Aleksandrov-Bakelman-Pucci maximum principles for a class of uniformly elliptic and parabolic integro-PDE

Andrzej Święch
(joint work with C. Mou)

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Classical ABP Maximum Principle

• **Aleksandrov-Bakelman-Pucci Maximum Principle:** If Ω be a bounded domain in $\mathbb{R}^n$ and $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a strong subsolution of the Pucci extremal PDE

$$\mathcal{P}^-(D^2u) - \gamma|Du| \le f(x) \qquad \text{in } \Omega,$$

where $\mathcal{P}^-(D^2u)$ is the Pucci extremal operator, $\gamma \ge 0$ and $f \in L^n(\Omega)$, then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + C(\operatorname{diam}(\Omega))\|f^+\|_{L^n(\Gamma^+_\Omega)},$$

where C is a constant depending only on $n, \lambda, \gamma \operatorname{diam}(\Omega)$ and Γ^+_Ω is the so-called upper contact set of u.

Classical Generalized ABP Maximum Principle

• **Generalized Aleksandrov-Bakelman-Pucci Maximum Principle:** (introduced by Fabes-Stroock)

There exists an exponent $p_0 = p_0(n, \Lambda/\lambda)$ such that if $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ for $p_0 < p < n$ is a strong subsolution of

$$\mathcal{P}^-(D^2 u) - \gamma |Du| \leq f(x) \qquad \text{in } \Omega,$$

then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C(\mathrm{diam}(\Omega))^{2-\frac{n}{p}} \|f^+\|_{L^p(\Omega)},$$

for some constant $C = C(n, p, \gamma\, \mathrm{diam}(\Omega), \lambda, \Lambda)$.

Here the L^n norm of f^+ over the upper contact set is replaced by the L^p norm of f^+ over Ω.

Classical ABP Maximum Principle

Versions and Generalizations:

• Semiconvex subsolutions, viscosity subsolutions and L^p-viscosity subsolutions.

• Pointwise versions of ABP maximum principle: for $W^{2,n}$ functions - Bony maximum principle; for semiconvex functions - Jensen's lemma; for L^p-viscosity solutions; pointwise maximum principle for classical viscosity solutions - "maximum principle for semicontinuous functions".

• Degenerate/singular equations.

• Unbounded domains.

Classical Parabolic ABP Maximum Principle

• **Aleksandrov-Bakelman-Pucci-Krylov-Tso Maximum Principle:**
If $Q = (-T, 0] \times \Omega$ and $u \in W^{1,2,n+1}_{\mathrm{loc}}(Q) \cap C(\overline{Q})$ is a strong subsolution of the Pucci extremal PDE

$$u_t + \mathcal{P}^-(D^2 u) - \gamma|Du| \le f(t,x) \qquad \text{in } Q,$$

where $\gamma \ge 0$ and $f \in L^{n+1}(Q)$, then

$$\sup_Q u \le \sup_{\partial_p Q} u + C(\mathrm{diam}(\Omega))^{\frac{n}{n+1}} \|f^+\|_{L^{n+1}(\Gamma_Q^+)},$$

where C is a constant depending only on $n, \lambda, \gamma^{n+1}|Q|/\mathrm{diam}(\Omega)$ and Γ_Q^+ is the so-called parabolic upper contact set of u.

Classical Parabolic ABP Maximum Principle

Versions and Generalizations:

• Generalized ABP-Krylov-Tso Maximum Principle:
$u \in W^{1,2,p}_{\text{loc}}(Q) \cap C(\overline{Q}), f \in L^p(Q)$ for $p_1 < p < n+1$ for some $p_1 = p_1(n, \Lambda/\lambda)$, then estimate holds with L^{n+1} norm of f^+ over the upper contact set replaced by the L^p norm of f^+ over Q.

• More general (not necessarily cylindrical) domains.

• L^p-viscosity solutions.

• Pointwise versions: for $W^{1,2,n+1}$ functions (Tso), for L^p-viscosity solutions, parabolic version of a maximum principle for semicontinuous functions.

