L^p Hardy inequality on $C^{1,\gamma}$ domains

Yehuda Pinchover

Department of Mathematics, Technion
32000 Haifa, ISRAEL

E-mail: pincho@technion.ac.il

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The Hardy inequality

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with compact nonempty boundary. Let $\delta(x) := d(x, \partial\Omega)$ be the distance function to the boundary. Fix $p \in (1, \infty)$. The L^p *Hardy inequality* is satisfied in Ω if there exists $c > 0$ s.t.

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \geq c \int_\Omega \frac{|u|^p}{\delta^p} \,\mathrm{d}x \qquad \text{for all } u \in C_0^\infty(\Omega).$$

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The L^p *Hardy constant* $H_p(\Omega)$ of Ω is given by the Rayleigh-Ritz variational problem

$$H_p(\Omega) := \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega \frac{|u|^p}{\delta^p} \,\mathrm{d}x}.$$

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The associated Euler-Lagrange equation is given by

$$\left(-\Delta_p - \frac{H_p(\Omega)}{\delta^p}\mathcal{I}_p\right) u = 0 \qquad \text{in } \Omega,$$

$-\Delta_p v := -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right)$ is the p-Laplacian, and $\mathcal{I}_p v := |v|^{p-2}v$.

Existence of minimizer

Theorem (Marcus-Mizel-YP (1998), Marcus-Shafrir (2000))

Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2, and denote $c_p := \left(\frac{p-1}{p}\right)^p$.
Then $0 < H_p(\Omega) \leq c_p$.

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Furthermore, if $\alpha \in ((p-1)/p, 1)$ is such that $\lambda_\alpha := (p-1)\alpha^{p-1}(1-\alpha) = H_p(\Omega)$, then

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Remark

The proof relies heavily on the assumption $\Omega \in C^2$, which implies the *tubular neighbourhood theorem* and also that $\delta \in C^2$ in a neighbourhood of the boundary, so $|\Delta\delta|$ is bounded. Both properties do not hold for $\Omega \in C^{1,\gamma}$ (δ is not necessarily differentiable near $\partial\Omega$!).

Properties of the Hardy constant

Theorem (Lewis-J. Li-Yanyan Li (2012))

If Ω *is convex, or* weakly mean convex C^2 domain *(i.e.* $-\Delta\delta \geq 0$ *in* Ω*), then* $H_p(\Omega) = c_p = \left(\frac{p-1}{p}\right)^p$.

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Remark

1. If $\Omega = \mathbb{R}^n \setminus \{0\}$, then $\int_\Omega |\nabla u|^p \,\mathrm{d}x \geq H_p(\Omega) \int_\Omega \frac{|u|^p}{|x|^p} \,\mathrm{d}x$ for all $u \in C_0^\infty(\Omega)$, where

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2. For C^2 domains such that $H_p(\Omega) < c_p$, the Hardy constant $H_p(\Omega)$ depends continuously on p and on domain perturbations (Barbatis and Lamberti).

$C^{1,\gamma}$ bounded domains

Theorem (Lamberti-YP (2016))

Let $\Omega \in C^{1,\gamma}$ be a bounded domain in $\mathbb{R}^n$. Then $H_p(\Omega) < c_p$ iff the Rayleigh-Ritz variational problem admits a (unique) minimizer $u \in W_0^{1,p}(\Omega)$.

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Moreover, if $\alpha \in [(p-1)/p, 1]$ is such that

$$0 \leq \lambda_\alpha := (p-1)\alpha^{p-1}(1-\alpha) \leq c_p = \left(\frac{p-1}{p}\right)^p,$$

then any positive solution u of the equation

$$\left(-\Delta_p - \frac{\lambda_\alpha}{\delta^p}\mathcal{I}_p\right) v = 0 \qquad \text{in } \Omega'$$

of minimal growth in a neighbourhood Ω' of $\partial\Omega$ satisfies

$$u(x) \asymp \delta^\alpha(x) \qquad \forall x \in \Omega_\beta.$$

Minimal Growth

Definition

Let $K_0 \Subset \Omega$. A positive solution u of the equation $Q(u) = 0$ in $\Omega \setminus K_0$ is *of minimal growth in a neighborhood of infinity in* Ω, if for all smooth K s.t. $K_0 \Subset \mathrm{int}(K) \Subset \Omega$ and any positive supersolution $v \in C((\Omega \setminus K) \cup \partial K)$ of $Q(u) = 0$ in $\Omega \setminus K$ we have

$$u \leq v \ \text{ on } \ \partial K \quad \Rightarrow \quad u \leq v \ \text{ in } \ \Omega \setminus K.$$

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$$u \le v \quad \text{on} \quad \partial K \quad \Rightarrow \quad u \le v \quad \text{in} \quad \Omega \setminus K.$$

Theorem

Let $\mathcal{Q} \ge 0$ on $C_0^\infty(\Omega)$. Then $\forall x_0 \in \Omega$ the E-L equation $Q(u) = 0$ admits a unique positive solution u_{x_0} in $\Omega \setminus \{x_0\}$ of minimal growth in a neighborhood of infinity in Ω.

