An overview of difference Galois theory

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Some transcendence results for special functions

Proposition

The function $\Gamma(x)$ satisfying $\Gamma(x+1) = x\Gamma(x)$ is transcendental over $C_1(x)$ where C_1 is the field of 1-periodic meromorphic functions over $\mathbb{C}$.

Theorem

Let $b_1, \ldots, b_r \in \mathbb{C}(x)$ and $f_1, \ldots, f_r$ meromorphic functions over $\mathbb{C}$ solutions of

$$f_i(x+1) = f_i(x) + b_i(x) \text{ for all } i = 1, \ldots, r$$

The f_i's are algebraically dependent over $C_1(x)$ if and only if there exist $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ not all zero and $g \in \mathbb{C}(x)$ such that

$$\gamma_1 b_1(x) + \cdots + \gamma_r b_r(x) = g(x+1) - g(x).$$

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Theorem (Roques 2007)

For $q \in \mathbb{C}$ with $|q| > 1$. Let $y_1(x), y_2(x)$ two linearly independent solutions of

$$y(q^2x) - \frac{2ax - 2}{a^2x - 1}y(qx) - \frac{x - 1}{a^2x - q^2x}y(x) = 0$$

with $a \notin q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$. Then, $y_1(x), y_2(x), y_1(qx)$ are algebraically independent.

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Elementary Proof of Transcendence of Γ

Suppose to the contrary that $\Gamma(x)$ is algebraic.

Let

$$\Gamma(x)^n + a_{n-1}(x)\Gamma(x)^{n-1} + \ldots + a_0(x) = 0, \qquad (1.1)$$

be a minimal relation with $a_i(x) \in \mathbb{C}(x)$ and $a_0(x) \neq 0$.
Change x into $x+1$ in (1.1) to find

$$\Gamma(x+1)^n + a_{n-1}(x+1)\Gamma(x+1)^{n-1} + \ldots + a_0(x+1) = 0. \quad (1.2)$$

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By minimality, we must have $a_0(x+1) = x^n a_0(x)$ with $a_0(x) \in \mathbb{C}(x)^*$. Absurd !

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Galois theoretic proof

- Set $K = C_1(x)$ and $K_\Gamma = C_1(x)(\Gamma) \subset \mathrm{Mer}(\mathbb{C})$. These fields are closed under $\sigma(f(x)) = f(x+1)$ and

$$\sigma(\Gamma) = x\Gamma$$

- $K_\Gamma^\sigma := \{f \in K_\Gamma | \sigma(f) = f\} = C_1 = K^\sigma$
- Consider

$$\mathrm{Gal}(K_\Gamma | K) = \{\tau \in \mathrm{Aut}(K_\Gamma) \ \ \tau|_K = id_K \ \ \tau \circ \sigma = \sigma \circ \tau\}.$$

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- Application to transcendence and differential transcendence
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Examples (Endomorphism of the complex variable)

- $K = \mathbb{C}(x)$, $\sigma(f(x)) = f(x+1)$;
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  Replace K *by* $\hat{K} = \cup_{n=0}^{\infty} \mathbb{C}(x^{1/p^n})$ *and set* $\sigma(x^{1/p^n}) = x^{1/p^{n-1}}$.

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Examples (Automorphisms of an elliptic curve)

- *Let* $(E, \oplus) \subset \mathbb{P}^2\mathbb{C}$ *be an elliptic curve and let* $\Omega \in E$.
  Let C_E *be the field of elliptic functions.*
  Then, $(C_E, \sigma : C_E \rightarrow C_E, f(P) \mapsto f(P \oplus \Omega))$ *is* σ*-field.*
- *Let* $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ *a lattice such that* $\mathbb{C}/\Lambda \simeq E$.
  The application $\mathbb{C} \rightarrow E, \omega \mapsto (x(\omega), y(\omega))$ *identifies*
  - $C_E = \{(\omega_1, \omega_2)$ *-periodic functions* $\} \subset \mathcal{M}er(\mathbb{C})$
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Difference equations

Let (K,σ) be a σ-field and g a solution of

$$\sigma^n(y)+a_{n-1}\sigma^{n-1}(y)+\cdots+a_0y=0 \qquad (\mathcal{L}),$$

with $a_0\neq 0$, $a_i\in K$. Then, $Z:=\begin{pmatrix} g\\ \sigma(g)\\ \vdots \\ \sigma^{n-1}(g)\end{pmatrix}$ is solution of

$$\sigma(Y)=A_{\mathcal{L}}Y$$

with

$$A_{\mathcal{L}}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 1\\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n}\end{pmatrix}\in \mathrm{GL}_n(K). \qquad (2.1)$$

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An equation $\sigma(Y) = AY$ with $A \in \mathrm{GL}_n(K)$ is called difference system.