Nonlocal Maximum Principles:

• Maximum principle estimates for classical and strong subsolutions of elliptic Integro-PDE giving estimates with $\|f^+\|_{L^n(\Omega)}$ replaced by $\|f^+\|_{L^\infty(\Omega)}$: Garroni-Menaldi, Gimbert-P.L. Lions. It was mentioned in that arguments to prove the classical ABP maximum principle for elliptic PDE can be adapted to obtain such estimate for elliptic integro-PDE but precise result was never stated and no proof was given.

• A nonlocal Bony maximum principle for elliptic integro-PDE: Gimbert-P.L. Lions.

• Versions of maximum principles for semicontinuous functions for integro-differential equations: Barles-Imbert, Jakobsen-Karlsen.

• Estimates of ABP type for purely nonlocal equations of elliptic and parabolic types: Caffarelli-Silvestre, Chang Lara-Dávila, ... More quantitative results and detailed study of the ABP maximum principle for uniformly elliptic nonlocal equations was done by Guillen-Schwab. Purely nonlocal case is very challenging and the area is still largely open.

Nonlocal ABP Maximum Principles

GOAL: Prove versions of ABP maximum principles for elliptic/parabolic integro-PDE where ellipticity comes from the PDE part of the equation.

Elliptic Case: We consider a strong subsolution $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C_b(\mathbb{R}^n)$ of the following extremal integro-PDE

$$\mathcal{P}^-(D^2u) - \gamma|Du| - \sup_{\alpha\in\mathcal{A}} \left\{ \int_{\mathbb{R}^n} \big[u(x+z) - u(x) - \langle Du(x), z\rangle \mathbf{1}_{\{|z|<1\}}(z)\big] N_\alpha(x,z)dz \right\} \le f(x) \quad \text{in } \Omega,$$

where $\gamma \ge 0, f \in L^p(\Omega)$, $\mathcal{A}$ is countable and the functions $N_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty), \alpha \in \mathcal{A}$, are measurable and there exists a measurable function $K : \mathbb{R}^n \to [0,+\infty)$ such that for every $\alpha \in \mathcal{A}, x \in \Omega$, $N_\alpha(x,\cdot) \le K(\cdot)$ and K satisfies

$$\int_{\mathbb{R}^n} \min(|z|^2, 1)K(z)dz < +\infty.$$

Nonlocal ABP Maximum Principles: Elliptic Case

Nonlocal Upper Contact Set: For a function $w : \mathbb{R}^n \to \mathbb{R}$ we define the nonlocal upper contact set

$$\Gamma_{\Omega}^{n,+}(w) := \{x \in \Omega : \ w(x) > \sup_{\Omega^c} w, \ \exists p \text{ such that } \\ w(y) \le w(x) + \langle p, y - x \rangle \text{ for } y \in \Omega_d\},$$

where $\Omega_d := \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, \Omega) < d\}$, $d := \mathrm{diam}(\Omega)$.

Difference compared to the usual upper contact set: Inequality is required to hold on the larger set Ω_d and the whole Ω^c plays the role of the boundary.

Non-local Classical ABP Maximum Principle

Theorem (Non-local Classical ABP)

Let Ω be a bounded domain in $\mathbb{R}^n$, $f \in L^n(\Omega)$ and let R be a number such that $\mathrm{diam}(\Omega) \leq R$. Then there exists a constant $C = C(n, \lambda, \gamma, R, K(\cdot))$ such that if $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C_b(\mathbb{R}^n)$ is a strong subsolution of the extremal integro-PDE then

$$\sup_{\Omega} u \leq \sup_{\Omega^c} u + C \mathrm{diam}(\Omega) \|f^+\|_{L^n(\Gamma^{n,+}_{\Omega}(u))}.$$

The proof is an adaptation of the classical proof (e.g. from Gilbarg-Trudinger), using some technical estimates of the non-local terms.

