Polytopes of maximal volume product

Matt Alexander
Kent State University

(based on a joint work with Matthieu Fradelizi and Artem Zvavitch)

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Notice that for a non-degenerate linear transform T, $\mathcal{P}(TK) = \mathcal{P}(K)$.

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Mahler's conjecture for $\mathcal{P}(K) = |K||K^\circ|$:

For any convex symmetric body $K \subset \mathbb{R}^n$: $\mathcal{P}(K) \geq \mathcal{P}(B_\infty^n) = \frac{4^n}{n!}$, where B_∞^n-cube.

True if $n = 2$ (Mahler,1939); Open if $n \geq 3$.
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- **Polytopes with at most a few vertices** (Lopez, Reisner 1998), (Meyer, Reisner, 2006).
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Other results

- **Curvature Conditions**: If a body has a point of positive curvature then it is not a minimizer. (Stancu, 2009), (Reisner, Schütt, Werner, 2010), (Gordon, Meyer, 2011).

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- **Functional forms** (for log-concave functions): (Klartag, Milman, 2005), (Fradelizi, Meyer, 2008, 2010), (Gordon, Fradelizi, Meyer, Reisner, 2010).

Upper bound for volume product $\mathcal{P}(K) = |K||K^{s(K)}|$:

Blaschke - Santaló Inequality: Let $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

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- (Blaschke, 1923) for $n \leq 3$, (Santaló,1948) for $n > 3$.
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- Stability Results: (Böröczky 2010), (Barthe, Böröczky, Fradelizi, 2012).
- Functional forms (for log-concave functions): (Ball,1986), (Artstein, Klartag, Milman,2004), (Fradelizi, Meyer, 2007).

Our Direction

Some notation:

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- We denote by M_m^n the supremum of the volume product of polytopes with at most m vertices and non-empty interior in $\mathbb{R}^n$

$$M_m^n := \sup_{K\in\mathcal{P}_m^n} \mathcal{P}(K).$$

Exact solution in $\mathbb{R}^2$

Theorem (Meyer, Reisner '11 / A., Fradelizi, Zvavitch '16+)

Let $N \geq 3$. The regular N-gon has maximal volume product among all polygons with at most N vertices, that is, polygons in $\mathcal{P}_N^2$. More precisely, for every polygon K with at most N vertices, one has

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Note, $\mathcal{P}(R_N) = N^2 \sin^2\left(\frac{\pi}{N}\right)$

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Theorem (A., Fradelizi, Zvavitch)

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$$\lim_{\epsilon \to 0} \mathcal{P}(\mathrm{conv}\{B_\infty^2, x_\epsilon\}) = \mathcal{P}(\mathrm{conv}\{(1,-1); (-1,-1); (-1,1); (10,1)\}) < \mathcal{P}(B_\infty^2).$$

Behavior of M_m^n

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The final inequality follows from the previous slide.

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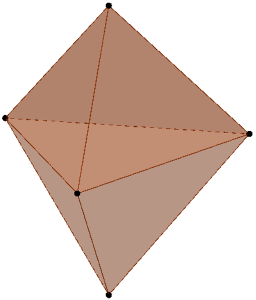
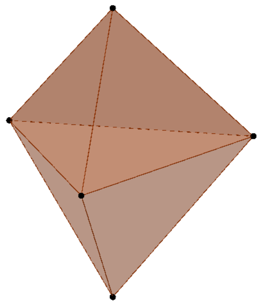
Let $n \geq 1$ and $m \geq n+1$. Let K be of maximal volume product among polytopes with at most m vertices. Then K is a simplicial polytope.



Exact Solution for $m = n+2$

Theorem (A., Fradelizi, Zvavitch)

Let K be the convex hull of $n+2$ points. Let $m = \lfloor \frac{n}{2} \rfloor$ and $p = \lceil \frac{n}{2} \rceil = n - m$. Then K is the convex hull of two simplices Δ_m and Δ_p living in supplementary affine subspaces of dimensions m and p respectively.



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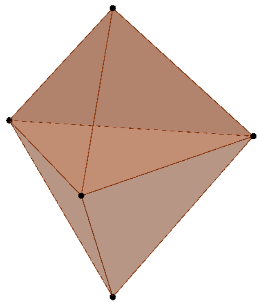
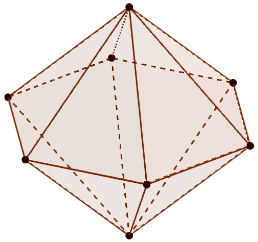
Open for $m > n + 2$



Exact Solution for symmetric case in $\mathbb{R}^3$ with 8 points

Theorem (A., Fradelizi, Zvavitch)

Let K be an origin symmetric body in $\mathcal{P}_8^3$. Then the maximal volume product of such bodies is the double cone on a regular hexagonal base.