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Fundamental solution matrix

Definition

Let (K, σ) be a σ-field and $A \in \mathrm{GL}_n(K)$. Let L be a σ-field extension of K.
An matrix $U \in \mathrm{GL}_n(L)$ such that $\sigma(U) = AU$ is called fundamental solution matrix of $\sigma(Y) = AY$

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Examples

Consider the difference field extension
$\left(\mathbb{C}(x), \sigma(f(x)) = f(x+1)\right) \subset \left(\mathcal{M}er(\mathbb{C}), \sigma(f(x)) = f(x+1)\right)$.

- Then $\Gamma \in \mathrm{Mer}(\mathbb{C})^*$ is a fundamental solution matrix for $\sigma(y) = xy$.
- Let $\psi(x)$ be the digamma function $\frac{\Gamma'}{\Gamma}$. Then, $\sigma(\psi(x)) = \psi(x) + \frac{1}{x}$.This correspond to the difference system

$$\sigma(Y) = \begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix} Y$$

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More generally, the difference equations $\sigma(y_i) = y_i + \frac{(-1)^{i-1}}{x^i}$ for $i = 1, \ldots, r$ are encoded by the difference system

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Picard-Vessiot field extensions

Definition

Let (K, σ) be a σ-field and let $A \in \mathrm{GL}_n(K)$.
A σ-field extension $K_A|K$ is called
*a **Picard-Vessiot field extension** of $\sigma(Y) = AY$ over K if*

- $K_A^{\sigma} = K^{\sigma}$;
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*The K-σ-algebra $R_A = K[U, \frac{1}{\det(U)}] \subset K_A$ is called a **PV- ring**.*

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Existence and uniqueness ?

Existence of a Picard-Vessiot extension for $\sigma(Y) = AY$:

No general results for arbitrary difference fields

BUT via analytic resolution, we get

- $\Big(\mathcal{M}er(\mathbb{C}), \sigma(f(x)) = f(x+1)\Big)$ with $\mathcal{M}er(\mathbb{C})^{\sigma} = C_1$. For any $A \in \mathrm{Gl}_n(\mathcal{M}er(\mathbb{C}))$, there exists $U \in \mathrm{GL}_n(\mathcal{M}er(\mathbb{C}))$ such that $\sigma(U) = AU$ (Praagman). Then, $K_A = C_1(x)(U) \subset \mathrm{Mer}(\mathbb{C})$ is a PV-field extension for $\sigma(Y) = AY$ over $K = C_1(x)$.
- Similar result for $\mathcal{M}er(\mathbb{C}^*)$ and $\sigma(f(x)) = f(qx)$

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Let (K, σ) be a σ-field and $A \in \mathrm{GL}_n(K)$.

Definition

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Group representation

Theorem

The application

$$\rho_U : \quad \mathrm{Gal}(K_A|K) \longrightarrow \mathrm{GL}_n(K^\sigma)$$

$$\tau \longrightarrow [\tau]_U$$

where $\tau(U) = U[\tau]_U$ *identifies* $\mathrm{Gal}(K_A|K)$ *with an algebraic subgroup of* $\mathrm{GL}_n(K^\sigma)$.

Algebraic groups

H is an algebraic subgroup of $\mathrm{GL}_n(k)$ if

- H subgroup of $\mathrm{GL}_n(k)$
- $H = \{M | P(M) = 0 \text{ for all } P \in S\}$ with $S \subset k[X, \frac{1}{\det(X)}]$

Examples

- $\mu_n = \{x_{1,1}^n = 1\} \subset \mathrm{Gl}_1(k)$
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$(K, \sigma) = (C_1(x), \sigma(f(x)) = f(x+1))$, $b \in \mathbb{C}(x)$ and $\sigma(y) = y + b$;

Matrix form $\sigma(Y) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} Y$ and $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ where $u \in \mathcal{M}er(\mathbb{C})$ solution of $\sigma(y) = y + b$.