$C^{1,\gamma}$ exterior domains

Theorem (Lamberti-YP (2016))

Let $\Omega \subset \mathbb{R}^n$ be an $C^{1,\gamma}$ exterior domain, and $p \neq n$. Let

$$c_{p,n} := \min\{c_p, c^*_{p,n}\} = \min\left\{\left(\frac{p-1}{p}\right)^p, \left|\frac{p-n}{p}\right|^p\right\}.$$

$$\widetilde{W}^{1,p}(\Omega) := \{u \in W^{1,p}_{\mathrm{loc}}(\Omega) \mid \|u\|_{L^p(\Omega;\delta^{-p})} + \|\nabla u\|_{L^p(\Omega)} < \infty\}.$$

Then $H_p(\Omega) < c_{p,n}$ iff the Rayleigh-Ritz variational problem admits a (unique) minimizer $u \in \widetilde{W}^{1,p}(\Omega)$.

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Remark

Chabrowski and Willem (2006) proved that if Ω is a C^2 exterior domain and $H_p(\Omega) < c_{p,n}$, then a minimizer exists (no asymptotic of the minimizer is given).

$C^{1,\gamma}$ exterior domains

Theorem (Lamberti-YP (2016))

Let $\alpha, \alpha_1 \in [(p-1)/p, 1)$ and $\alpha_2 \in (0, (p-1)/p]$ be s.t. $\lambda = \lambda_\alpha := (p-1)\alpha^{p-1}(1-\alpha)$, $\lambda_{\alpha_1} = \lambda_{\alpha_2} = |(p-1)/(p-n)|^p \lambda$. If $p \neq n$ and $0 \le \lambda \le c_{p,n} := \min\{c_p, c^*_{p,n}\}$, then any positive solution u of the equation

$$-\Delta_p v - \frac{\lambda}{\delta^p}\mathcal{I}_p v = 0 \quad \text{in } \Omega' = \Omega \setminus K, K \Subset \Omega,$$

of minimal growth in a neighbourhood of infinity in Ω satisfies

(i) $u(x) \asymp \delta^\alpha(x)$ near $\partial\Omega$.

(ii) If $p < n$, then $u(x) \asymp |x|^{\frac{\alpha_1(p-n)}{p-1}}$ for all $|x| > M$.

(iii) If $p > n$, then $u(x) \asymp |x|^{\frac{\alpha_2(p-n)}{p-1}}$ for all $|x| > M$.

The supersolution construction

Lemma

Let G be a positive p-harmonic function in $U \subset \mathbb{R}^n$. Let $W := |\nabla G/G|^p$. Then for every $\alpha \in (0,1)$ we have

$$(-\Delta_p - \lambda_\alpha W \mathcal{I}_p) G^\alpha = 0, \quad \text{in } U,$$

$0 < \lambda_\alpha = (p-1)\alpha^{p-1}(1-\alpha) \leq c_p$.

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Lemma

Let $\Omega \in C^1$. Let $\Omega' \subset \Omega$ be a nbd of $\partial\Omega$ and $0 \leq G \in C^1(\overline{\Omega'})$ s.t. $G(x) = 0$, $\nabla G(x) \neq 0$ on $\partial\Omega$. Then

$$\lim_{x \to \partial\Omega} \frac{|\nabla G(x)|}{G(x)} \delta(x) = 1.$$

Moreover, if ω is the modulus of continuity of ∇G near $\partial\Omega$, then

$$\left| \frac{\nabla G(x)}{G(x)} \right| = \frac{1}{\delta(x)} + \frac{O(\omega(\delta(x)))}{\delta(x)} \quad \text{as } x \to \partial\Omega.$$

Hopf's boundary point lemma

Lemma (Mikayelyan-Shahgholian (2015) (Li-Nirenberg (2007)))

Hopf lemma holds for the p-Laplacian if $\partial\Omega$ is of class $C^{1,\gamma}$ or even $C^{1,\mathrm{Dini}}$. In particular, if G is positive p-harmonic function in $\Omega \in C^{1,\gamma}$, and $G = 0$ on $\partial\Omega$, then $\nabla G(x) \neq 0$ on $\partial\Omega$.

