The Persistent Homotopy Type Distance

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Metrics on functional spaces

Supremum Distance

For $\varphi_X, \varphi'_X : X \to \mathbb{R}$,

$$\|\varphi_X - \varphi'_X\|_\infty := \sup_{x \in X} |\varphi_X(x) - \varphi'_X(x)|.$$

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Natural Pseudo-Distance

For $\varphi_X : X \to \mathbb{R}$, $\varphi_Y : Y \to \mathbb{R}$, X and Y homeomorphic,

$$d_{NP}(\varphi_X, \varphi_Y) := \inf_{h \in Homeo(X,Y)} \|\varphi_X - \varphi_Y \circ h\|_\infty.$$

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1st Goal:

Extend d_{NP} to the case when X and Y are only homotopy equivalent.

Lifting stability results

Stability of Persistence (I)

For $\varphi_X, \varphi'_X : X \to \mathbb{R}$, $d_B(dgm(\varphi_X), dgm(\varphi'_X)) \leq \|\varphi_X - \varphi'_X\|_\infty$.

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For $\varphi_X : X \to \mathbb{R}$, $\varphi_Y : Y \to \mathbb{R}$, with X homeomorphic to Y,

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2nd Goal: Stability of Persistence (III)

For $\varphi_X : X \to \mathbb{R}$, $\varphi_Y : Y \to \mathbb{R}$, with X homotopy equivalent to Y,

$$d_B(dgm(\varphi_X), dgm(\varphi_Y)) \leq d_{HT}(\varphi_X, \varphi_Y)$$

and, for X, Y homeomorphic,

$$d_{HT}(\varphi_X, \varphi_Y) \leq d_{NP}(\varphi_X, \varphi_Y)$$

Motivation

Definition and stability of d_{HT}

Connection to interleaving distance

α-maps and α-homotopies

Let **S** be the category such that:

- objects are bounded continuous functions $\varphi_X : X \to \mathbb{R}$,
- morphisms from φ_X to φ_Y are all continuous maps $f : X \to Y$ such that $\varphi_Y \circ f \leq \varphi_X$.

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Let $\alpha \geq 0$. Any $f : X \to Y$ such that $\varphi_Y \circ f \leq \varphi_X + \alpha$ is called an α-map with respect to (φ_X, φ_Y).

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Given two α-maps $f_1 : X \to Y$ and $f_2 : X \to Y$, an α-homotopy between f_1 and f_2 with respect to the pair (φ_X, φ_Y) is a homotopy that is an α-map at every instant.

α-homotopy equivalences

φ_X and φ_Y are α-homotopy equivalent if there exist α-maps

$$f : X \to Y \text{ and } g : Y \to X$$

w.r.t. (φ_X, φ_Y) and (φ_Y, φ_X) such that:

- $g \circ f : X \to X$ is 2α-homotopic to id_X with respect to (φ_X, φ_X);
- $f \circ g : Y \to Y$ is 2α-homotopic to id_Y with respect to (φ_Y, φ_Y).

The persistent homotopy type distance

Definition

$$d_{HT}(\varphi_X, \varphi_Y) := \inf\left\{\alpha \in \mathbb{R} : \varphi_X \text{ and } \varphi_Y \text{ are } \alpha\text{-homotopy equivalent}\right\}$$

Proposition

- d_{HT} is an extended pseudo-metric.
- If X and Y are homeomorphic, then

$$d_{HT}(\varphi_X, \varphi_Y) \leq d_{NP}(\varphi_X, \varphi_Y)$$

.

- X and Y are homotopy equivalent iff $d_{HT}(\varphi_X, \varphi_Y) < \infty$.

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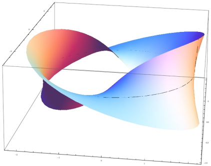
- What about defining α-homotopies w/out the condition to be an α-map at each instant?
  - The stability property would not lift.
- Is it possible to define d_{HT} via a minimum instead of an infimum? Is d_{HT} only a pseudo-metric or actually a metric?
  - Examples where the infimum is 0 and it is not attained.
- Is d_{HT} different from d_{NP}?
  - X contractible, $x \in X$, $c \in \mathbb{R}$: $d_{HT}((X,c),(\{x\},c)) = 0$, $d_{NP} = \infty$.
  - Cylinder C and strip M twisted of 2π radians: $d_{HT}(C,M) = 1$, $d_{NP}((C,z),(M,z)) = 2$.



Lifting stability results via d_{HT}

Stability Theorem

X, Y compact polyhedra, φ_X, φ_Y continuous functions.

$$d_B\big(dgm(\varphi_X), dgm(\varphi_Y)\big) \leq d_{HT}\big(\varphi_X, \varphi_Y\big).$$

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Idea of the proof

- Any α-homotopy equivalence induces an α-interleaving of q-tame persistence modules.
- Any α-interleaving of q-tame persistence modules induces a bottleneck matching with cost $\leq \alpha$ (Algebraic Stability Theorem).

Restricting d_{HT} to subcategories

We can restrict d_{HT} to any sub-category $\mathbf{C}$ of $\mathbf{S}$ closed with respect to the α*-shift functor*. Denote it $d_{HT}^{\mathbf{C}}$.

