On the p-ranks of Prym varieties

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p-rank

- k algebraically closed field of characteristic $p > 0$,
- A abelian variety of dimension g over k,
- p-rank of A is the number f_A such that $\#A[p](k) = p^{f_A}$,
- If C is a curve of genus g over k then its p-rank is the p-rank of $\mathrm{Jac}(C)$ and
- $0 \leq f_A \leq g$,
- C is called *ordinary* if p-rank of C is g and *almost ordinary* if p-rank of C is $g - 1$.

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Fundamental Questions

Question

Given (p, g, f) is there a curve C of genus g with p-rank f defined over an algebraically closed field of characteristic p?

YES, by Faber and Van der Geer

- Stratify by p-rank: $\mathcal{M}_g^0 \subset \ldots \subset \mathcal{M}_g^{g-1} \subset \mathcal{M}_g^g$
  Every component of $\mathcal{M}_g^f$ has codimension $g - f$ in $\mathcal{M}_g$(i.e. has dim 2g-3+f).

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Similarly:

- Faber and Van der Geer : Every component of $\mathcal{M}_g^f$ has codimension $g - f$ in $\mathcal{M}_g$(dim 2g-3+f).
- Norman and Oort: $\mathcal{A}_g^f$ has codimension $g - f$ in $\mathcal{A}_g$
- Glass and Pries, Pries and Zhu: Every component of $\mathcal{H}_g^f$ has codimension $g - f$ in $\mathcal{H}_g$(dim g-1+f).

where $\mathcal{A}_g$ abelian varieties, $\mathcal{H}_g$ hyperelliptic curves

By Chai and Oort $\mathcal{A}_g^f$ is irreducible for $g \geq 3$

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Similar Questions about Prym varieties

Suppose :
X has genus $\geq 2, \ell \neq p$, prime,
$\pi : Y \to X$ an unramified $\mathbb{Z}/\ell\mathbb{Z}$-cover.

Definition

The *Prym variety* P_π is the connected component containing 0 of the norm map on Jacobians i.e.
if σ generates $\mathrm{Gal}(Y/X)$ then $P_\pi = \ker(1 + \sigma + \ldots + \sigma^{\ell-1})^0$.

- If $X \in \mathcal{M}_g$ then $Y \in \mathcal{M}_{\ell(g-1)+1}$ and
- $\mathrm{Jac}(Y) \sim \mathrm{Jac}(X) \oplus P_\pi$, so $P_\pi \in \mathcal{A}_{(g-1)(\ell-1)}$ and
- If $X \in \mathcal{M}_g^f, P_\pi \in \mathcal{A}_{(g-1)(\ell-1)}^{f'}$ then $Y \in \mathcal{M}_g^{f+f'}$.

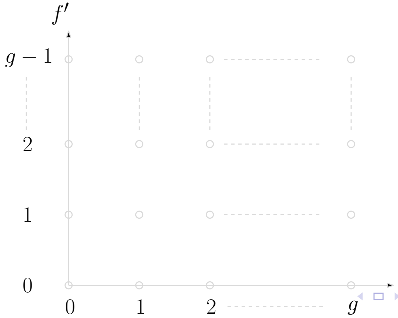
Notation

$\pi : Y \to X, \mathrm{Jac}(Y) \sim \mathrm{Jac}(X) \oplus P_\pi$

- $\mathcal{R}_{g,\ell} = \{(\pi : Y \to X), X \in \mathcal{M}_g, \pi$ unramified $\mathbb{Z}/\ell\mathbb{Z} -$ cover$\}$
- $\Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g$, natural projection, $(\pi : Y \to X) \mapsto X$
  - Π_ℓ is finite
  - $\dim \mathcal{R}_{g,\ell} = \dim \mathcal{M}_g = 3g - 3$

Question

What is the interaction between the p-ranks f and f'?



