Chaotic orbits for systems of nonlocal equations

Stefania Patrizi

UT Austin

"Mostly Maximum Principle"
B.I.R.S.
April 02-07, 2017

S. Dipierro, S. P. and E. Valdinoci, Chaotic orbits for systems of nonlocal equations, *Commun. Math. Phys.*, **349** (2017), no. 2, 583-626.

We consider a system of nonlocal equations driven by a perturbed periodic potential.

We construct multibump solutions that connect one integer point to another one in a prescribed way.

In particular: heteroclinic, homoclinic and chaotic trajectories are constructed.

Mathematical framework

Given $s \in (\frac{1}{2}, 1)$, we consider an interaction kernel $K : \mathbb{R} \to [0, +\infty]$, satisfying the structural assumptions

- $K(-x) = K(x)$,
- $$\frac{\theta_0 \, (1-s) \, \chi_{[-\rho_0,\rho_0]}(x)}{|x|^{1+2s}} \leq K(x) \leq \frac{\Theta_0 \, (1-s)}{|x|^{1+2s}}$$

  for some $\rho_0 \in (0, 1]$ and $\Theta_0 \geq \theta_0 > 0$, and
- $$|\nabla K(x)| \leq \frac{\Theta_1}{|x|^{2+2s}}$$

  for some $\Theta_1 > 0$.

Mathematical framework

We consider the energy associated to such interaction kernel: namely, for any measurable function $Q : \mathbb{R} \to \mathbb{R}^n$, with $n \in \mathbb{N}$, $n \geq 1$, we define

$$E(Q) := \iint_{\mathbb{R}\times\mathbb{R}} K(x-y)\left|Q(x)-Q(y)\right|^2 dx\, dy.$$

We consider the energy associated to such interaction kernel: namely, for any measurable function $Q : \mathbb{R} \to \mathbb{R}^n$, with $n \in \mathbb{N}$, $n \geq 1$, we define

$$E(Q) := \iint_{\mathbb{R}\times\mathbb{R}} K(x-y)\left|Q(x)-Q(y)\right|^2 dx\, dy.$$

Given an interval $J \subseteq \mathbb{R}$, a measurable function $Q : \mathbb{R} \to \mathbb{R}^n$, with $E(Q) < +\infty$, and $f \in L^1(J, \mathbb{R}^n)$ we say that Q is a solution of

$$\mathcal{L}(Q)(x) + f(x) = 0, \quad x \in J$$

if,

$$2\iint_{\mathbb{R}\times\mathbb{R}} K(x-y)\big(Q(x)-Q(y)\big)\cdot\big(\psi(x)-\psi(y)\big)dxdy + \int_{\mathbb{R}} f(x)\cdot\psi(x)dx = 0,$$

for any $\psi \in C_0^\infty(J, \mathbb{R}^n)$.

Mathematical framework

In the strong version, the operator $\mathcal{L}(Q)$ may be interpreted as the integrodifferential operator

$$4 \int_{\mathbb{R}} K(x-y)\,(Q(x)-Q(y))\,dy,$$

with the singular integral taken in its principal value sense.

Prototype:

$$K(x) := \frac{1-s}{|x|^{1+2s}}, \quad s \in \left(\frac{1}{2}, 1\right)$$

In this case, the operator $\mathcal{L}(Q)$ is (up to multiplicative constants) the fractional Laplacian $(-\Delta)^s Q$.

The setting considered is very general, since it comprises operators which are not necessarily homogeneous or isotropic.

