The vanishing discount problem for fully nonlinear degenerate elliptic PDEs

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Vanishing discount problem

We consider the second-order PDE

$$\lambda v(x) + F(x, Dv(x), D^2 v(x)) = 0 \quad \text{in } \mathbb{T}^n.$$

Here

$$\begin{cases} v = v^\lambda \text{ denotes the unknown function on } \mathbb{T}^n \\ \lambda > 0 \text{ is a given constant,} \\ F \text{ is a given function of } (x, Dv(x), D^2 v(x)). \end{cases}$$

Problem: asymptotic behavior of v^λ as $\lambda \to 0$.

Class of PDEs

Hypotheses:

(F1) F has the form

$$F(x,p,X) = \sup_{\alpha\in\mathcal{A}} \left(-\operatorname{tr} a(x,\alpha)X - b(x,\alpha)\cdot p - L(x,\alpha)\right)$$

for $(x,p,X)\in\mathbb{T}^n\times\mathbb{R}^n\times\mathbb{S}^n$,

where $\mathcal{A}$ is a σ-compact, locally compact metric space ($\neq\emptyset$), $\mathbb{S}^n$ denotes the space of $n\times n$ real symmetric matrices,

$$a\in C(\mathbb{T}^n\times\mathcal{A},\mathbb{S}^n_+),\quad b\in C(\mathbb{T}^n\times\mathcal{A},\mathbb{R}^n),\quad L\in C(\mathbb{T}^n\times\mathcal{A},\mathbb{R}),$$

and $\mathbb{S}^n_+$ denotes the subset of $\mathbb{S}^n$ consisting of non-negative matrices.

(F2) F is a continuous function on $\mathbb{T}^n\times\mathbb{R}^n\times\mathbb{S}^n$.

Property: F is degenerate elliptic. That is,

$$X \leq Y \quad \Longrightarrow \quad F(x,p,X) \geq F(x,p,Y)$$

Property: F is a convex function. More precisely,

$$(p,X) \mapsto F(x,p,X) \text{ is convex on } \mathbb{R}^n \times \mathbb{S}^n.$$

Notation : $\mathcal{L}u = \mathcal{L}u(x,\alpha) := -\operatorname{tr} aD^2u - b \cdot Du.$

Ergodic problem

(DP) $$\lambda v^\lambda + F(x, Dv^\lambda, D^2 v^\lambda) = 0 \quad \text{in } \mathbb{T}^n.$$

A classical observation regarding the behavior of the solutions v^λ, as $\lambda \to 0+$, is the following (P.-L. Lions-G. Papanicolaou-S. R. S. Varadhan).

Under suitable assumptions (the comparison principle and equicontinuity), for some constant $c \in \mathbb{R}$ and function $u \in C(\mathbb{T}^n)$, as $\lambda \to 0+$, we have

$$\begin{cases} -\lambda v^\lambda(x) \to c & \text{uniformly on } \mathbb{T}^n, \\ v^\lambda(x) - m^\lambda \to u(x) & \text{uniformly on } \mathbb{T}^n \text{ along a subsequence}, \end{cases}$$

where m_λ is chosen as $m^\lambda = \min_{\mathbb{T}^n} v^\lambda$, for instance.

Furthermore, the pair (u, c) is a solution of

$$\text{(EP)} \qquad F(x, Du(x), D^2u(x)) = c \quad \text{in } \mathbb{T}^n.$$

The problem

$$\text{(EP)} \qquad F(x, Du(x), D^2u(x)) = c \quad \text{in } \mathbb{T}^n.$$

is called the ergodic problem or additive eigenvalue problem . Here the problem is to find a pair (u, c) of a solution u of PDE (EP) and a constant c.

Main question: In recent years there has been a growing interest in the question if the whole family $\{v^\lambda - m^\lambda\}_{\lambda>0}$ converges to a function as $\lambda \to 0+$.

