Abel–Jacobi map

$$A^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J_a^{2n+1}(X_{\mathbb{C}}).$$

Lemma

The model J/K of $J_a^{2n+1}(X_{\mathbb{C}})$ makes AJ $\mathrm{Gal}(K)$*-equivariant on torsion.*

$$\begin{tikzcd}
A^{n+1}(X_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\mathrm{Lecomte}"', "\sim"] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "="] \\
A^{n+1}(X_{\overline{K}})[N] \arrow[r] \arrow[d, "\mathrm{Bloch}"', "\lambda^{n+1}"] & J_{\overline{K}}[N] \arrow[dl, hook] \\
H^{2n+1}(X_{\overline{K}}, \mathbb{Z}/N(n+1)) &
\end{tikzcd}$$

Corollary

J is a phantom for X in degree $2n+1$.

- Still want to show

$$\mathrm{A}^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J(\mathbb{C})$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant.

- Rigidity fails for non-torsion points (on abelian varieties) and cycles (on arbitrary varieties).

Key Tool

AJ : $A^{n+1}(X_{\mathbb{C}}) \to J_a^{2n+1}(X)(\mathbb{C})$ is *regular* (in the sense of Samuel).

Regular maps

- $X/k = \bar{k}$, A/k an abelian variety.
- An abstract group homomorphism

$$\mathrm{A}^i(X) \xrightarrow{\phi} A(k)$$

is regular if for every pointed variety (T, t_0), and every family of cycles $Z \in \mathrm{CH}^i(T \times X)$, the map of sets

$$T(k) \xrightarrow{w_Z} \mathrm{A}^i(X) \xrightarrow{\phi} A(k)$$

$$t \longmapsto [Z_t] - [Z_{t_0}]$$

Regular maps

- $X/k = \bar{k}$, A/k an abelian variety.
- An abstract group homomorphism

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is regular if for every pointed variety (T, t_0), and every family of cycles $Z \in \mathrm{CH}^i(T \times X)$, the map of sets is induced by a morphism

$$\begin{array}{ccccc} T(k) & \xrightarrow{w_Z} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\ t & \longmapsto & [Z_t] - [Z_{t_0}] & & \\ T & & \overset{\psi_Z}{\dashrightarrow} & & A \end{array}$$

Ω/k

Lemma

Ω/k an extension of algebraically closed fields of characteristic zero, X/k smooth projective, A/Ω an abelian variety,

$$\mathrm{A}^i(X_\Omega) \xrightarrow{\phi} A(\Omega)$$

regular and surjective. Then $A = (\underline{\underline{A}})_\Omega$; $\phi = (\underline{\underline{\phi}})_\Omega$; and

$$\mathrm{A}^i(X) \xrightarrow{\underline{\underline{\phi}}} \underline{\underline{A}}(k)$$

is regular and surjective.

Key Idea

Use rigidity; $\mathrm{A}^i(X_\Omega)[N] \cong \mathrm{A}^i(X_{\overline{K}})[N]$.

$\overline{K}/K$

Proposition

K perfect, X/K smooth and projective, A/K an abelian variety. Suppose

$$\mathrm{A}^i(X_{\overline{K}}) \xrightarrow{\phi} A(\overline{K})$$

is regular and surjective.
If $\phi[\ell^n]$ is $\mathrm{Gal}(K)$-equivariant for all n, then ϕ is $\mathrm{Gal}(K)$-equivariant.

Key Idea

For test varieties (T, t_0), abelian varieties are enough.

Weil's lemma

Algebraically trivial cycles are witnessed by abelian varieties:

Lemma

Let X/K be a scheme of finite type over a field, and let $\alpha \in \mathrm{A}^i(X_{\overline{K}})$ be an algebraically trivial cycle class.
Then there exist an abelian variety B/K, a cycle class $Z \in \mathrm{CH}^i(B \times X)$, and a $t \in Z(\overline{K})$ such that

$$\alpha = [Z_t] - [Z_0].$$

- Weil (and Lang) prove this for $K = \overline{K}$.
- Their proof breaks down over arbitrary K; may not be enough Brill-Noether generic K-rational points.

For regular maps, $\mathrm{Gal}(K)$-equivariance on torsion implies equivariance:

- Weil's lemma: Find B/K abelian variety, $Z \in \mathrm{CH}^i(B \times X)$,

$$B(\overline{K}) \xrightarrow{w_Z} \mathrm{A}^i(X_{\overline{K}}) \longrightarrow A(\overline{K})$$

  surjective.
- On torsion, have

$$B(\overline{K})[\ell^\infty] \xrightarrow{w_Z[\ell^\infty]} \mathrm{A}^i(X_{\overline{K}})[\ell^\infty] \xrightarrow{\phi[\ell^\infty]} A(\overline{K})[\ell^\infty]$$

- $\phi[\ell^\infty]$ $\mathrm{Gal}(K)$-equivariant by hypothesis.
- $w_Z[\ell^\infty]$ is $\mathrm{Gal}(K)$-equivariant since $Z/K, 0 \in B(K)$.