Non-local Generalized ABP Maximum Principle

Theorem (Non-local Generalized ABP)

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $p_0 < p < n$, where $p_0 = p_0(n, \Lambda/\lambda)$ is the exponent from the Classical Generalized ABP Maximum Principle. Suppose that $f \in L^p(\Omega)$ and R is a number such that $\operatorname{diam}(\Omega) \le R$. Then there exists a constant $C = C(n, p, \lambda, \Lambda, \gamma, R, K(\cdot))$, such that if $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C_b(\mathbb{R}^n)$ is a strong subsolution of the extremal integro-PDE, then

$$\sup_{\Omega} u \le \sup_{\Omega^c} u + C(\operatorname{diam}(\Omega))^{2-\frac{n}{p}} \|f^+\|_{L^p(\Omega)}.$$

Non-local Generalized ABP Maximum Principle

IDEA OF PROOF: Perform infinite sequence of perturbations of the subsolution u by solutions of Pucci extremal equations to obtain a subsolution of similar equation with right hand side g in L^q for $q > n$, with L^q norm of g controlled by the L^p norm of the original f^+. Then use the first non-local ABP maximum principle.

• Scaling and Smoothing: By rescaling $v(x) := u(rx), r = \mathrm{diam}(\Omega)$ and approximation of v by mollification we reduce the problem to a problem where v is smooth, Ω is contained in the unit ball, the new kernels $M_\alpha(x,z) := r^{n+2}N_\alpha(rx,rz)$, the new $K_1(z) := r^{n+2}K(rz)$, and the new right hand side function f (which is an approximation of $f(x) := r^2 f^+(rx)$) is in L^q for every $q < \infty$. We only control the L^p norms of f, the L^q norms of the approximations may blow up. This requires careful estimates for the approximations for the non-local terms. We set

$$q = p^* = np/(n-p).$$

Non-local Generalized ABP Maximum Principle

- Iteration: Take $u_1 \in W^{2,q}_{\mathrm{loc}}(B_2) \cap C(\overline{B}_2)$ to be the solution of

$$\begin{cases} \mathcal{P}^+(D^2 u_1) + \gamma R |Du_1| = -f(x) \quad \text{in } B_2, \\ u_1 = 0 \quad \text{on } \partial B_2, \end{cases}$$

where we extended f by 0 outside of Ω. We have

$$\|u_1\|_{W^{2,p}(B_{\frac{3}{2}})} \le C_1 \|f\|_{L^p(\Omega)},$$

$$\|u_1\|_{W^{2,q}(B_{\frac{3}{2}})} \le C_1 \|f\|_{L^q(\Omega)}$$

$$\sup_{B_2} |u_1| \le C_2 \|f\|_{L^p(\Omega)}.$$

Extend u_1 to $\mathbb{R}^n$ by setting $u_1 = 0$ on B_2^c. Then the function $v_1 = v + u_1 \in W^{2,q}(B_{\frac{3}{2}}) \cap C_b(\mathbb{R}^n)$ satisfies

Non-local Generalized ABP Maximum Principle

$$
\begin{aligned}
&\mathcal{P}^{-}(D^2 v_1(x)) - \gamma R|Dv_1(x)| \\
&- \sup_{\alpha\in\mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[v_1(x+z) - v_1(x) - \langle Dv_1(x), z\rangle \mathbf{1}_{\{|z|<1/r\}}(z) \right] M_\alpha(x,z)dz \right\} \\
&\le \int_{\mathbb{R}^n} \left| u_1(x+z) - u_1(x) - \langle Du_1(x), z\rangle \mathbf{1}_{\{|z|<1/r\}}(z) \right| K_1(z)dz \\
&= \int_{B_\delta} + \int_{B_\delta^c} =: g_1(x) + h_1(x),
\end{aligned}
$$

where $\delta > 0$ depends on constants C_1, C_2 above and other absolute quantities and is chosen so that we can get

$$\|g_1\|_{L^p(\Omega)} \le \frac{1}{2}\|f\|_{L^p(\Omega)}, \quad \|g_1\|_{L^q(\Omega)} \le \frac{1}{2}\|f\|_{L^q(\Omega)}$$

$$\|h_1\|_{L^q(\Omega)} \le C_3\|f\|_{L^p(\Omega)}.$$

Non-local Generalized ABP Maximum Principle

We extend g_1 by 0 outside of Ω and take $u_2 \in W^{2,q}_{\mathrm{loc}}(B_2) \cap C(\overline{B}_2)$ to be the unique solution of

$$\begin{cases} \mathcal{P}^+(D^2 u_2) + \gamma R |Du_2| = -g_1(x) \quad \text{in } B_2, \\ u_2 = 0 \quad \text{on } \partial B_2. \end{cases}$$

