Surface Superconductivity in 3D

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Motivation from 3D Ginzburg-Landau

Consider the standard 3D GL functional,

$$\mathcal{G}^{\mathrm{3D}}_{\kappa,H}(\psi,\mathbf{A}) = \int_\Omega \Big[|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \Big]\,dx \\ + \kappa^2 H^2 \int_{\mathbb{R}^3} |\mathrm{curl}\,\mathbf{A} - \mathbf{e}_{x_3}|^2\,dx\,,$$

with ground state energy $g_0(\kappa,H)$.

Theorem [F-Kachmar-Persson]
Suppose $H-\kappa \geq o(\kappa)$ as $\kappa\to\infty$. Then the GL ground state energy satisfies,

$$g_0(\kappa,H) = \sqrt{\kappa H}\int_{\partial\Omega} E\left(\mathfrak{b},\nu(x)\right)\,d\sigma(x) + E_2|\Omega|\,[\kappa - H]_+^2 \\ + o(\max\left(\kappa^2, \kappa[\kappa-H]_+^2\right)\,.$$

Here $\mathfrak{b} = \min\left(\kappa/H, 1\right)$, $d\sigma(x)$ is the surface measure on the boundary of Ω and $\nu(x)$ is the angle of the tangent plane to $\mathbf{e}_{x_3}$.

Setup of surface energy I

Let $\nu \in [0, \frac{\pi}{2}]$, and $\ell > 0$. We introduce the set

$$\mathcal{D}_\ell = (0, \infty) \times (-\ell, \ell) \times (-\ell, \ell),$$

and the magnetic potential $\mathbf{A}_\nu$ defined on $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1 > 0\}$ by

$$\mathbf{A} = \mathbf{A}_\nu = \begin{pmatrix} 0 \\ 0 \\ -x_1 \cos\nu + x_2 \sin\nu \end{pmatrix},$$

for which the associated magnetic field is the constant unit vector that makes an angle ν with the $x_2 x_3$ plane

$$\mathbf{B} = \mathbf{B}_\nu = \nabla \times \mathbf{A}_\nu = \begin{pmatrix} \sin\nu \\ \cos\nu \\ 0 \end{pmatrix}.$$

Setup of surface energy II

We consider the following reduced Ginzburg-Landau-type energy functional

$$\mathcal{E}_{\mathfrak{b},\nu,\ell}(\varphi) = \int_{\mathcal{D}_\ell} \left(|(-i\nabla + \mathbf{A}_\nu)\varphi|^2 - \mathfrak{b}|\varphi|^2 + \frac{\mathfrak{b}}{2}|\varphi|^4 \right) \mathrm{d}x,$$

for φ in the space

$$\mathcal{S}_\ell = \left\{ \varphi \in L^2(\mathcal{D}_\ell), (-i\nabla + \mathbf{A}_\nu)\varphi \in L^2(\mathcal{D}_\ell), \varphi = 0 \text{ on } \partial\mathcal{D}_\ell \setminus \{x_1 = 0\} \right\}.$$

Furthermore, we define

$$E(\mathfrak{b},\nu,\ell) = \inf_{\varphi \in \mathcal{S}_\ell} \mathcal{E}_{\mathfrak{b},\nu,\ell}(\varphi),$$

and (for those values of $\mathfrak{b}$ where the limit exists, i.e. $\mathfrak{b} \leq 1$) :

$$e(\mathfrak{b},\nu) = \lim_{\ell \to \infty} \frac{1}{4\ell^2} E(\mathfrak{b},\nu,\ell).$$

The spectral quantity Θ_0

Consider the harmonic oscillator on the half-axis $\mathbb{R}_+ = \{t \in \mathbb{R}, t > 0\}$

$$H(\xi) = -\frac{d^2}{dt^2} + (t-\xi)^2 \quad \text{in } L^2(\mathbb{R}_+),$$

with Neumann boundary condition $u'(0) = 0$, and for $\xi \in \mathbb{R}$.
This operator has compact resolvent and its eigenvalues are simple. Let $\mu_1(\xi)$ denote the first eigenvalue of $H(\xi)$. Then, Θ_0 is defined as

$$\Theta_0 = \inf_{\xi \in \mathbb{R}} \mu_1(\xi).$$

The linear problem

Consider the Schrödinger operator with constant magnetic field on the half-space $\mathbb{R}_+$,

$$\mathcal{L}(\nu) = (-i\nabla + \mathbf{A}_\nu)^2 \quad \text{in } L^2(\mathbb{R}^3_+)\,,$$

with Neumann realization.
Let $\zeta(\nu) = \inf \operatorname{Spec} \mathcal{L}(\nu)$.