A few previous work:

1) A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique,
First-order HJ equation on $\mathbb{T}^n$ (closed manifold), with coercive and convex Hamiltonians. Invent. Math. (2016)

2) E. S. Al-Aidarous, E. O. Alzahrani, HI, A. M. M. Younas,
First-order HJ equation with the Neumann type BC, with coercive and convex Hamiltonian. Proc. Roy. Soc. Edinburgh Sect. A (2016)

3) H. Mitake, H. V. Tran
Viscous HJ equation on $\mathbb{T}^n$, with smooth, coercive and convex Hamiltonian.

....

- Use of Mather measures , the adjoint method due to L. C. Evans.

Classical observations

Two more hypotheses:

(CP) $\left\{\right.$ **The comparison principle holds for** (DP). **More precisely, if $\lambda > 0$ and if $v \in \mathrm{USC}(\mathbb{T}^n)$, $w \in \mathrm{LSC}(\mathbb{T}^n)$ are a subsolution and a supersolution of** (DP), **respectively, then $v \le w$ in $\mathbb{T}^n$.**

Proposition 1

Assume (F1), (F2) and (CP). **Let $\lambda > 0$. Problem** (DP) **has a unique solution in $C(\mathbb{T}^n)$.**

(EC) $\left\{\right.$ **For every $\lambda > 0$, there exists a solution $v^\lambda \in C(\mathbb{T}^n)$ of** (DP), **and the family $\{v^\lambda\}_{\lambda>0}$ is equi-continuous on $\mathbb{T}^n$.**

Proposition 2 (classical results)

Assume (F1), (F2), (CP), **and** (EC). **(i) Problem** (EP) **has a solution $(u, c) \in C(\mathbb{T}^n) \times \mathbb{R}$, and the constant c is determined uniquely.**
(ii) For $\lambda > 0$, let $v^\lambda \in C(\mathbb{T}^n)$ be a unique solution of (DP)**, Then**

$$c = -\lim_{\lambda \to 0+} \lambda v^\lambda(x) \text{ in } C(\mathbb{T}^n),$$

and, for any sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ converging to zero, there exists its subsequence, which is denoted by the same symbol, such that $\{v^{\lambda_j} - \min_{\mathbb{T}^n} v^{\lambda_j}\}_{j\in\mathbb{N}}$ converges in $C(\mathbb{T}^n)$. Moreover, the pair of the function

$$u(x) := \lim_{j\to\infty} (v^{\lambda_j}(x) - \min_{\mathbb{T}^n} v^{\lambda_j}) \in C(\mathbb{T}^n),$$

and the constant c is a solution of (EP).

If $(u, c) \in C(\mathbb{T}^n) \times \mathbb{R}$ is a solution of (EP)**, then the constant c is called a critical value or an additive eigenvalue.**

Main theorem

We introduce two more hypotheses.

(AC) the set $\mathcal{A}$ is compact.

For simplicity, we write $F[u](x)$ for $F(x, Du(x), D^2u(x))$.
For any $\phi \in C(\mathbb{T}^n \times \mathcal{A})$, we consider the function defined by

$$F_\phi(x, p, X) = \sup\left(-\operatorname{tr} a(x,\alpha)X - b(x,\alpha)\cdot p - \phi(x,\alpha)\right),$$

and the problem

$$(\mathrm{DP}_\phi) \qquad \lambda v(x) + F_\phi[v] = 0 \quad \text{in } \mathbb{T}^n.$$

(CP′) The comparison principle holds for (DP_ϕ). More precisely, let $\lambda > 0$, $\phi \in C(\mathbb{T}^n \times \mathcal{A})$, and U be any open subset of $\mathbb{T}^n$. If $v,\ w \in C(U)$ are a subsolution and a supersolution of $\lambda u + F_\phi[u] = 0$ in U, respectively, and $v \le w$ on ∂U, then $v \le w$ in U.

Main theorem

Assume (F1), (F2), (AC), (CP$'$), **and** (EC)**. Let c be the critical value of** (EP) **and, for each $\lambda > 0$, let $v^{\lambda} \in C(\mathbb{T}^n)$ be the unique solution of** (DP)**. Then, the family $\{v^{\lambda} + \lambda^{-1}c\}_{\lambda>0}$ converges to a function u in $C(\mathbb{T}^n)$ as $\lambda \to 0$. Furthermore, the pair (u, c) is a solution of** (EP)**.**

The comparison principle (CP$'$) is strong enough to guarantees the conclusion of Proposition 2.