We have

$$\sup_{B_2} |u_2| \le C_2 \|g_1\|_{L^p(\Omega)} \le \frac{C_2}{2} \|f\|_{L^p(\Omega)}.$$

We extend u_1 to $\mathbb{R}^n$ by $u_1 = 0$ on B_2^c. Then $v_2 = v_1 + u_2$ is a subsolution of

$$\begin{aligned} &\mathcal{P}^-(D^2 v_2(x)) - \gamma R |Dv_2(x)| \\ &- \sup_{\alpha \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[v_2(x+z) - v_2(x) - \langle Dv_2(x), z \rangle \mathbf{1}_{\{|z|<1/r\}}(z) \right] M_\alpha(x,z) dz \right\} \\ &\le g_2(x) + h_2(x) \end{aligned}$$

in Ω, where

Non-local Generalized ABP Maximum Principle

$$\|g_2\|_{L^p(\Omega)} \leq \frac{1}{2}\|g_1\|_{L^p(\Omega)} \leq \frac{1}{4}\|f\|_{L^p(\Omega)},$$

$$\|g_2\|_{L^q(\Omega)} \leq \frac{1}{2}\|g_1\|_{L^q(\Omega)} \leq \frac{1}{4}\|f\|_{L^q(\Omega)}$$

$$\|h_2\|_{L^q(\Omega)} \leq C_3\left(1+\frac{1}{2}\right)\|f\|_{L^p(\Omega)}, \quad \|h_1 - h_2\|_{L^q(\Omega)} \leq \frac{C_3}{2}\|f\|_{L^p(\Omega)}.$$

We continue the process. This way we construct a sequence $u_m \in W^{2,q}_{\mathrm{loc}}(B_2) \cap C(\overline{B}_2)$ of solutions of

$$\begin{cases} \mathcal{P}^+(D^2 u_m) + \gamma R|Du_m| = -g_{m-1}(x) & \text{in } B_2, \\ u_m = 0 \quad \text{on } \partial B_2, \end{cases}$$

$$\|u_m\|_{W^{2,q}(B_{\frac{3}{2}})} \leq \frac{C_1}{2^{m-1}}\|f\|_{L^q(\Omega)},$$

$$\sup_{B_2}|u_m| \leq \frac{C_2}{2^{m-1}}\|f\|_{L^p(\Omega)}$$

Non-local Generalized ABP Maximum Principle

such that $v_m = v_{m-1} + u_m$ is a subsolution of

$$\mathcal{P}^-(D^2 v_m(x)) - \gamma R|Dv_m(x)| - \sup_{\alpha\in\mathcal{A}} \Big\{ \int_{\mathbb{R}^n} \big[v_m(x+z) - v_m(x)$$

$$- \langle Dv_m(x), z\rangle \mathbf{1}_{\{|z|<1/r\}}(z) \big] M_\alpha(x,z)dz \Big\} \le g_m(x) + h_m(x)$$

in Ω, where

$$\|g_m\|_{L^p(\Omega)} \le \frac{1}{2^m}\|f\|_{L^p(\Omega)}$$

$$\|g_m\|_{L^q(\Omega)} \le \frac{1}{2^m}\|f\|_{L^q(\Omega)}$$

$$\|h_m\|_{L^q(\Omega)} \le C_3 \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-1}}\right)\|f\|_{L^p(\Omega)}$$