Lemma Let Θ_0 be the universal constant introduced above. The function $[0, \pi/2] \ni \nu \mapsto \zeta(\nu)$ is monotone non-decreasing, and we have that $\zeta(0) = \Theta_0$ and $\zeta(\pi/2) = 1$.

This lemma tells us that only the range $\mathfrak{b} \in (\Theta_0, 1)$ is interesting for the non-linear problem.
More precisely,

$$E(\mathfrak{b}, \nu, \ell) = 0 \text{ for } \mathfrak{b} \leq \zeta(\nu).$$

Theorem [F-Miqueu-Pan]
For all $\mathfrak{b} \in (\Theta_0, 1)$, the function $(0, \frac{\pi}{2}) \ni \nu \mapsto e(\mathfrak{b}, \nu)$ is monotone non-decreasing.

Remark That $\mathfrak{b} \mapsto e(\mathfrak{b}, \nu)$ is monotone non-decreasing is obvious by differentiation.

The special case $\nu = 0$

Notice that for $\nu = 0$

$$\mathcal{E}_{\mathfrak{b},\nu=0,\ell}(\varphi) = \int_{[-\ell,\ell]^2} \int_0^{+\infty} \left\{ |(-i\nabla - x_1 \mathbf{e}_{x_3})\varphi|^2 - \mathfrak{b}|\varphi|^2 + \frac{\mathfrak{b}}{2}|\varphi|^4 \right\}.$$

Here, one can apply the similar argument from the 2D-Ginzburg-Landau functional by Correggi and Rougerie to obtain a dimensional reduction by a 'separation of variables'

Theorem

For $\nu = 0$ and $\mathfrak{b} \in (\Theta_0, 1]$ we have $e(\mathfrak{b}, \nu = 0) = E_0^{1\mathrm{D}}$.
Here $E_0^{1\mathrm{D}}$ is defined by

$$E_0^{1\mathrm{D}} = \inf_{\xi \in \mathbb{R}} \left(\inf_{f \in H^1(\mathbb{R}_+)} \mathcal{E}_{\mathfrak{b},\xi}^{1\mathrm{D}}(f) \right),$$

with

$$\mathcal{E}_{\mathfrak{b},\xi}^{1\mathrm{D}}(f) := \int_0^\infty |f'(t)|^2 + (t-\xi)^2 |f(t)|^2 - \mathfrak{b}|f(t)|^2 + \frac{\mathfrak{b}}{2}|f(t)|^4 \, \mathrm{d}t.$$

Mononicity Proof

Recall/generalize

$$\mathcal{E}_{\mathfrak{b},\nu,\ell}(\varphi) = \int_{\mathcal{D}_\ell} \left(|(-i\nabla + \mathbf{A}_\nu)\varphi|^2 - \mathfrak{b}|\varphi|^2 + \frac{\mathfrak{b}}{2}|\varphi|^4 \right) \mathrm{d}x,$$

Here $\mathcal{D}_\ell := \mathbb{R}_+ \times \ell A$, and $A := [-1,1]^2$. The boundary energy density is

$$e(\mathfrak{b},\nu) = \lim_{\ell\to\infty} \frac{1}{4\ell^2} E(\mathfrak{b},\nu,\ell) = \lim_{\ell\to\infty} \frac{1}{|\mathcal{D}_\ell \cap \{x_1 = 0\}|} E(\mathfrak{b},\nu,\ell).$$

Generalizes (with unchanged limit!) to

- Cylinders $\mathbb{R}_+ \times A_\ell$ with 'general' A (thermodynamic limit).
- Cylinders, where the cylinder axis has a fixed non-zero angle to the plane $\{x_1 = 0\}$.