Viscosity Mather measures

We introduce sort of a generalized Mather measure, that applies to second-order degenerate elliptic PDEs. This generalization is inspired by

D. Gomes, Duality principle for fully nonlinear ... 2005.

Henceforth we assume that the critical value c is equal to zero .

This can be always realized by replacing F by $F - c$ as well as v^λ by $v^\lambda + \lambda^{-1}c$.

Let $\phi \in C(\mathbb{T}^n \times \mathcal{A})$ and consider subsolutions $u \in C(\mathbb{T}^n)$ of the PDE

$$(\mathrm{EP}_\phi) \qquad F_\phi[u] = 0 \quad \text{in } \mathbb{T}^n.$$

We denote the set of all such pairs (ϕ, u) by $\mathcal{F}_\pi(0)$. The set $\mathcal{F}_\pi(0)$ is positively homogeneous , that is,

$$t > 0,\ (\phi, u) \in \mathcal{F}_\pi(0) \implies t(\phi, u) \in \mathcal{F}_\pi(0).$$

This set is also convex , thanks to the convexity of $(p, X) \mapsto F(x, p, X)$.

Lemma 1

Under hypotheses (F1), (F2) and (CP′), the set $\mathcal{F}_\pi(0)$ is a convex set in $C(\mathbb{T}^n \times \mathcal{A})$.

Thus, $\mathcal{F}_\pi(0)$ is a convex cone with vertex at the origin.
Consider the dual cone $\mathcal{F}_\pi{}'(0)$ of $\mathcal{F}_\pi(0)$ in the space of all Radon measures. That is, a Radon measure μ is in $\mathcal{F}_\pi{}'(0)$ if and only if

$$0 \leq \langle \mu, \phi \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(0),$$

where

$$\langle \mu, \phi \rangle := \int_{\mathbb{T}^n \times \mathcal{A}} \phi(x, \alpha) \mu(dx d\alpha) \quad (C^*(\mathbb{T}^n \times \mathcal{A}) \overset{\text{duality}}{\longleftrightarrow} C(\mathbb{T}^n \times \mathcal{A})).$$

We set

$$\mathcal{P}_\pi(0) = \{\mu \in \mathcal{F}_\pi{}'(0) : \mu \text{ is a probability measure on } \mathbb{T}^n \times \mathcal{A}\}.$$

The next claim ensures the existence of "Mather measures".

Theorem 1

Assume **(F1)**, **(F2)**, **(AC)**, (CP′) and (EC). Also, assume that $c = 0$. Then,

$$\min_{\mu \in \mathcal{P}_\pi(0)} \langle \mu, L \rangle = 0.$$

Here the role of (EC) is to guarantees that (EP) has a solution.

We call $\mu \in \mathcal{P}_\pi(0)$ a **viscosity Mather measure** if it attains the minimum value of the left hand side of the identity above.

Let $\mathcal{M}_\pi$ denote the set of viscosity Mather measures.

The key idea for the proof of the identity above is to use Sion's minimax theorem, which differs from the use of the convex duality by Diogo Gomes.

A crucial property of $\mathcal{P}_\pi(0)$ is introduced here as the dual cone of $\mathcal{F}_\pi(0)$, which corresponds to the closedness (or holonomy) property of Mather measures.

In the work of D. Gomes, he chooses, for $\mathcal{F}_\pi(0)$, the linear space of the pairs $(\phi, \psi) \in C(\mathbb{T}^n \times \mathcal{A}) \times C^2(\mathbb{T}^n)$, where

$$\phi : (x, \alpha) \mapsto \mathcal{L}\psi = -\operatorname{tr} a(x, \alpha) D^2\psi(x) - b(x, \alpha) \cdot D\psi(x).$$

Note that this pair (ϕ, ψ) belongs to $\mathcal{F}_\pi(0)$. Indeed,

$$\boldsymbol{F_\phi[\psi] = \max_{\alpha \in \mathcal{A}}\{\mathcal{L}\psi(x,\alpha) - \phi(x,\alpha)\} = 0.}$$

In the approach by D. Gomes, the dual cone property can be stated as

$$\boldsymbol{\langle \mu, \mathcal{L}\psi \rangle = 0 \ \ \forall \psi \in C^2(\mathbb{T}^n).}$$

This explains why we call our measures "viscosity" Mather measures.

