Optimal transport between unequal dimensions

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- Setting: probability measures $d\mu = fdx$ on $X \subset \mathbb{R}^m$ and $d\nu = gdy$ on $Y \subseteq \mathbb{R}^n$ and a surplus $s(x, y)$.
- $\Pi(\mu, \nu)$ is the set of probability measures on $X \times Y$ which *project* to μ and ν; that is $\gamma(B \times Y) = \mu(B)$, $\gamma(X \times A) = \nu(A)$ for all $B \subset X, A \subset Y$.
- Monge-Kantorovich problem: maximize the linear functional

$$\int_{X \times Y} s(x, y) d\gamma(x, y)$$

over the convex set $\gamma \in \Pi(\mu, \nu)$.

- Usually, take $m = n$.

Motivation for unequal dimensional problem

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  - Firms and employees in a labour market
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- These may well be different.
- Other applications?

Duality

- The problem is **dual** to minimizing
$$\int_X u(x)d\mu(x) + \int_Y v(y)d\nu(y)$$
among $u \in L^1(\mu)$, $v \in L^1(\nu)$ such that $u(x) + v(y) \geq s(x, y)$.
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- We have $u(x) + v(y) - s(x,y) = 0$ γ a.e. and so

$$Du(x) = D_x s(x,y), \quad Dv(y) = D_y s(x,y)$$
$$D^2 u(x) \geq D^2_{xx} s(x,y), \quad D^2 v(y) \geq D^2_{yy} s(x,y)$$

Optimal transport preliminaries

- We will assume:
  Twist: $y \mapsto D_x s(x, y)$ is injective (for each fixed x); therefore $m \geq n$.
  Non-degeneracy: $D^2_{xy} s(x, y)$ has full rank at each (x, y); that is, the rank is n.

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- **When m = n**, the map solves the change of variables equation $|\det DT(x)| = f(x)/g(T(x))$, which leads to a second order partial differential equation for u.
- Ex. When $s(x,y) = x \cdot y$, u solves the elliptic Monge-Ampere equation, $\det D^2u(x) = f(x)/g(Du(x))$.

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- Ex. When $s(x,y) = x \cdot y$, u solves the elliptic Monge-Ampere equation, $\det D^2u(x) = f(x)/g(Du(x))$.
- Caffarelli '92: For $s(x,y) = x \cdot y$, regularity holds for convex target Y and good f, g.
- Ma-Trudinger-Wang '05, Loeper '10...: Same holds for other $s(x,y)$ satisfying a **deep structural condition**. If not, one can find densities for which the optimal map is discontinuous.

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- Can they be characterized by a PDE?
- Do we expect smoothness?

Smoothness fails

Theorem (P '12)

There exists smooth densities f, g, bounded above and below, for which the optimal map is discontinuous, UNLESS s is of index form; that is,

$$s(x, y) = b(I(x), y) + \alpha(x)$$

where $I : \mathbb{R}^m \to \mathbb{R}^n$.

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- Example: If Y is an n-dimensional submanifold of $\mathbb{R}^m$, and $s(x,y) = x \cdot y$, regularity theory requires Y to be convex, which can only happen if Y is affine. In this case, $x \cdot y = x_Y \cdot y$, where $x_Y \in \mathbb{R}^n$ is the projection of x onto Y.

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- Consequence: any regularity result will have to depend on the **interaction** between s and μ, ν.

A partial differential equation for unequal dimensions

- When $m > n$, we guess:

$$g(y) = \int_{T^{-1}(y)} \frac{f(x)}{\sqrt{\det[DT(x)DT(x)^T]}} d\mathcal{H}^{m-n}(x)$$

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- Except for the domain of integration, T is eliminated, (almost) leaving a PDE for v.

Partial differential equation cont.

- Neglecting null sets

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for $p = Dv(y)$, $Q = D^2 v(y)$.

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- Replacing $T^{-1}(y)$ with $\partial_s v(y)$ yields a non-local PDE:

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- Replacing $T^{-1}(y)$ with either X_i results in a local PDE.

Characterizing solutions via a non-local PDE

Theorem (McCann -P. '17)

If $(u, v) = (v^s, u^s)$ are s-conjugate functions, then they maximize the dual problem if and only if v solves the non-local equation

$$g(y) = \int_{\partial_s v(y)} \frac{f(x)\det[D^2 v(y) - D^2_{yy} s(x,y)]}{\sqrt{\det[D^2_{yx} s(x,y) D^2_{yx} s(x,y)^T]}} d\mathcal{H}^{m-n}(x)$$

$\mathcal{H}^n$ almost everywhere.

Solving via local PDE

Local equation: $g(y) = G_i(y, Dv(y), D^2v(y))$, where

$$G_i(y,p,Q) = \int_{X_i(y,p,Q)} \frac{f(x)\det[Q - D^2_{yy}s(x,y)]}{\sqrt{\det[D^2_{yx}s(x,y)D^2_{yx}s(x,y)^T]}} d\mathcal{H}^{m-n}(x)$$

for $i = 1, 2$.

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Assume $(u, v) = (v^s, u^s)$ with $v \in C^2(Y)$. If v solves the local equation with $i = 2$, then (u, v) solves the dual problem.