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Monotonicity proof II

Define the functional

$$\widetilde{\mathcal{E}}_{\mathfrak{b},\nu,\alpha,L,L_3}(\tilde{\varphi}) = \int_{\widetilde{\mathcal{D}}_{L,L_3,\alpha}} |D_1\tilde{\varphi}|^2 + \tan^2(\nu)|D_2\tilde{\varphi}|^2 + |(D_3+v_1)\tilde{\varphi}|^2 \\ -\mathfrak{b}|\tilde{\varphi}|^2 + \frac{\mathfrak{b}}{2}\tan(\nu)|\tilde{\varphi}|^4 \,\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}v_3,$$

with

$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{L,L_3,\alpha} = \widetilde{\mathcal{D}}_{L,\alpha} \times (-L_3, L_3),$$

with

$$\widetilde{\mathcal{D}}_{L,\alpha} = \left\{ v_1 > -v_2, |(\tan\alpha)v_1 - v_2| \leq \frac{L}{\sqrt{2}}(1+\tan\alpha) \right\},$$

Let $\tilde{E}(\mathfrak{b},\nu,\alpha,L,L_3)$ be the corresponding ground state energy.

Monotonicity proof III

Composing the changes of variables

$$\begin{cases} x_1 &= -u_1\cos(\nu) - u_2\sin(\nu) \\ x_2 &= u_1\sin(\nu) - u_2\cos(\nu) \\ x_3 &= u_3 \end{cases} \quad \text{and} \quad \begin{cases} u_1 &= -v_1 \\ u_2 &= \frac{v_2}{-\tan(\nu)} \\ u_3 &= v_3 \end{cases}$$

on easily finds

Lemma

In the case where $\alpha = \arctan(\tan^2(\nu))$, $L_3 = \ell$, and $L = \sqrt{2}\ell\sin(\nu)$ we have

$$\widetilde{E}(\mathfrak{b}, \nu, \alpha, L, L_3) = E(\mathfrak{b}, \nu, \ell).$$

In particular, still with this special relation between the parameters,

$$\sqrt{2}\sin(\nu)\frac{\widetilde{E}(\mathfrak{b}, \nu, \alpha, L, L_3)}{4LL_3} = \frac{E(\mathfrak{b}, \nu, \ell)}{4\ell^2}.$$

Monotonicity proof - 'differentiation'

$$\begin{aligned}\Delta_{\mathfrak{b},\nu}(\varepsilon) &= e(\mathfrak{b},\nu+\varepsilon) - e(\mathfrak{b},\nu)\\ &= \frac{\sqrt{2}}{4}\lim_{L\to+\infty}\left(\sin(\nu+\varepsilon)\frac{\widetilde{E}(\mathfrak{b},\nu+\varepsilon,L)}{4L^2} - \sin(\nu)\frac{\widetilde{E}(\mathfrak{b},\nu,L)}{4L^2}\right).\end{aligned}$$

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We take $\varepsilon > 0$ and we are looking for a positive lower bound for $\Delta_{\mathfrak{b},\nu}(\varepsilon)$ in order to prove the monotonicity. We will use a minimizer (which exists) of $\widetilde{\mathcal{E}}_{\mathfrak{b},\nu+\varepsilon,L}$ that we denote $\varphi^{\min}$. Therefore we have

$$\widetilde{\mathcal{E}}_{\mathfrak{b},\nu,L}(\varphi^{\min}) \geq \widetilde{E}(\mathfrak{b},\nu,L) \quad \text{and} \quad \widetilde{\mathcal{E}}_{\mathfrak{b},\nu+\varepsilon,L}(\varphi^{\min}) = \widetilde{E}(\mathfrak{b},\nu+\varepsilon,L).$$

Differentiation II

Therefore,

$$\Delta_{\mathfrak{b},\nu}(\varepsilon) \geq \frac{\sqrt{2}}{4} \lim_{L\to+\infty}\Big(\frac{1}{L^2}(\sin(\nu+\varepsilon)-\sin(\nu))\widetilde{E}(\mathfrak{b},\nu+\varepsilon,L)$$

$$\frac{1}{L^2}\sin(\nu)(\tan^2(\nu+\varepsilon)-\tan^2(\nu))\int_{\widetilde{\mathcal{D}}_{\ell,\nu}}|D_2\varphi^{\min}|^2\,\mathrm{d}v$$

$$\frac{1}{L^2}\sin(\nu)\frac{\mathfrak{b}}{2}(\tan(\nu+\varepsilon)-\tan(\nu))\int_{\widetilde{\mathcal{D}}_{\ell,\nu}}|\varphi^{\min}|^4\,\mathrm{d}v\Big).$$

For $\varepsilon \geq 0$ and small enough, we have $\tan^2(\nu+\varepsilon)-\tan^2(\nu)\geq 0$ so that the term $\lim_{L\to+\infty}\frac{\sqrt{2}}{4L^2}\int_{\widetilde{\mathcal{D}}_{\ell,\nu}}(\nu+\varepsilon)(\tan^2(\nu+\varepsilon)-\tan^2(\nu))|D_2\varphi^{\min}|^2\,\mathrm{d}v$ is positive and we can discard it in the lower bound.