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The general idea

Defining equations of the Galois group

$$\Updownarrow$$

Algebraic relations between solutions

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Algebraic relations between solutions

Independence of discrete logarithms

Set $(K, \sigma) = (C_1(x), \sigma(f(x) = f(x+1))$.
Let $b_1, \ldots, b_r \in \mathbb{C}(x)$ and $u_1, \ldots, u_r \in \mathcal{M}er(\mathbb{C})$ solutions of

$$\sigma(y_1) = y_1 + b_1$$
$$\sigma(y_2) = y_2 + b_2$$
$$\vdots = \vdots$$
$$\sigma(y_r) = y_r + b_r$$

If the u_i are algebraically dependent over K then there exists $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ not all zero and $g \in \mathbb{C}(x)$ such that

$$\gamma_1 b_1 + \cdots + \gamma_r b_r = \sigma(g) - g. \quad (5.1)$$

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Ideas of proof

- $K_A = C_1(x)(u_1, \dots, u_r)$ and $\mathrm{Gal}(K_A|K) = \{(c_\tau^i)_{i=1..r} | \tau(u_i) = u_i + c_\tau^i\}$
- u_i alg. dep implies $\mathrm{Gal}(K_A|K) \subsetneq (C_1^r, +)$
- There exist $\gamma_1, \dots, \gamma_r \in C_1$ not all zero such that

$$\mathrm{Gal}(K_A|K) \subset \{c_i | \gamma_1 c_1 + \cdots + \gamma_r c_r = 0\}.$$

Group equation : $\gamma_1 c_1 + \cdots + \gamma_r c_r = 0$

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Relation for solutions : $\gamma_1 u_1 + \cdots + \gamma_r u_r = g$ with $g \in K$

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Differential independence of Γ

Let $K = \mathbb{C}(x)$ with $\sigma(f(x)) = f(x+1)$ and $\sigma(y) = xy$.

Is Γ differentially transcendental $\mathbb{C}(x)$, i.e.
$\Gamma, \frac{d}{dx}(\Gamma), \ldots, \frac{d^r}{dx^r}(\Gamma), \ldots$ are alg. independent over $\mathbb{C}(x)$.

With $\psi(x) = \frac{\frac{d}{dx}(\Gamma)}{\Gamma}$,

$$\Gamma \text{ is diff.alg } \mathbb{C}(x)$$
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Deriving the functional equation for Γ and using $\frac{d}{dx} \circ \sigma = \sigma \circ \frac{d}{dx}$ we find

$$\sigma(\psi(x)) = \psi(x) + \frac{1}{x}$$

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Thus $\Gamma(x)$ diff.alg iff $\exists r \ \psi(x), \ldots, \frac{d^r}{dx^r}(\psi(x))$ algebraically dependent

$\Rightarrow \exists r, \gamma_1, \ldots, \gamma_{r+1} \in \mathbb{C},$ not all zero $g \in \mathbb{C}(x)$ such that

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General criteria

Theorem

Let (K,σ) a σ-field with a derivation δ commuting with σ and let $b \in K$.
Let $f \in L$, a σ-δ-field extension of K , such that $\sigma(f) = f + b$.
Assume that K^σ is algebraically closed.
Then, f is differentially transcendental over K if there are no $r \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_{r+1} \in k$ and $g \in K$ such that

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A first difficulty

Examples

$(K,\sigma) = (\mathbb{C}(x), \sigma(f(x)) = f(x+1))$ and $\sigma(y) = -y$.
Suppose that there exists a solution $u \neq 0$ in a σ-field L with $L^\sigma = \mathbb{C}(x)^\sigma = \mathbb{C}$.
Then $\sigma(u^2) = (-u)^2 = u^2$ and $u^2 \in L^\sigma = \mathbb{C}$.
Then $u \in \mathbb{C}$ and $\sigma(u) = u$. Contradiction !

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A dichotomy

For difference systems, one has a kind of dichotomy

- either, one solves in a field extension L BUT $L^\sigma \neq K^\sigma$.
  $K = \mathbb{C}(x)$, $\sigma(x) = x + 1$. For any $\sigma(Y) = AY$ with $A \in \mathrm{GL}_n(\mathbb{C}(x))$ there exists a fundamental solution matrix $U \in \mathrm{GL}_n(\mathrm{Mer}(\mathbb{C}))$.
  But $\mathcal{M}er(\mathbb{C})^\sigma = C_1 \neq \mathbb{C}(x)^\sigma = \mathbb{C}$.
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Existence and uniqueness of Picard-Vessiot pseudofields

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Let (K, σ) a σ-field with K^σ algebraically closed and $A \in \mathrm{Gl}_n(K)$.
Then, there exists a unique Picard-Vessiot pseudofield K_A for the system $\sigma(Y) = AY$.

That is

- $K_A = K(U)$ with U a fundamental solution matrix
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And all the results for fields hold in this context

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And all the results for fields hold in this context

A concrete example

Let $\mathbb{C}^{\mathbb{N}}$ the ring of $\mathbb{C}$-valued sequences with addition and multiplication defined component by component.