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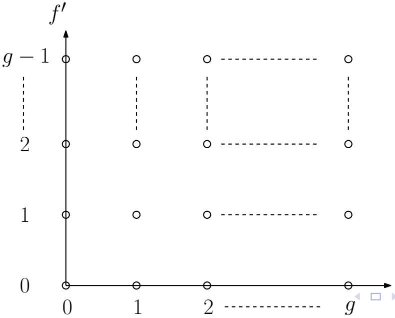

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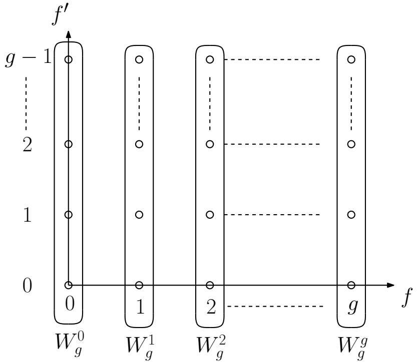
- $\mathcal{R}_{g,\ell} = \{(\pi : Y \to X), X \in \mathcal{M}_g, \pi$ unramified $\mathbb{Z}/\ell\mathbb{Z}$ – cover$\}$
- $W_g^f = \{(\pi : Y \to X) | (\pi : Y \to X) \in \mathcal{R}_{g,\ell}, X \in \mathcal{M}_g^f\}$
- ◇ $W_g^f = \Pi_\ell^{-1}(\mathcal{M}_g^f)$ and $\dim W_g^f = \dim \mathcal{M}_g^f = 2g - 3 + f$



Main Result 1

Let $g \geq 2$ and $0 \leq f \leq g$.

Theorem 1 (O., Pries)

Let $\ell \neq p$ and $(g, f) \neq (2, 0)$.

Prym varieties of all unramified cyclic degree ℓ covers of a generic curve X of $p-$rank f is ordinary.

For each irreducible component S of $\mathcal{M}_g^f$, $\Pi_\ell^{-1}(S)$ is irreducible of dimension $2g - 3 + f$ and the cover represented by the generic point of $\Pi_\ell^{-1}(S)$ has an ordinary Prym.

If Q is an irreducible component of W_g^f then the Prym of the cover represented by the generic point of Q is ordinary.

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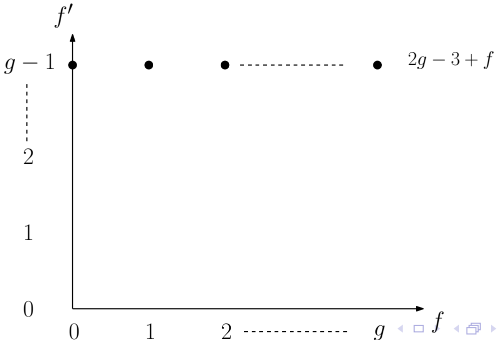
By Theorem 1 we know the dimension of the folowing stratum:



Comparing with previous work

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This generalizes the following theorem:
Nakajima, 1983: The cover represented by the generic point of $\mathcal{R}_{g,\ell}$ has an ordinary Prym.
and also:
Raynaud, 1982: For any genus g curve X and for sufficiently large ℓ, there is an unramified $\mathbb{Z}/\ell\mathbb{Z}$-cover π such that P_π is ordinary.

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Idea of the proof

Aim: is to produce unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ such that $X \in \mathcal{M}_g^f$ and P_π is ordinary.

Naive idea: Build a cover of singular curves, deform it to a smooth curve and proceed by induction

STP 1 Let S_0 be an irreducible component of $\overline{\mathcal{M}}_g^f$. Then $\Pi_\ell^{-1}(S_0)$ is also irreducible, follows from a result of Achter and Pries

STP 2 An irreducible component Q of W_g^f intersects a particular boundary

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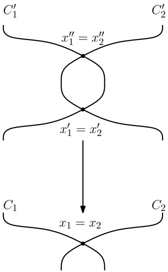
In fact, Q contains a component of $\kappa_{i,g-i}(W_{i,1}^{f_1} \times W_{g-i,1}^{f_2})$



$1 \to \mathbb{T} \to P_\pi \to P_{\pi_1} \oplus P_{\pi_2} \to 1$, where $\mathbb{T}$ is a torus of rank $\ell - 1$.