So $\psi : B_{\overline{K}} \to A_{\overline{K}}$ descends to K.

Consequence

Corollary

If $K \subset \mathbb{C}$, *then* $\mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J(\mathbb{C})$ *is* $\mathrm{Aut}(\mathbb{C}/K)$*-equivariant.*

What about the explicit moduli?

Classification

$X_n(a_1, \cdots, a_d) \subset \mathbb{P}^{n+d}$ a smooth complete intersection of dimension n, multidegree $\underline{a}$.

Rapoport's Classification

A smooth complete intersection has Hodge level one if and only if it belongs to the following list:

- $X_n(2,2)$ intersection of two quadrics in $\mathbb{P}^{n+2}$;
- $X_n(2,2,2)$ intersection of three quadrics;
- $X_3(3)$ cubic threefold;
- $X_3(2,3)$ a threefold, realized as the intersection of a quadric and cubic;
- $X_5(3)$ cubic fivefold;
- $X_3(4)$ quartic threefold.

Period maps for Hodge level one

Distinguished models give new proof of:

Theorem (Deligne)

Let $\mathcal{V}$ be a moduli space of complete intersection varieties of Hodge level one. The period map

$$\mathcal{V}(\mathbb{C}) \longrightarrow \mathcal{A}_{g(\mathcal{V})}(\mathbb{C})$$

is induced by a morphism

$$\mathcal{V}_{\mathbb{Q}} \longrightarrow \mathcal{A}_{g(\mathcal{V}),\mathbb{Q}}$$

over $\mathbb{Q}$.

From points to period maps

Proof.

- If $X \in \mathcal{V}(\mathbb{C})$,
$$\mathrm{CH}_0(X)_{\mathbb{Q}}, \cdots, \mathrm{CH}_{n-1}(X)_{\mathbb{Q}}$$
spanned by linear sections (Otwinoska).
- Decomposition of the diagonal; $\mathrm{A}^n(X) \to J^{2n+1}(X)$ surjective, so $J^{2n+1}(X) = J_a^{2n+1}(X)$ (Bloch-Srinivas).
- If $\Gamma \in \mathrm{CH}(J^{2n+1}(X) \times X)$ witnesses $J^{2n+1}(X)$ as intermediate Jacobian, and $\sigma \in \mathrm{Aut}(\mathbb{C})$, then Γ^σ witness $J^{2n+1}(X)^\sigma$ as intermediate Jacobian of X^σ.
- So,
$$\left\{(X, J^{2n+1}(X))\right\} \subset (\mathcal{V} \times \mathcal{A}_{g(\mathcal{V})})(\mathbb{C})$$
stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, and the period map descends.

□

Cubic surface

A cubic surface $X/\mathbb{C}$ has no periods:

- $H^0(X, \Omega^2) = 0$;
- Hodge filtration is trivial:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 6 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Nonetheless:

Cubic surfaces

Theorem (Allcock-Carlson-Toledo, Dolgachev-van Geemen-Kondō)

There is an open immersion

$$\mathcal{S}_{\mathbb{C}} \hookrightarrow \Gamma\backslash\mathbb{B}^4$$

where

- $\mathcal{S} = \mathcal{V}_2(3)$ *is the moduli space of cubic surfaces;*
- $\Gamma \cong \mathrm{SU}_{1,4}(\mathbb{Z}[\zeta_3])$.

How do they do it?

Given $X/\mathbb{C}$ a cubic surface, A-C-T:

- Let $Y \to \mathbb{P}^3$ be triple cover ramified along X.
- Y is a cubic threefold with $\boldsymbol{\mu}_3$-action.
- Compute the periods of Y. *Equivalently,* $J^3(Y)$.

Get diagram of spaces over $\mathbb{C}$

$$\begin{tikzcd} \widetilde{\mathcal{S}}_{\mathbb{C}} = \mathcal{H}(3,3,1)_{\mathbb{C}} \arrow[r] \arrow[d, "\boldsymbol{\mu}_3"'] & \mathcal{T}_{\mathbb{C}} \arrow[r] & \mathcal{A}_{5,\mathbb{C}} \\ \mathcal{S}_{\mathbb{C}} \end{tikzcd}$$

$\mathcal{S}$	cubic surfaces;
$\mathcal{T}$	cubic threefolds;
$\mathcal{H}(n,r,d)$	uniform cyclic covers of $\mathbb{P}^n$ of degree r, branch along divisor of degree rd.