Mathematical framework

The particular equation that we consider is

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}.$$

Mathematical framework

The particular equation that we consider is

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}.$$

We suppose that $W \in C^{1,1}(\mathbb{R}^n)$, that

- W is 1-periodic:

$$W(\tau + \zeta) = W(\tau) \text{ for any } \tau \in \mathbb{R}^n,\ \zeta \in \mathbb{Z}^n,$$

- the minima of W are attained at the integers:

$$W(\zeta) = 0 \text{ for any } \zeta \in \mathbb{Z}^n \text{ and that } W(\tau) > 0 \text{ for any } \tau \in \mathbb{R}^n \setminus \mathbb{Z}^n.$$

- the minima of W are “nondegenerate”: there exist $r \in (0, 1/4]$, $c_0 \in (0, 1)$ and $C_0 \in (1, +\infty)$ such that

$$c_0\,|\tau|^2 \le W(\tau) \le C_0\,|\tau|^2 \qquad \text{for any } \tau \in B_r.$$

The particular equation that we consider is

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}.$$

The particular equation that we consider is

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}.$$

- To make a simple and concrete example, we stick to the case in which

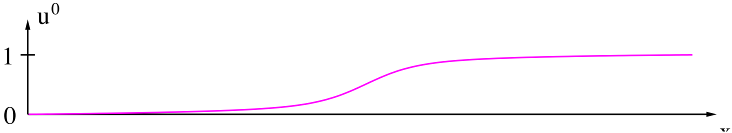
$$a(x) := a_1 + a_2 \cos(\epsilon x),$$

  with $\epsilon > 0$ to be taken suitably small and $a_1 > a_2 > 0$.

- When $n = 1$, $\mathcal{L} = (-\Delta)^s$ and $\epsilon = 0$:

$$(-\Delta)^s(Q) + W'(Q) = 0.$$

Heteroclinic solutions connecting two consecutive integers have been constructed in different papers by Cabré and Solà-Morales ($s = \frac{1}{2}$); Cabré and Sire ($s \in (0, 1)$); Palatucci, Savin and Valdinoci ($s \in (0, 1)$).



Main result

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}. \qquad (1)$$

Main result

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}. \tag{1}$$

Theorem

Let $\zeta_1 \in \mathbb{Z}^n$ and $N \in \mathbb{N}$. There exist $\zeta_2, \ldots, \zeta_N \in \mathbb{Z}^n$ and $b_1, \ldots, b_{2N-2} \in \mathbb{R}$, with $b_{i+1} \geq b_i + 3$ for all $i = 1, \ldots, 2N-3$, and a solution Q_* of (1) such that

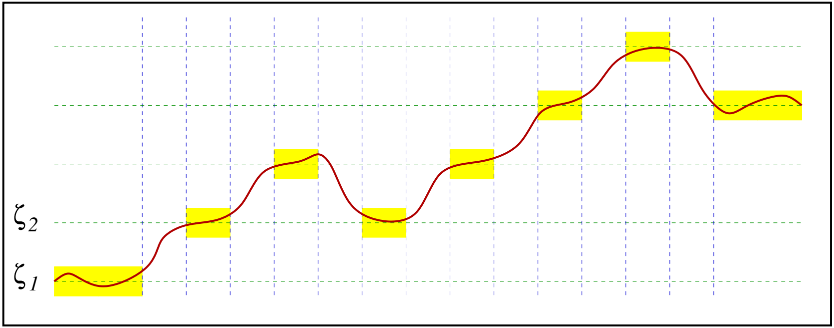
$$\zeta_{i+1} \neq \zeta_i \text{ for any } i \in \{1, \ldots, N-1\},$$

$$\lim_{x \to -\infty} Q_*(x) = \zeta_1, \qquad \sup_{x \in (-\infty, b_1]} |Q_*(x) - \zeta_1| \leq \frac{1}{4},$$

$$\sup_{x \in [b_{2i}, b_{2i+1}]} |Q_*(x) - \zeta_{i+1}| \leq \frac{1}{4} \qquad \text{for all } i = 1, \ldots, N-2,$$

$$\sup_{x \in [b_{2N-2}, +\infty)} |Q_*(x) - \zeta_N| \leq \frac{1}{4} \quad \text{and} \quad \lim_{x \to +\infty} Q_*(x) = \zeta_N.$$

A chaotic trajectory.