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Remark: G_2 is elliptic, so if $v \in C^{2,\alpha}$ one can bootstrap and get that v is as smooth as G_2 and g.

One dimensional targets (joint with McCann and Chiappori)

- Take $m > n = 1$. We (try to) construct the optimal map explicitly one level set at a time, assuming $T^{-1}(y) = X_1(y, p)$ for some p.

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- We say (s, μ, ν) is nested if, whenever $y_0 < y_1$,

$$X_{\leq}(y_0, p(y_0)) \subseteq X_{<}(y_1, p(y_1)).$$

Solution Theorem (roughly stated)

Theorem (Chiappori-McCann-P ('16))

If (s, μ, ν) is nested, then the map that assigns each $x \in X_1(y, p(y))$ to $y = T(x)$ is the unique optimizer.

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- In this case, the $i = 1$ version of the local equation holds:

$$g(y) = \int_{X_1(y,p(y))} \frac{p'(y) - s_{yy}(x,y)}{|D_x s_y|} f(x) d\mathcal{H}^{m-1}(x)$$

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- The velocity of $X_1(y, p(y))$ is given by $\frac{p' - s_{yy}}{|D_x s_y|^2} D_x s_y$. If the speed $(p'(y) - s_{yy}(y,x))/|D_x s_y|$ is positive for each $x \in X_1(y, p(y))$ and each y, the model is nested. Conversely, if nestedeness holds, the speed is always nonnegative, and positive for at least one $x \in X_1(y, p(y))$ for each y.

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- To ensure smoothness of h, we need smoothness and bounds on f, g, s, ∂X, as well as nondegeneracy, and transversality of the intersection of ∂X and the $X_1(y, p(y))$ (or at least $H^{m-1}(\partial X \cap X_1(y, p(y))) = 0$ locally). We can differentiate these using a generalized divergence theorem and the coarea formula.

Sample derivatives

Example:

$$\begin{aligned} h_p(y,p(y)) &= \int_{X_1(y,p)} f(x)\frac{d\mathcal{H}^{m-1}(x)}{|D_x s_y(x,y)|} \\ &= \int_{X_\leq(y,p)} \nabla \cdot \left(f(x)\frac{D_x s_y(x,y)}{|D_x s_y(x,y)|^2}\right) d\mathcal{H}^m(x) \\ &\quad - \int_{\partial X \cup \bar{X}_\leq(y,p)} f(x)\frac{D_x s_y(x,y)\cdot \hat{n}_X}{|D_x s_y(x,y)|^2} d\mathcal{H}^{m-1}(x) \end{aligned}$$

$$h_y(y,p(y)) = \int_{X_1(y,p)} f(x)\frac{-s_{yy}(x,y)d\mathcal{H}^{m-1}(x)}{|D_x s_y(x,y)|} - g(y).$$

Regularity of husband's payoff v

Theorem (Chiappori-McCann-P ('16))

Fix an integer $r \geq 1$. Suppose there is an interval $Y' = (y_0, y_1) \subset Y$ such that $X' \cap \partial X \in C^1$ intersects $\overline{X(y, k(y))}$ transversally for all $y \in \overline{Y'}$, where $X' = \bigcup_{y \in Y'} \overline{X_1(y, p(y))}$. Then $\|p\|_{C^{r,1}(Y')}$ is controlled by the following quantities, all assumed positive and finite: $\|\log f\|_{C^{r-1,1}(X')}$, $\|\log g\|_{C^{r-1,1}(Y')}$, $\|s_y\|_{C^{r,1}(X' \times Y')}$, $\|\hat{n}_X\|_{(C^{r-2,1} \cap W^{1,1})(X' \cap \partial X)}$, $\mathcal{H}^{m-1}[\partial^* X]$,

$$\inf_{y \in Y'} \mathcal{H}^{m-1}\left[X(y, k(y))\right] \qquad \text{(proximity to ends of } Y\text{)}, \tag{1}$$

$$\inf_{x \in X', y \in Y'} |D_x s_y(x, y)| \qquad \text{(non-degeneracy)}, \tag{2}$$

$$\inf_{x \in X' \cap \partial X, y \in Y'} 1 - (\hat{n}_X \cdot \hat{n}_{X_=})^2 \qquad \text{(transversality)} \tag{3}$$

where $\hat{n}_{X_=}(x, y) = D_x s_y / |D_x s_y|$, *and* $\mathcal{H}^{m-2}\left[\overline{X(y_0, k(y_0))} \cap \partial X\right]$.

- The optimal map T satisfies $\frac{\partial s}{\partial y}(x, T(x)) = p(T(x)) = v'(T(x))$. With a *speed limit condition*, $p'(y) - s_{yy}(x, y) > 0$, this is as smooth as $\frac{\partial s}{\partial y}$ and $v' = p$ via the implicit function theorem.

Regularity cont'd

- The optimal map T satisfies $\frac{\partial s}{\partial y}(x, T(x)) = p(T(x)) = v'(T(x))$. With a *speed limit condition*, $p'(y) - s_{yy}(x, y) > 0$, this is as smooth as $\frac{\partial s}{\partial y}$ and $v' = p$ via the implicit function theorem.
- The potential u is then one derivative smoother than T via $Du(x) = D_x s(x, T(x))$.

Thanks

- Thank you!