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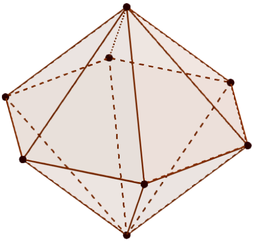

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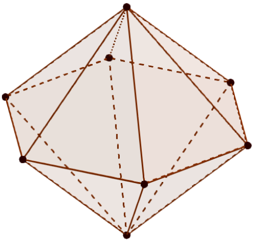
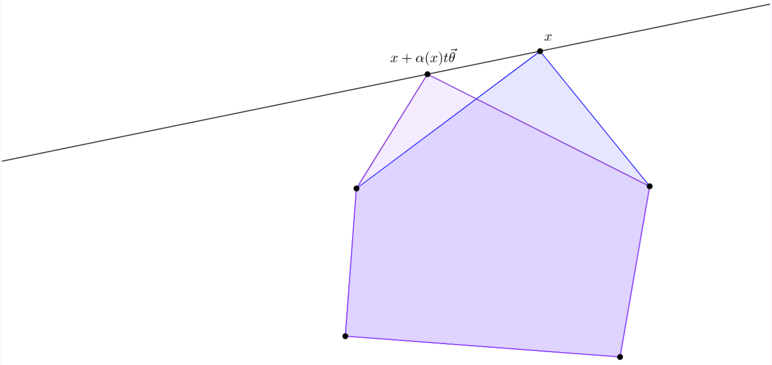


Definition (Shadow System)

A shadow system in direction $\vec{\theta} \in S^{n-1}$ is given by

$$K_t = \mathrm{conv}\{x + \alpha(x) t\vec{\theta} | x \in M\}$$

where $M \subset \mathbb{R}^n$ is bounded, $\alpha : M \to \mathbb{R}$ is bounded, and $t \in [a, b] \subset \mathbb{R}$.



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Let K_t, $t \in [0, 1]$ be a shadow system of origin symmetric convex bodies in $\mathbb{R}^n$ then $|K_t^o|^{-1}$ is convex in t.

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Theorem (Fradelizi, Meyer, & Zvavitch)

Let K_t, $t \in [0,1]$ be a shadow system of convex bodies in $\mathbb{R}^n$ with $|K_t|$ an affine function on $[-a,a]$, then $\left(|K_t||K_t^{s(K_t)}|\right)^{-1}$ is quasi-convex in t. That is, for any $[c,d] \subset [-a,a]$, $\min_{[c,d]} \mathcal{P}(K_t) = \min\{\mathcal{P}(K_c), \mathcal{P}(K_d)\}$

Lemma

Let $n, m \in \mathbb{N}$ with $m \geq n+1$ and $K \in \mathcal{P}_m^n$. Let F be a facet of K with exterior normal $u \in S^{n-1}$, let x_F be in the relative interior of F and let $K_t = \mathrm{conv}(K, x_F + tu)$, for $t > 0$. Then for t small enough the volume product of K_t is strictly larger than the volume product of K:

$$\mathcal{P}(K_t) > \mathcal{P}(K).$$

Adding one point

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- Take the point x_F and move it slightly adding volume to K.
- This move cuts some volume from $K^{s(K)}$

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Let $n, m \in \mathbb{N}$ with $m \geq n+1$ and $K \in \mathcal{P}_m^n$. Let F be a facet of K with exterior normal $u \in S^{n-1}$, let x_F be in the relative interior of F and let $K_t = \mathrm{conv}(K, x_F + tu)$, for $t > 0$. Then for t small enough the volume product of K_t is strictly larger than the volume product of K:

$$\mathcal{P}(K_t) > \mathcal{P}(K).$$

- Take the point x_F and move it slightly adding volume to K.
- This move cuts some volume from $K^{s(K)}$
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$$\mathcal{P}(K_t) \geq \mathcal{P}(K) + t|K^{s(K)}||F|/n + o(t) > \mathcal{P}(K).$$

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- Essential to use result of Kim and Reisner on stability of the volume product with respect to small changes to the center of duality

Theorem

Let $n \geq 1$ and $m \geq n+1$. Then the supremum M_m^n is achieved at some polytope with exactly m vertices and the sequence M_m^n is strictly increasing in m.

- $$M_m^n := \sup_{K \in \mathcal{P}_m^n} \mathcal{P}(K) = \sup\{\mathcal{P}(K) : K \in \mathcal{P}_m^n, B_2^n \subset K \subset nB_2^n\}.$$

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Behavior of M_m^n

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- So the supremum is attained.
- We induct using the previous theorem.