Occult periods

- $\boldsymbol{\mu}_3$ action on Y gives $\mathbb{Z}[\zeta_3]$ action on $J^3(Y)$.
- Let
$$\mathcal{A}_{\mathbb{Z}[\zeta_3],(1,4)}$$
be the moduli space (over $\mathbb{Z}[\zeta_3, 1/3]$) of principally polarized abelian fivefolds with action by $\mathbb{Z}[\zeta_3]$ of signature $(1,4)$.
- Image of $\widetilde{\mathcal{S}}_{\mathbb{C}} \to \mathcal{A}_{5,\mathbb{C}}$ lands in $\mathcal{A}_{\mathbb{Z}[\zeta_3],(1,4),\mathbb{C}}$.

Note: $\mathcal{A}_{\mathbb{Z}[\zeta_3],(1,4),\mathbb{C}}$ is an arithmetic quotient of $\mathbb{B}^4$, the complex 4-ball.

Occult period map descends

Theorem (Kudla–Rapoport,A.-, A–C-M–V)

The Allcock-Carlson-Toledo map is the base change of a morphism

$$\widetilde{\mathcal{S}}_{\mathbb{Q}(\zeta_3)} \longrightarrow \mathcal{A}_{\mathbb{Z}[\zeta_3],(1,4),\mathbb{Q}(\zeta_3)}$$

of stacks over $\mathbb{Q}(\zeta_3)$.

Proofs.

K–R Deligne strategy (irreducibility of monodromy);

A.– Construct intermediate Jacobians geometrically;

A–C-M–V Distinguished models.

□

In fact, spreads to $\mathbb{Z}[\zeta_3, 1/6]$.

Three views of six points

Let $\mathcal{M}_{0,6}$ be the moduli space of six points on a line.

Proposition

$\mathcal{M}_{0,6}(\mathbb{C})$ is open in $\Gamma\backslash\mathbb{B}^3$, an arithmetic quotient of the 3-ball.

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Proposition

$\mathcal{M}_{0,6}(\mathbb{C})$ is open in $\Gamma \backslash \mathbb{B}^3$, an arithmetic quotient of the 3-ball.

Three reasons:

- Picard curves;
- K3 surfaces;
- Cubic surfaces.

Curves

- $$\begin{array}{ll} D = \{P_1, \cdots, P_6\} & f(x) \\ C \to \mathbb{P}^1 \text{ cyclic triple cover} & y^3 = f(x) \\ \quad \text{ramified along } D & \\ J = \mathrm{Jac}(C) & \end{array}$$

  Then J has action by $\mathbb{Z}[\zeta_3]$ of signature $(1,3)$.
- Torelli map factors:

$$\mathcal{M}_{0,6} \longrightarrow \mathcal{A}_{\mathbb{Z}[\zeta_3],(1,3)} \hookrightarrow \mathcal{A}_4$$

- Count dimensions:

$$\dim \mathcal{M}_{0,6} = 6 \dim \mathbb{P}^1 - \dim \mathrm{Aut}(\mathbb{P}^1) = 6 - 3 = 3$$
$$\dim \mathcal{A}_{\mathbb{Z}[\zeta_3],(1,3)} = 1 \cdot 3 = 3$$

K3 surfaces (d'apres Kondō)

- $$\begin{array}{ll} D = \{P_1, \cdots, P_6\} & f(x) \\ C \to \mathbb{P}^1 \text{ cyclic triple cover} & y^3 = f(x) \\ \quad\quad \text{ramified along } D & \\ Y \to \mathbb{P}^2 \text{cyclic triple cover} & z^3 = (y^3 - f(x)) \\ \quad\quad \text{ramified along } Y & \\ Z \text{ minimal resolution of} Y & \end{array}$$

- Then Z is a K3 surface with (diagonal) $\boldsymbol{\mu}_3$-action.

Lattice polarizations

- Consider lattices

$$\begin{aligned} L_{K3} &= U^3 \oplus E_8(-1)^2 \\ L &= U \oplus E_6(-1) \oplus A_2(-1)^3 \\ L_{K3} &\cong L \oplus L^\perp \\ L^\perp &= A_2 \oplus A_2(-1)^3 \end{aligned}$$

- Cycles from construction give primitive $L \hookrightarrow \mathrm{Pic}(Z)$.
- $(L \otimes \mathbb{Q}_\ell)^\perp \subset H^2(Z, \mathbb{Q}_\ell)$ free over $\mathbb{Z}[\zeta_3] \otimes \mathbb{Q}_\ell$, Hermitian form of signature $(1,3)$.
- $Z \in \mathcal{K}_{L,\boldsymbol{\mu}_3,(1,3)}$, moduli space of L-polarized K3 surfaces with action by $\boldsymbol{\mu}_3$, $\cdots$.
- Get map

$$\mathcal{M}_{0,6} \longrightarrow \mathcal{K}_{L,\boldsymbol{\mu}_3,(1,3)}$$

Periods for K3 surfaces

- $\mathcal{S}h^{L}$ Shimura variety attached to $\mathrm{SO}_{L^\perp}$.