We have a theorem, similar to the above, for discount problem (DP)**.**

We fix $(z,\lambda) \in \mathbb{T}^n \times (0,\infty)$. Define $\mathcal{F}_\pi(\lambda) \subset C(\mathbb{T}^n \times \mathcal{A}) \times C(\mathbb{T}^n)$ by

$$\mathcal{F}_\pi(\lambda) = \{(\phi,u) \in C(\mathbb{T}^n \times \mathcal{A}) \times C(\mathbb{T}^n) : u \text{ is a subsolution of } (\mathrm{DP}_\phi)\},$$

and $\mathcal{P}_\pi(z,\lambda)$ as the set of Radon probability measures μ on $\mathbb{T}^n \times \mathcal{A}$ having the property

$$0 \le \langle \mu, \phi - \lambda u(z) \rangle \quad \text{for all} \quad (\phi,u) \in \mathcal{F}_\pi(\lambda).$$

Theorem 2

Assume (F1), (F2), (AC) and (CP′). Let $\lambda \ge 0$ and $v^\lambda \in C(\mathbb{T}^n)$ be the (unique) solution of (DP). Then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P}_\pi(z,\lambda)} \langle \mu, L \rangle.$$

This is a representation formula for solutions of (DP).

If $\mu \in \mathcal{P}_\pi(z,\lambda)$ is a minimizer of the following minimization problem

$$\min_{\mu \in \mathcal{P}_\pi(z,\lambda)} \langle \mu, L \rangle,$$

then we call $\lambda^{-1}\mu$ a viscosity Green measure . We denote by $\mathcal{G}_\pi(z,\lambda)$ the set of viscosity Green measures.

Following the argument by Davini-Fathi-Iturriaga-Zavidovique and using Theorems 1 and 2, the proof of Main theorem is now easy.

Proof of the Main Theorem (Convergence).

Normalize so that $c = 0$. By comparison, we see that $\{v^\lambda\}_{\lambda>0}$ is uniformly bounded on $\mathbb{T}^n$. Thus, $\{v^\lambda\}_{\lambda>0}$ is precompact in $C(\mathbb{T}^n)$.

We select $\lambda_j \to 0+$ so that for some $v \in C(\mathbb{T}^n)$,

$$v^{\lambda_j} \to v \quad \text{in } \; C(\mathbb{T}^n).$$

It is enough to show that for any $x \in \mathbb{T}^n$,

$$v(x) = \max\{w(x) \mid F[w] = 0 \; \text{ in } \mathbb{T}^n, \; \langle \mu, w\rangle \leq 0 \; \forall \mu \in \mathcal{M}_\pi\}.$$

First note that

$$0 = \lambda_j v^{\lambda_j} + F[v^{\lambda_j}] = F_{L-\lambda_j v^{\lambda_j}}[v^{\lambda_j}].$$

In particular,

$$(L - \lambda_j v^{\lambda_j}, \, v^{\lambda_j}) \in \mathcal{F}_\pi(0),$$

To repeat,

$$(L-\lambda_j v^{\lambda_j},\, v^{\lambda_j}) \in \mathcal{F}_\pi(0),$$

and hence, if $\mu \in \mathcal{M}_\pi$, then

$$0 \le \langle \mu, L-\lambda_j v^{\lambda_j}\rangle = -\lambda_j \langle \mu, v^{\lambda_j}\rangle,$$

and, in the limit as $j \to \infty$,

$$\langle \mu, v\rangle \le 0,$$

which shows that for all $x \in \mathbb{T}^n$,

$$v(x) \le \max\{w(x) \mid F[w]=0 \ \text{ in } \mathbb{T}^n,\ \langle \mu, w\rangle \le 0\ \forall \mu \in \mathcal{M}_\pi\}.$$

Next, fix any $w \in C(\mathbb{T}^n)$ so that

$$F[w] = 0 \quad \text{in } \mathbb{T}^n \quad \text{and} \quad \langle \mu, w \rangle \leq 0 \ \forall \mu \in \mathcal{M}_\pi.$$