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- Using the fact that this holds for all vertices and some combinatorics we find that K must be simplicial.

Lemma

Let $K \in \mathbb{R}^n$ be a convex body, and F a concave continuous function $F : K \to \mathbb{R}$. Assume that K and F are invariant under linear isometries $T_1, ..., T_m$. Then there is $x_0 \in K$ such that $T_i(x_0) = x_0$, for all $i = 1, \ldots, m$ and $F(x_0) \geq F(x)$ for all $x \in K$.

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Theorem (Radon's Theorem)

A set of points with cardinality greater than $n + 2$ in $\mathbb{R}^n$ can be separated into two disjoint sets whose convex hulls intersect.

Non-symmetric with $m = n + 2$

Theorem

Let K be the convex hull of $n+2$ points. Let $m = \lfloor \frac{n}{2} \rfloor$ and $p = \lceil \frac{n}{2} \rceil = n - m$. Then

$$\mathcal{P}(K) \leq \frac{(p+1)^{p+1}(m+1)^{m+1}}{n!p!m!},$$

with equality if and only if K is the convex hull of two simplices Δ_m and Δ_p living in supplementary affine subspaces of dimensions m and p respectively.

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- Use Radon's to separate the set of vertices into two supplementary subspaces.
- Notice that the number of vertices do not allow for the intersection of the subspaces to be larger.

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Theorem

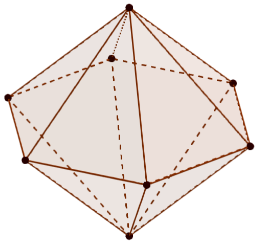
Let $1 \leq k \leq n-1$ be integers and let E and F be two supplementary subspaces of $\mathbb{R}^n$ of dimensions k and $n-k$ respectively. Let $L \subset E$ and $M \subset F$ be convex bodies of the appropriate dimensions such that $Fix(L) = Fix(M) = \{0\}$. Then for every $x \in L$ and $y \in M$

$$\mathcal{P}(\mathrm{conv}(L-x, M-y)) \leq \mathcal{P}(\mathrm{conv}(L, M)) = \frac{\mathcal{P}(L)\mathcal{P}(M)}{\binom{n}{k}},$$

with equality if and only if $x = y = 0$.

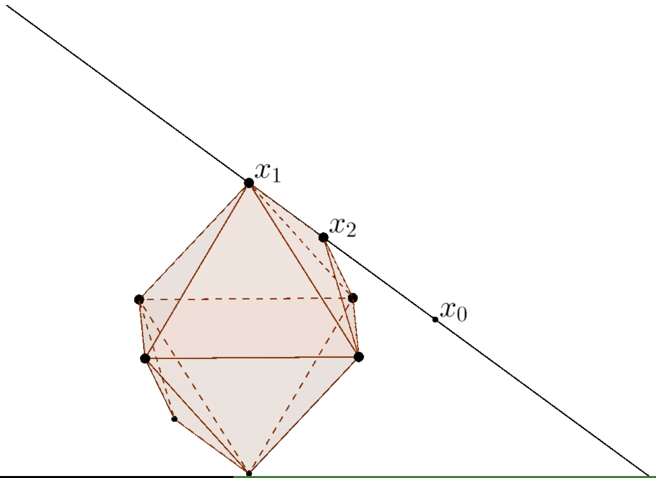
Corollary

Let $L \subset \mathbb{R}^{n-1}$ be a convex body such that Fix(L) is one point. Then among all double pyramids $K = \operatorname{conv}(L, x, y)$ in $\mathbb{R}^n$ with base L separating apexes x and y, the volume product $\mathcal{P}(K)$ is maximal when x and y are symmetric with respect to the Santaló point of L.



Theorem

Let K be an origin symmetric body in $\mathcal{P}_8^3$. Then the maximal volume product of such bodies is the double cone on a regular hexagonal base.



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We consider several cases

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- Using the same symmetry we have either a double cone again, or a line in the direction of the edges.
- Compare these two cases directly:

$$\mathcal{P}(CP) = \frac{4}{3}\mathcal{P}(H) = \frac{4}{3} \times 9 = 12 > \frac{100}{9}$$

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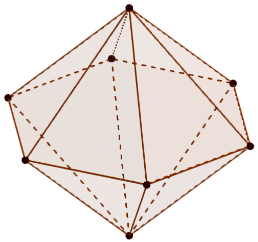

Reminder: Soccer is mandatory

Thank You!