Example

$\mathcal{S}h^{L(2d)}(\mathbb{C}) \cong \Gamma\backslash\mathbb{X}^{L(2d)}$, an arithmetic quotient of a 19-dimensional Hermitian symmetric domain of type IV.

Theorem

The period map gives an open embedding $\mathcal{K}_{L(2d)}(\mathbb{C}) \hookrightarrow \mathcal{S}h^{L(2d)}(\mathbb{C})$.

Theorem (Rizov, Madapusi-Pera)

The period map descends to $\mathbb{Z}[1/2d]$.

Integral period maps

Proposition

The period map descends to maps over $\mathbb{Z}[\zeta_3, 1/6d]$

$$\begin{tikzcd}
\mathcal{K}_{L,\boldsymbol{\mu}_3,(1,3)} \arrow[r, hook] \arrow[d] & \mathcal{K}_L \arrow[r] \arrow[d] & \mathcal{K}_{L(2d)} \arrow[d] \\
\mathcal{A}_{\mathbb{Z}[\zeta_3],(1,3)} \arrow[r, hook] & \mathcal{S}h^L \arrow[r] & \mathcal{S}h^{L(2d)}
\end{tikzcd}$$

where horizontal arrows are closed, vertical are étale.

Integral period maps

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where horizontal arrows are closed, vertical are étale.

Proof.

- Moduli spaces of structured K3 surfaces are smooth.
- Integral canonical models of Shimura varieties (Milne, Vasiu, Kisin).

□

Integral period maps

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where horizontal arrows are closed, vertical are étale.

Now, compose with $\mathcal{M}_{0,6} \to \mathcal{K}_{L,\boldsymbol{\mu}_3,(1,3)}$.

Cubics

Since a cubic surface is the blowup of a projective plane at six points, consider the following moduli spaces:

$\mathcal{S}$ Smooth cubic surfaces;

$\mathcal{S}^{\mathrm{st}}$ Stable cubic surfaces;

$\mathcal{S}^n = \mathcal{S}^{\mathrm{st}} \smallsetminus \mathcal{S}$ nodal cubic surfaces;

$\mathcal{M}^{\circ}_{\mathbb{P}^2,6}$ 6 points in the projective plane, general position;

$\mathcal{M}^{\mathrm{st}}_{\mathbb{P}^2,6}$ allow points to lie on smooth conic.

Geometry:

$$\mathcal{M}^{\circ}_{\mathbb{P}^2,6} \longrightarrow \mathcal{S}$$

Occult period:

$$\mathcal{S} \longrightarrow \mathcal{A}_{\mathbb{Z}[\zeta_3],(1,4)}$$

Finer analysis shows:

$$
\begin{tikzcd}
\mathcal{M}^{\circ}_{\mathbb{P}^2,6} \arrow[r] \arrow[d, hook] & \mathcal{S} \arrow[r] \arrow[d, hook] & \mathcal{A}_{(1,4)} \smallsetminus \mathcal{A}_{(1,3)} \times \mathcal{A}_{(0,1)} \arrow[d, hook] \\
\mathcal{M}^{\mathrm{st}}_{\mathbb{P}^2,6} \arrow[r] & \mathcal{S}^{\mathrm{st}} \arrow[r] & \mathcal{A}_{(1,4)} \\
\mathcal{M}^{\mathrm{st}}_{\mathbb{P}^2,6} \smallsetminus \mathcal{M}^{\circ}_{\mathbb{P}^2,6} \arrow[r] \arrow[u, hook] & \mathcal{S}^{n} \arrow[r] \arrow[u, hook] & \mathcal{A}_{(1,3)} \times \mathcal{A}_{(0,1)} \arrow[u, hook]
\end{tikzcd}
$$

Finer analysis shows:

$$
\begin{tikzcd}
\mathcal{M}^{\circ}_{\mathbb{P}^2,6} \arrow[r] \arrow[d, hook] & \mathcal{S} \arrow[r] \arrow[d, hook] & \mathcal{A}_{(1,4)} \smallsetminus \mathcal{A}_{(1,3)} \times \mathcal{A}_{(0,1)} \arrow[d, hook] \\
\mathcal{M}^{\mathrm{st}}_{\mathbb{P}^2,6} \arrow[r] & \mathcal{S}^{\mathrm{st}} \arrow[r] & \mathcal{A}_{(1,4)} \\
\mathcal{M}_{0,6} \arrow[r] \arrow[u, hook] & \mathcal{S}^{n} \arrow[r] \arrow[u, hook] & \mathcal{A}_{(1,3)} \times \mathcal{A}_{(0,1)} \arrow[u, hook]
\end{tikzcd}
$$

$\boxed{\text{Thanks!}}$