Agmon trick for bounded domains

Lemma

Consider a C^1-domain $\Omega \subset \mathbb{R}^n$ with compact boundary, and a neighbourhood $U \subset \Omega$ of $\partial\Omega$. Let $0 < G \in C^{1,\gamma}(\overline{\Omega \cap U})$ be p-harmonic in U s.t. $G = 0$ and $\nabla G(x) \neq 0$ on $\partial\Omega$. Let $\frac{(p-1)}{p} \leq \alpha < \beta < \alpha + \gamma < 1$.

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$$\Big(-\Delta_p v - \frac{\lambda_\alpha}{\delta^p} \mathcal{I}_p \Big) \big(G^\alpha \pm G^\beta \big) \quad \substack{\leq \\ \geq} \quad 0 \qquad \text{in } \mathcal{U},$$

where $0 < \lambda_\alpha = (p-1)\alpha^{p-1}(1-\alpha) \leq c_p$.

Allegretto-Piepenbrink theory

The functional

$$\mathcal{Q}_V(u) := \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V|u|^p \,\mathrm{d}x,$$

is nonnegative on $C_0^\infty(\Omega)$ iff the corresponding Euler-Lagrange equation admits a positive (super)solution.

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$$C_H(\Omega) = \lambda_{p,0}(\Omega) := \sup \Big\{ \lambda \in \mathbb{R} \mid \exists u \in W^{1,p}_{\mathrm{loc}}(\Omega) \text{ s.t.}$$
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Define the *Hardy constant at infinity*

$$\lambda_{p,\infty}(\Omega) := \sup\Big\{\lambda \in \mathbb{R} \mid \exists K \Subset \Omega \text{ and } u \in W^{1,p}_{\mathrm{loc}}(\Omega \setminus \bar{K}) \text{ s.t. } u > 0 \text{ and } \big(-\Delta_p - \frac{\lambda}{\delta^p}\mathcal{I}_p\big)u \underset{\geq}{=} 0 \text{ in } \Omega \setminus \bar{K}\Big\},$$

Hardy constant at infinity

Under a mild smoothness assumption $\lambda_{p,\infty}(\Omega) \le c_p$. Hence, Agmon's trick implies:

Corollary

If Ω is a C^1 bounded domain, then

$$0 < C_H(\Omega) = \lambda_{p,0}(\Omega) \le \lambda_{p,\infty}(\Omega) = c_p.$$

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Question: What happens if $\lambda_{p,0} < \lambda_{p,\infty}(\Omega)$ i.e. if there is a spectral gap.
Answer: The corresponding operator is critical.

Spectral gap

Let Ω is a $C^{1,\gamma}$ bounded domain.
Any positive solution u in a nbd U of $\partial\Omega$ which has minimal growth at infinity in Ω satisfies

$$u \leq G^{\alpha_\lambda} \asymp \delta^{\alpha_\lambda} \quad \text{in a nbd of } \partial\Omega.$$

Note that $\delta^{\alpha_\lambda} \in L^p(U, \delta^{-p})$ iff $\lambda < c_p$.

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Recall that if there is a spectral gap $0 < H(\Omega) < \lambda_{p,\infty}(\Omega) = c_p$, then $\Delta_p v - \frac{H_p(\Omega)}{\delta^p}\mathcal{I}_p$ is critical in Ω i.e. the equation $\left(\Delta_p v - \frac{H_p(\Omega)}{\delta^p}\mathcal{I}_p\right) u = 0$ in Ω admits unique positive (super)solution φ called the Agmon ground state, it has minimal growth at infinity in Ω.

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Hence, $\varphi \leq C\delta^{\alpha_\lambda}$, where $\lambda = C_H(\Omega) < c_p$. Thus, φ is a minimizer.

Comparison principle or Phragmén-Lindelöf principle (Agmon, Marcus-Mizel-YP and Marcus-Shafrir)

If a positive subsolution near $\partial\Omega$ of the Euler-Lagrange equation does not grow too fast (i,e, it satisfies a certain growth condition),then it is bounded (up to a multiplicative constant) by any positive supersolution.

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Hence, any minimizer u satisfies

$$\delta^{\alpha_\lambda} \asymp G^{\alpha_\lambda} \leq Cu.$$

But $\lambda = c_p$ iff $\delta^{\alpha_\lambda} \notin L^p(U, \delta^{-p})$. Hence, if $C_H(\Omega) = c_p$, then there is no minimizer.

Thank you for your attention!