Differentiation III

Using a Ginzburg-Landau equation

$$\Delta_{\mathfrak{b},\nu}(\varepsilon) \geq \frac{\sqrt{2}}{4}\frac{\mathfrak{b}}{2}\Big(\sin(\nu)(\tan(\nu+\varepsilon)-\tan(\nu)) \\ - (\sin(\nu+\varepsilon)-\sin(\nu))\tan(\nu+\varepsilon)\Big)\lim_{L\to+\infty}\frac{1}{L^2}\int_{\widetilde{\mathcal{D}}_{\ell,\nu}} |\varphi^{\min}|^4\,\mathrm{d}v.$$

But by differentiation,

$$\sin(\nu)(\tan(\nu+\varepsilon)-\tan(\nu)) - (\sin(\nu+\varepsilon)-\sin(\nu))\tan(\nu+\varepsilon) \approx \tan^2(\nu)\varepsilon,$$

for small ε. $\square$

Lattice states

Consider the square $D_R = (0,R)^2$ in the $\{x_1 = 0\}$-plane. Flux through D_R,

$$\Phi := \int_{D_R} \mathbf{B}\cdot \mathbf{e}_{x_1} = R^2 \sin\nu \overset{assume}{\in} 2\pi\mathbb{Z}.$$

Consider the magnetic periodic boundary conditions on D_R :

$$\psi(x_1, x_2 + R, x_3) = \psi(x_1,x_2,x_3)e^{iRx_3\sin\nu}, \quad \psi(x_1,x_2,x_3+R) = \psi(x_1,x_2,x_3).$$

Let $H^{\rm per}$ be the operator $(-i\nabla + \mathbf{A}_\nu)^2$ on $\mathbb{R}_+ \times D_R$ with Neumann boundary condition at $x_1 = 0$ and periodic magnetic boundary conditions on $\mathbb{R}_+ \times \partial D_R$.

Lemma For $\nu \in (0, \pi/2)$ and assuming $\frac{R^2 \sin \nu}{2\pi} \in \mathbb{Z}$ we have that

$$\zeta(\nu, R) := \inf \mathrm{Spec}\, H^{\mathrm{per}}$$

is a discrete eigenvalue of H^{per}.
Furthermore,

$$\zeta(\nu, R) = \zeta(\nu).$$

Let Ψ^{per} be an associated eigenfunction. We can extend it to $\mathbb{R}^3_+$ by magnetic periodicity.
Then we see that

$$\begin{aligned}
|D_{nR}|^{-1}\mathcal{E}_{\mathfrak{b},\nu,nR}(\Psi) &= |\mathcal{D}_{nR}|^{-1}\int_{\mathcal{D}_{nR}}\left(|(-i\nabla+\mathbf{A}_\nu)\Psi|^2-\mathfrak{b}|\Psi|^2+\frac{\mathfrak{b}}{2}|\Psi|^4\right)\mathrm{d}x\\
&= |D_R|^{-1}\int_{\mathcal{D}_R}\left(|(-i\nabla+\mathbf{A}_\nu)\Psi|^2-\mathfrak{b}|\Psi|^2+\frac{\mathfrak{b}}{2}|\Psi|^4\right)\mathrm{d}x\\
&= |D_R|^{-1}\left(\frac{\mathfrak{b}}{2}\|\Psi\|_4^4-(\mathfrak{b}-\zeta(\nu))\|\Psi\|_2^2\right).
\end{aligned}$$

By replacing Ψ by $\lambda\Psi$ and optimizing in λ, we get for $\mathfrak{b}\in(\zeta(\nu),1)$,

$$e(\mathfrak{b},\nu)\leq -\frac{(\mathfrak{b}-\zeta(\nu))^2}{2\mathfrak{b}}\frac{\|\Psi\|_2^4}{|D_R|\|\Psi\|_4^4}.$$

- Of course, we can also study other lattice geometries than squares.
- The projection of the magnetic field on the plane $\{x_1 = 0\}$ specifies a direction. Therefore, the orientation of the lattices is important.

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