Idea of the proof

STP 3 Inductive Step:
Choose C_1, C_2 with p-ranks f_1, f_2 s.t. $f_1 + f_2 = f$ such that there exists $\pi_1 : C_1' \to C_1, \pi_2 : C_2' \to C_2$ s.t. P_{π_1}, P_{π_2} are ordinary. Then

$$f_\pi = f_{\pi_1} + f_{\pi_2} + \ell - 1$$

$$f_\pi = (\ell - 1)(g_1 - 1), f_{\pi_2} = (\ell - 1)(g_2 - 1),$$

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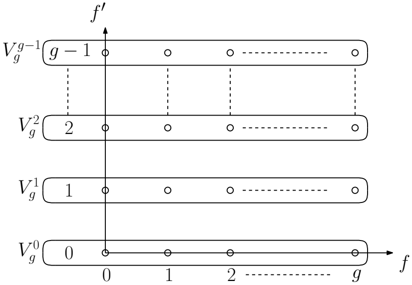
- $\mathcal{R}_g = \{(\pi : Y \to X), X \in \mathcal{M}_g, \pi$ unramified $\mathbb{Z}/2\mathbb{Z}$ – cover$\}$
- $V_g^{f'} = \{(\pi : Y \to X) | (\pi : Y \to X) \in \mathcal{R}_g, P_\pi$ has p-rank $f'\}$



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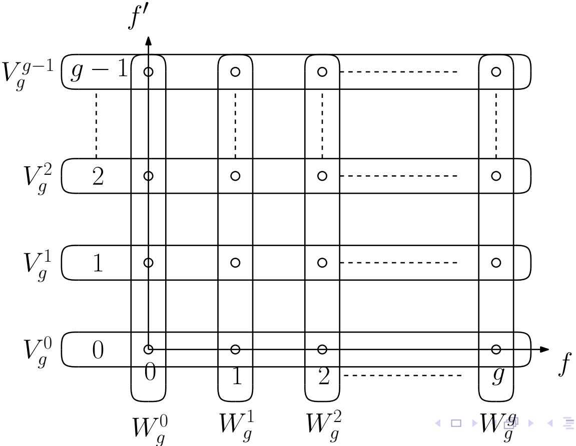
- $\mathcal{R}_g^{(f,f')} = \{(\pi : Y \to X) \in \mathcal{R}_g, X \in \mathcal{M}_g^f, P_\pi \in \mathcal{A}_{g-1}^{f'}\}$

◇ $\mathcal{R}_g^{(f,f')} = W_g^f \cap V_g^{f'}$



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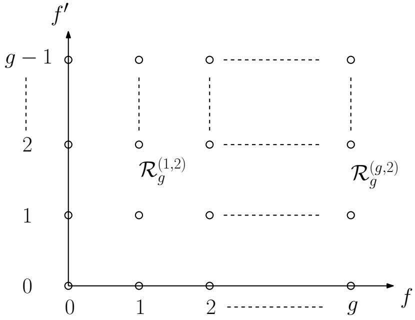

Question

What is the interaction between the p-ranks f and f'?
What can be said about the dimension of $\mathcal{R}_g^{(f,f')}$?

Main Result 2

Let $g \geq 2$ and $0 \leq f \leq g$. For $\ell = 2$ and $p \geq 5$.

Theorem 2 (O., Pries)

For a curve of genus g and p-rank f there is an unramified double cover π such that $P_\pi$is almost ordinary(has p-rank $g-2$)

For each irreducible component S of $\mathcal{M}_g^f$, the locus of points for which there exists an unramified double cover π with P_π almost ordinary is nonempty with codimension one in S.

$$\dim \mathcal{R}_g^{(f,g-2)} = 2g - 4 + f$$

Raynaud, 2000: For any genus g curve X there is an unramified solvable cover $Z \to X$ s.t. Z is not ordinary.
Pop, Saidi, 2003: If X is non-ordinary or if $\mathrm{Jac}(X)$ is simple then there is an unramified $\mathbb{Z}/\ell\mathbb{Z}$-cover π such that P_π is not ordinary for infinitely many ℓ.