Note that

$$0 = F[w] = \delta_j w + F_{L+\delta_j w}[w],$$

which says

$$(L + \delta_j w,\, w) \in \mathcal{F}_\pi(\delta_j).$$

Fix any $z \in \mathbb{T}^n$ and $\nu_j \in \mathcal{G}_\pi(z, \delta_j)$, and set $\mu_j = \delta_j \nu_j$. From the above observation,

$$0 \leq \langle \mu_j,\, L + \delta_j w - \delta_j w(z) \rangle = \delta_j v^{\delta_j}(z) + \delta_j(\langle \mu_j, w \rangle - w(z)).$$

$\square$

Passing to a subsequnce, we may assume that for some $\mu \in \mathcal{M}_\pi$,

$$\mu_j \to \mu \quad \text{weakly in the sense of measures.}$$

(It is easy to see that $\mu \in \mathcal{M}_\pi$.)
The previous observation that

$$0 \le \langle \mu_j,\, L + \delta_j w - \delta_j w(z) \rangle = \delta_j v^{\delta_j}(z) + \delta_j(\langle \mu_j, w \rangle - w(z))$$

yields

$$0 \le v(z) + \langle \mu, w \rangle - w(z).$$

Since $\langle \mu, w \rangle \le 0$, we see that

$$w(z) \le v(z),$$

which shows that

$$v(z) \ge \max\{w(x) \mid F[w] = 0 \ \text{ in } \mathbb{T}^n,\ \langle \mu, w \rangle \le 0\ \forall \mu \in \mathcal{M}_\pi\}.$$

Because z is arbitrary, we conclude the proof. $\square$

Further remarks

1. The case when $\mathcal{A}$ is non-compact. We assume that

(L) $$L = +\infty \quad \text{at infinity.}$$

We introduce

$$\Phi^{+} = \{tL + \chi : t > 0,\ \chi \in C(\mathbb{T}^n)\}.$$

We replace $\mathcal{F}_\pi(0)$ (resp., $\mathcal{F}_\pi(z,\lambda)$) by the set of $(\phi, u) \in \Phi^{+} \times C(\mathbb{T}^n)$ such that u is a subsolution of (EP$_\phi$) **(resp.,** (DP$_\phi$)**).**

Let $\mathcal{P}_L$ denote the space of Radon probability measures μ such that L is integrable on $\mathbb{T}^n \times \mathcal{A}$ with respect to μ. We replace $\mathcal{P}_\pi(0)$ (resp., $\mathcal{P}_\pi(z,\lambda)$) by the set of $\mu \in \mathcal{P}_L$ with the property

$$0 \leq \langle \mu, \phi \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(0)$$

$$\big(\text{resp., } 0 \leq \langle \mu, \phi - \lambda u(z) \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(z,\lambda)\big).$$

(CP″) $\left\{\begin{array}{l}\text{The comparison principle holds for } \lambda u + F[u] = \eta \text{ in } \mathbb{T}^n\text{, where } \eta \in C(\mathbb{T}^n)\text{. More precisely, let } \lambda > 0 \text{ and } U \\ \text{be any open subset of } \mathbb{T}^n\text{. If } v,\ w \in C(U) \text{ are a subsolution and a supersolution of } \lambda u + F(x, Du, D^2u) = \eta \\ \text{in } U\text{, respectively, and } v \le w \text{ on } \partial U\text{, then } v \le w \text{ in } U.\end{array}\right.$

Theorem 3

Assume (F1), (F2), (L), (CP″) **and** (EC). **Assume that the critical value c is zero. Then,**

$$\min_{\mu \in \mathcal{P}_\pi(0)} \langle \mu, L \rangle = 0.$$

Fix $(z, \lambda) \in \mathbb{T}^n \times (0,\ \infty)$ and let v^λ be the solution of (DP)**. Then**

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P}_\pi(z,\lambda)} \langle \mu, L \rangle.$$

2. **With a generality similar to the case of $\mathbb{T}^n$, we can treat the state-constraint, Neumann, and Dirichlet problems on bounded domains.**