- Let X be fixed. Take $\mathbf{C}$ whose objects are functions $\varphi : X \to \mathbb{R}$, and between any two objects φ, φ' there is at most one morphism, $\mathrm{id}_X : X \to X$, then

$$d_{HT}^{\mathbf{C}}(\varphi, \varphi') = \|\varphi - \varphi'\|_\infty.$$

- Take $\mathbf{C}$ whose objects are those of $\mathbf{S}$, while morphisms from φ_X to φ_Y are the homeomorphisms f such that $\varphi_Y \circ f \leq \varphi_X$. Then

$$d_{HT}^{\mathbf{C}}(\varphi_X, \varphi_Y) = d_{NP}(\varphi_X, \varphi_Y).$$

- Take $\mathbf{C}$ be the PL or C^∞ subcategory. Then

$$d_{HT}^{\mathbf{C}}(\varphi_X, \varphi_Y) = d_{HT}^{\mathbf{S}}(\varphi_X, \varphi_Y).$$

An application

Use d_{HT} to compare merge trees: $mrg(\varphi_X)$ is the Reeb graph of

$$\bar{\varphi}_X : epi(\varphi_X) = \{(x,t) \in X \times \mathbb{R} : \varphi_X(x) \leq t\} \to \mathbb{R}, \qquad \bar{\varphi}_X(x,t) = t$$

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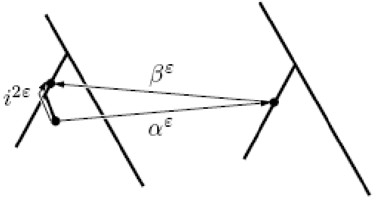

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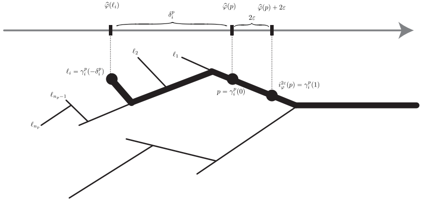

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The category $\mathbf{hTop}/\mathbb{R}^{\leq}$

Let $\mathbf{hTop}/\mathbb{R}^{\leq}$ be the category with

- objects: topological spaces X endowed with functions $\varphi_X : X \to \mathbb{R}$
- morphisms: 0-homotopy classes of 0-maps between X and Y
- composition of morphisms: the 0-homotopy class of the composition of 0-maps:

$$[g]_{(\varphi_Y,\varphi_Z)} \circ [f]_{(\varphi_X,\varphi_Y)} = [g \circ f]_{(\varphi_X,\varphi_Z)}.$$

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Every $\varphi_X : X \to \mathbb{R}$ defines a functor $hT^{\varphi_X} : \mathbf{R} \to \mathbf{hTop}/\mathbb{R}^{\leq}$:

- For $u \in \mathbb{R}$, $hT^{\varphi_X}(u) := (X^u, \varphi_X^u)$;
- For $u \leq v \in \mathbb{R}$, $hT^{\varphi_X}(u \leq v) := [i_X^{u,v}]_{(\varphi_X^u,\varphi_X^v)}$.

0-maps and natural transformations

Let φ_X, φ_Y be bounded functions and let hT^{φ_X}, hT^{φ_Y} be the induced functors.

Lemma

Every map $f : X \to Y$ such that $\varphi_Y \circ f \leq \varphi_X$ induces a natural transformation

$$h\xi^f : hT^{\varphi_X} \Rightarrow hT^{\varphi_Y}$$

such that for every $u \in \mathbb{R}$.

$$h\xi^f_u = [f^{|Y^u}_{|X^u}]_{(\varphi^u_X, \varphi^u_Y)}$$

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Reciprocally, for every natural transformation $h\xi : hT^{\varphi_X} \Rightarrow hT^{\varphi_Y}$ there exists a continuous map $f : X \to Y$ such that $\varphi_Y \circ f \leq \varphi_X$ and, for every $u \in \mathbb{R}$, $h\xi_u = [f_{|X^u}^{|Y^u}]_{(\varphi_X^u, \varphi_Y^u)}$.

d_{HT} as interleaving distance

Define interleaving distance $d_I^{\mathbf{hTop}/\mathbb{R}^{\leq}}$ on functors

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following [Bubenik & Scott].

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Theorem

For every bounded functions $\varphi_X : X \to \mathbb{R}$, $\varphi_Y : Y \to \mathbb{R}$,

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Corollary

$$d_I^{\mathbf{Vect}_{\mathbb{F}}}(PM(\varphi_X), PM(\varphi_Y)) \leq d_I^{\mathbf{hTop}/\mathbb{R}^{\leq}}(hT^{\varphi_X}, hT^{\varphi_Y}).$$

Open questions

- Possible applications: quantify error in passing from a grayscale 3D image as a cubical complex to its skeleton?
- Utility of subcategories: applications to Frosini's group invariant persistence?
- Further lifting of stability, i.e. tighter upper bounds for bottleneck distance?
- Tighter lower bounds for d_{HT}?