The morphism

$$\sigma : \mathbb{C}^{\mathbb{N}} \rightarrow \mathbb{C}^{\mathbb{N}}, (a(0), a(1), \ldots, a(n), \ldots) \mapsto (a(1), \ldots, a(n), \ldots)$$

is not injective.

Set

$$a \sim b \text{ iff } \exists N | a(n) = b(n) \text{ for all } n > N.$$

Then σ induces on $\mathcal{S} = \mathbb{C}^{\mathbb{N}} / \sim$ an injective morphism.

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$$\sigma : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}, (a(0), a(1), \ldots, a(n), \ldots) \mapsto (a(1), \ldots, a(n), \ldots)$$

is not injective.
Set

$$a \sim b \text{ iff } \exists N | a(n) = b(n) \text{ for all } n > N.$$

Then σ induces on $\mathcal{S} = \mathbb{C}^{\mathbb{N}} / \sim$ an injective morphism.
The σ-ring $\mathcal{S}$ is called the ring of germs of sequences.

A concrete example

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Consider $\mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x+1)$.
The application

$$\mathbb{C}(x) \longrightarrow \mathcal{S}$$

$$f \longmapsto (f(0), \ldots, f(n), \ldots)$$

is an injective ring morphism, the identity on $\mathbb{C}$, commutes with σ.
The difference field $(\mathbb{C}(x), \sigma)$ is a σ-subring of $\mathcal{S}$.

A universal Picard-Vessiot ring

An example

Examples

$K = (\mathbb{C}(x), \sigma(f(x)) = f(x+1)) \subset \mathcal{S}$ and $\sigma(y) = -y$. Then $u = ((-1)^n)_{n \in \mathbb{N}}$ is a fundamental solution matrix. In $\mathcal{S}$, we have

- *$u - 1 \neq 0$ and $u + 1 \neq 0$ but $(u-1)(u+1) = u^2 - 1 = 0$.*
- *$\mathbb{C}(x)(u) = L_1 \times L_2 \subset \mathcal{S}$ with $L_1 = \mathbb{C}(x).(u-1)$ and $L_2 = \mathbb{C}(x).(u+1)$.*

A general result : Let $\mathbb{C}(x) \subset \mathcal{S}$ and $A \in \mathrm{GL}_n(\mathbb{C}(x))$. There exists a Picard-Vessiot ring $R_A := \mathbb{C}(x)[U, \frac{1}{\det(U)}] \subset \mathcal{S}$.

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Application of the pseudofield structure

Theorem (Larson-Tarft-1990)

Let $u, v \in \mathcal{S}$ two sequences each of them satisfying a difference equation over $\mathbb{C}(x)$ and such that $uv = 0$. Then, there exist $u_0, \dots, u_{t-1}, v_0, \dots, v_{t-1} \in \mathcal{S}$ such that

- *u (resp.v) is the interlacing of the u_i(resp. v_i)*
- *for all i either $u_i = 0$ or $v_i = 0$*

Theorem (Wibmer 2012)

Let $u \in \mathcal{S}$ satisfying a linear difference equation $\mathcal{L}$ over $\mathbb{C}(x)$. The following are equivalent

- *Skolem Mahler Lech problem : the set $\{i | u(i) = 0\}$ is a finite union of arithmetic progressions*
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Comparing "special functions" and "abstract solutions"

Let (K, σ) be a σ-field with K^{σ} algebraically closed and $A \in \mathrm{GL}_n(K)$. Let L be a σ-field and let $Z \in L^n$ anon zero solution of $\sigma(Y) = AY$.

First case : $L^{\sigma} = K^{\sigma}$

Then, there exists a Picard-Vessiot extension for $\sigma(Y) = AY$ containing Z.

Examples

$K = \cup_{n \in \mathbb{N}} \mathbb{C}(z^{1/p^n})$ and $\sigma(f(z)) = f(z^p)$. Generating series for automatic sequences belong to some $L = \mathbb{C}((z^{1/p^k}))$. One has $L^{\sigma} = \mathbb{C}$.

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Second case : $L^\sigma \neq K^\sigma$

If Z satisfies an algebraic relation over K, there exists a Picard-Vessiot extension for $\sigma(Y) = AY$ where a solution vector satisfies the same relation.

Examples

$K = \mathbb{C}(x) \subset \mathrm{Mer}(\mathbb{C}) = L$ *with* $\sigma(f(x)) = f(x+1)$. *Then* $\mathbb{C}(x)^\sigma = \mathbb{C} \neq \mathrm{Mer}(\mathbb{C})^\sigma$. *If* $\Gamma(x)$ *is algebraic over* $\mathbb{C}(x)$ *there exists a Picard-Vessiot extension for* $\sigma(y) = xy$ *containing a non zero solution algebraic solution.*

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