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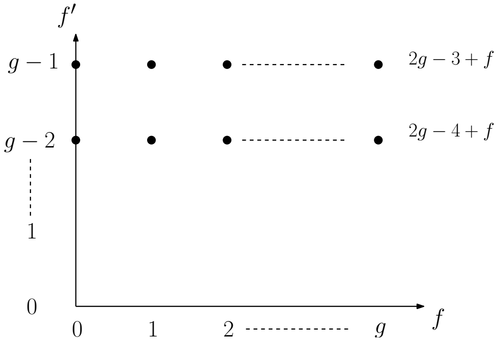
This gives us:



An application

Let $g \geq 2$ and $0 \leq f \leq g$.

Corollary (O., Pries)

*Let $\ell = 2, g \geq 4$ and $p \geq 5$ and $\frac{g}{2} - 1 \leq f' \leq g - 3$.
Then there exits a smooth curve X over $\mathbb{F}_p$ of genus g and p-rank f having an unramified double cover $\pi : Y \to X$ for which P_π has p-rank f'.*

Summary and Further Directions

Question

Given g, f, f' such that $g \geq 2, 0 \leq f \leq g, 0 \leq f' \leq g-1$, does there exists a curve X over $\overline{\mathbb{F}}_p$ of genus g and p-rank f having an unramified double cover $\pi : Y \to X$ with p-rank of P_π being f'?

The answer is YES for $p \geq 3$ and $0 \leq f \leq g$ when:

- $g = 2$, unless $p = 3$ and $f = 0, 1$ and $f' = 0$, in which case the anser is NO by Faber and van der Geer.
- $g \geq 3$ and $f' = g - 1$ by Theorem 1
- $g \geq 3$ and $f' = g - 2$ (with $f \geq 2$ when $p = 3$) by Theorem 2
- when $p \geq 5$ and $g \geq 4$ and $\frac{g}{2} - 1 \leq f' \leq g - 3$ by Corollary

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First open case: $g = 3$, P_π has p-rank 0 studied as part of WINE 2 project and the answer is yes for $3 \leq p \leq 19$,moreover

Theorem (CEGNOPT)

If $3 \leq p \leq 19$, the answer to the question above is YES for all $g \geq 2$ as long as f is bigger than (appr.) $\frac{2g}{3}$ and f′ bigger than (appr.) $\frac{g}{3}$.

Summary and Further Directions

Thm:[O., Pries] Once we know that $\mathcal{R}_g^{(f,f')} \neq \emptyset$ then each of its components has dimension at least $g - 2 + f + f'$ (an application of purity)

This lower bound is realized when:

- [Thm 1] $f' = g - 1, \dim \mathcal{R}_g^{(f,f')} = 2g - 3 + f$
- [Thm 2] $f' = g - 2$, with $f \geq 2$ when $p = 3$, $\dim \mathcal{R}_g^{(f,f')} = 2g - 4 + f$
- [Cor.] $p \geq 5$ and $\frac{g}{2} - 1 \leq f' \leq g - 3$, at least one component of $\mathcal{R}_g^{(f,f')}$ has dimension $g - 2 + f + f'$

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- [Cor.] $p \geq 5$ and $\frac{g}{2} - 1 \leq f' \leq g - 3$, at least one component of $\mathcal{R}_g^{(f,f')}$ has dimension $g - 2 + f + f'$

Summary and Further Directions

Similarly:

Corollary (CEGNOPT)

If $3 \leq p \leq 19$, $\mathcal{R}_g^{(f,f')}$ has a nonempty component of dimension $g-2+f+f'$ for all $g \geq 2$ as long as f is bigger than (appr.) $2g/3$ and f' bigger than (appr.) $g/3$.

Remark

Condition on p is needed to show that $\mathcal{R}_3^{(2,0)}$ has dimension 3

Open Question: What is the dimension of $\mathcal{R}_3^{(2,0)}$? Is there a 3-dimensional family of smooth plane quartics X with p-rank 2 having an unramified double cover π such that P_π has p rank 0.

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