Motivations

- For local equations, the local counterpart of our result is a celebrated result in:

  Paul H. Rabinowitz, Periodic and heteroclinic orbits for a periodic Hamiltonian system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 6(5):331–346, 1989.

  Important related results by: Rabinowitz, Coti-Zelati, Séré, Bessi, Maxwell; Bolotin, MacKay, Berti, Bolle....

- Our estimates are uniform for $s \in (s_0, 1)$, for any $s_0 > \frac{1}{2}$, so that we recover the results by Rabinowitz as $s \to 1^-$.

Strategy of the proof

Both local and nonlocal case share the variational idea of looking for constrained minimal orbits and then proving that they are in fact unconstrained.

Constrained minimizer:

- Define the energy functional

$$I(Q) := \iint_{\mathbb{R}\times\mathbb{R}} K(x-y)\big|Q(x)-Q(y)\big|^2 dxdy + \int_{\mathbb{R}} a(x)W(Q(x))dx.$$

- Given $\zeta_1, \zeta_2 \in \mathbb{Z}^n$, $b_1, b_2 \in \mathbb{R}$ with $b_2 > b_1 + 3$, and a small $r > 0$ one minimizes the energy $I(Q)$ in the set

$$\Gamma(\zeta_1, \zeta_2, b_1, b_2) := \Big\{ Q : \mathbb{R} \to \mathbb{R}^n \text{ s.t. } Q \text{ is measurable, } \\ Q(x) \in \overline{B_r(\zeta_1)} \text{ for a.e. } x \in (-\infty, b_1], \\ Q(x) \in \overline{B_r(\zeta_2)} \text{ for a.e. } x \in [b_2, +\infty) \Big\}.$$

$N = 2$: Heteroclinic solutions

- The goal is to construct solutions that emanate from a fixed $\zeta_1 \in \mathbb{Z}^n$ as $x \to -\infty$ and approach a suitable $\zeta_2 \in \mathbb{Z}^n \setminus \{\zeta_1\}$ as $x \to +\infty$.
- More precisely, fixed $\zeta_1 \neq \zeta_2 \in \mathbb{Z}^n$ we consider the minimizer $Q_* = Q_*^{\zeta_1,\zeta_2}$
  Let
  $$I_{\zeta_1} := \inf_{\zeta_2 \in \mathbb{Z}^n \setminus \{\zeta_1\}} I(Q_*^{\zeta_1,\zeta_2}).$$
  and define $\mathcal{A}(\zeta_1)$ the family of all $\zeta_2 \in \mathbb{Z}^n$ attaining such minimum. Then $\mathcal{A}(\zeta_1)$ consists of a finite number of integer.

$N = 2$: Heteroclinic solutions

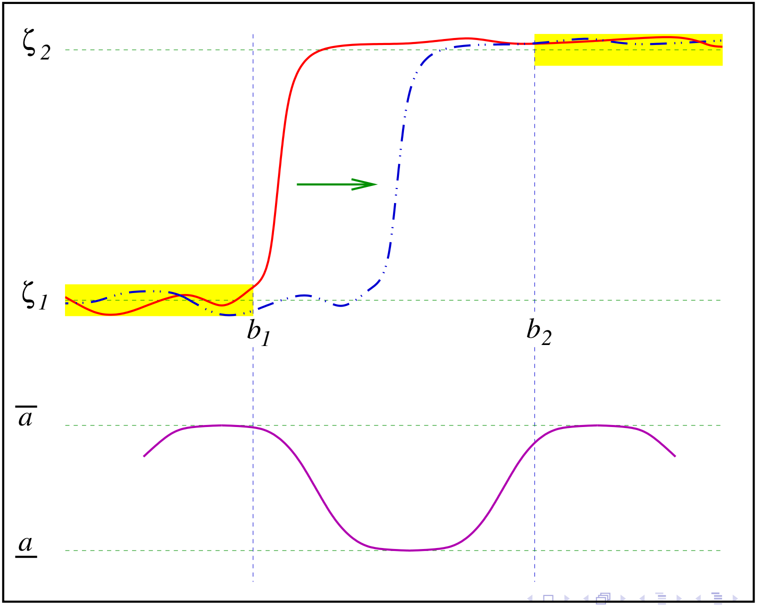
If $\zeta_2 \in \mathcal{A}(\zeta_1)$, then the constrained minimizer that connects ζ_1 and ζ_2 is actually a free minimizer:

Theorem

Let $s_0 \in \left(\frac{1}{2}, 1\right)$ and $s \in [s_0, 1)$. There exist $\epsilon_* > 0$ and $b_2 > b_1 \in \mathbb{R}$, possibly depending on n, s_0 and the structural constants of the kernel and the potential, such that if $\epsilon \in (0, \epsilon_*]$, the following statement holds.
Let $\zeta_1 \in \mathbb{Z}^n$ and $\zeta_2 \in \mathcal{A}(\zeta_1)$.
Then $Q_*^{\zeta_1, \zeta_2}$ is a solution of

$$\mathcal{L}(Q)(x) + a(x)\,\nabla W(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}.$$

$N = 2$: Heteroclinic solutions



Glueing functions

To prove that the constrained minimizer is unconstrained, we have to build competitors, by glueing functions.

Let $L : (T_1, x_0] \to \mathbb{R}^n$ and $R : [x_0, T_2) \to \mathbb{R}^n$. Define

$$V(x) := \begin{cases} L(x), & x \in (T_1, x_0] \\ R(x) & x \in (x_0, T_2) \end{cases}$$

How do we estimate the nonlocal energy of V in (T_1, T_2) in terms of the nonlocal energies of L and R respectively in (T_1, x_0) and (x_0, T_2)?

Glueing functions

- Given an interval $J \subseteq \mathbb{R}$, it is convenient to introduce the notation

$$E_J(Q) := \iint_{J\times J} K(x-y)\,\big|Q(x)-Q(y)\big|^2\,dx\,dy. \quad (2)$$

For instance, we have that $E_{\mathbb{R}} = E$.

Glueing functions

- Given an interval $J \subseteq \mathbb{R}$, it is convenient to introduce the notation

$$E_J(Q) := \iint_{J\times J} K(x-y)\,\big|Q(x)-Q(y)\big|^2\,dx\,dy. \qquad (2)$$

For instance, we have that $E_{\mathbb{R}} = E$.

- With this notation, we are able to glue two functions L and R at a point x_0:

$$V(x) := \begin{cases} L(x), & x \in (T_1, x_0] \\ R(x) & x \in (x_0, T_2) \end{cases}$$

under the additional assumption that

$$[L]_{C^{0,1}([x_0-\beta,x_0])} \le \eta \quad \text{and} \quad [R]_{C^{0,1}([x_0,x_0+\beta])} \le \eta,$$

for some $\eta > 0$, where

$$\beta \in (0, \min\{T_2 - x_0, x_0 - T_1\}]$$

Indeed, in this case, one can prove the following estimate

$$E_{(T_1,T_2)}(V) - E_{(T_1,x_0)}(L) - E_{(x_0,T_2)}(R) \leq C \left(\eta^2 \beta^{3-2s} + \frac{\|L\|_{L^\infty((T_1,x_0),\mathbb{R}^n)} + \|R\|_{L^\infty((x_0,T_2),\mathbb{R}^n)}}{\beta^{2s-1}} \right),$$

for some $C > 0$.

A notion of clean intervals and clean points

Definition

Given $\rho > 0$ and $Q : \mathbb{R} \to \mathbb{R}^n$, we say that an interval $J \subseteq \mathbb{R}$ is a "clean interval" for (ρ, Q) if $|J| \geq |\log \rho|$ and there exists $\zeta \in \mathbb{Z}^n$ such that

$$\sup_{x \in J} |Q(x) - \zeta| \leq \rho.$$

Definition

If J is a bounded clean interval for (ρ, Q), the center of J is called a "clean point" for (ρ, Q).

Glueing functions at clean points

Let Q_* be an optimal trajectory connecting the integers ζ_1 and ζ_2.

Lemma

There exists $\rho_* > 0$, possibly depending on n and the structural assumptions of the kernel and the potential, such that if $\rho \in (0, \rho_*]$ the following statement holds.
Let $\zeta_1 \in \mathbb{Z}^n$ and $\zeta_2 \in \mathcal{A}(\zeta_1)$. Assume that there exists $\zeta \in \mathbb{Z}^n$ and a clean point $x_0 \in (b_1, b_2 - 1)$ such that $Q_*(x_0) \in \overline{B_\rho(\zeta)}$, then

$$\zeta \in \{\zeta_1, \zeta_2\}.$$

Glueing functions at clean points

We glue the optimal trajectory Q_* to a linear interpolation with the integer ζ, close to $Q_*(x_0)$, namely consider

$$V(x) := \begin{cases} Q_*(x) & \text{if } x \le x_0 - 1, \\ Q_*(x_0)\,(x_0 - x) + \zeta\,(x - x_0 + 1) & \text{if } x \in (x_0 - 1, x_0), \\ \zeta & \text{if } x \ge x_0. \end{cases}$$

In this way, and taking $\rho > 0$ suitably small, we know that Q_* is ρ-close to an integer in $[x_0 - 2\beta, x_0 + 2\beta]$, with

$$\beta = \beta(\rho) = \frac{|\log \rho|}{2}.$$

In particular, Q_* is solution of our equation in $[x_0 - 2\beta, x_0 + 2\beta]$. Consequently,

$$[Q_*]_{C^{0,1}([x_0-\beta, x_0+\beta])} \le C\left(\frac{1}{\beta^{2s}} + \rho\right),$$

Glueing functions at clean points

This says that in this case we can take $\eta := C\left(\frac{1}{\beta^{2s}} + \rho\right)$ and get the bound

$$E_{(T_1,T_2)}(V) \leq E_{(T_1,x_0)}(Q_*) + o_\rho(1),$$

where $o_\rho(1) \to 0$ as $\rho \to 0$. If

$$\zeta \notin \{\zeta_1, \zeta_2\},$$

then

$$\int_{-\infty}^{+\infty} a(x)W(V(x))dx < \int_{-\infty}^{+\infty} a(x)W(Q_*(x))dx - c.$$

Therefore, we obtain

$$I(V) < I(Q_*)$$

which contradicts the fact that Q_* is an optimal trajectory.

Chaotic orbits

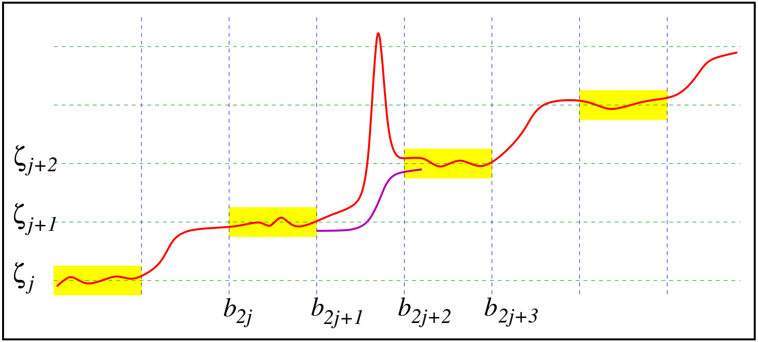


Figure: Glueing Q_* with the heteroclinic joining ζ_{j+1} to ζ_{j+2}.

A chaotic trajectory.