$$\|h_m - h_{m-1}\|_{L^q(\Omega)} \le \frac{C_3}{2^{m-1}}\|f\|_{L^p(\Omega)}.$$

Non-local Generalized ABP Maximum Principle

• Passage to the limit: There exists $w \in W^{2,q}(B_{\frac{3}{2}}) \cap C_b(\mathbb{R}^n), w = v$ on B_2^c, such that $v_m \to w$ uniformly in $\mathbb{R}^n$ and $\|v_m - w\|_{W^{2,q}(B_{\frac{3}{2}})} \to 0$ as $m \to +\infty$. We prove that w is a subsolution of

$$\begin{aligned}&\mathcal{P}^-(D^2w(x)) - \gamma R|Dw(x)| \\ &- \sup_{\alpha\in\mathcal{A}}\left\{\int_{\mathbb{R}^n}\left[w(x+z) - w(x) - \langle Dw(x), z\rangle \mathbf{1}_{\{|z|<1/r\}}(z)\right] M_\alpha(x,z)dz\right\} \\ &\quad \le h(x) \qquad \text{in } \Omega.\end{aligned}$$

for some $h \in L^q(\Omega)$ such that

$$\|h\|_{L^q(\Omega)} \le 2C_3\|f\|_{L^p(\Omega)}.$$

Non-local Generalized ABP Maximum Principle

• Conclusion: Applying the first non-local ABP maximum principle we obtain

$$\sup_{\Omega} w \le \sup_{\Omega^c} w + C\|h\|_{L^q(\Omega)} \le \sup_{\Omega^c} w + 2CC_3\|f\|_{L^p(\Omega)}.$$

It remains to use the L^∞ estimates for the functions u_m to conclude

$$\sup_{\Omega} v \le \sup_{\Omega^c} v + 2C_2\|f\|_{L^p(\Omega)} \sum_{m=1}^{\infty} \frac{1}{2^{m-1}} + 2CC_4\|f\|_{L^p(\Omega)}.$$

Non-local ABP Maximum Principle: Parabolic Case

$Q = (-T, 0] \times \Omega$ for some $T > 0$. The non-local parabolic boundary of Q is defined by

$$\partial_{pn} Q := (\{-T\} \times \mathbb{R}^n) \cup ([-T, 0] \times \Omega^c).$$

Parabolic Non-local Upper Contact Set: For a function $v : [-T, 0] \times \mathbb{R}^n \to \mathbb{R}$ we define the non-local parabolic upper contact set

$$\Gamma_Q^{n,+}(v) := \{(t, x) \in Q : v(t, x) > \sup_{\partial_{pn} Q} v,$$
$$\exists p \text{ such that } v(s, y) \leq v(t, x) + \langle p, y - x \rangle \text{ for } (s, y) \in [-T, t] \times \Omega_d\}.$$

Compared to the usual parabolic upper contact set, inequality is required to hold on the larger set $[-T, t] \times \Omega_d$ and the standard parabolic boundary $\partial_p Q$ is replaced by $\partial_{pn} Q$.

Non-local ABP Maximum Principle: Parabolic Case

Theorem (Parabolic Non-local Classical ABP)

Let Ω *be a bounded domain in* $\mathbb{R}^n, T>0$, $f\in L^{n+1}(Q)$ *and let* R *be a number such that* $\mathrm{diam}(Q)\le R$. *Then there exists a constant* $C=C(n,\lambda,\gamma,R,K(\cdot))$ *such that if* $u\in W^{1,2,n+1}_{\mathrm{loc}}(Q)\cap C_b([-T,0]\times\mathbb{R}^n)$ *is a strong subsolution of the extremal integro-PDE*

$$u_t+\mathcal{P}^-(D^2u)-\gamma|Du|-\sup_{\alpha\in\mathcal{A}}\left\{\int_{\mathbb{R}^n}\Big[u(t,x+z)-u(t,x)\right.$$
$$\left.-\langle Du(t,x),z\rangle\mathbf{1}_{\{|z|<1\}}(z)\Big]N_\alpha(t,x,z)dz\right\}\le f(t,x)\quad \textit{in } Q$$

then

$$\sup_Q u\le\sup_{\partial_{pn}Q}u+C\mathrm{diam}(\Omega)^{\frac{n}{n+1}}\|f^+\|_{L^{n+1}(\Gamma_Q^{n,+}(u))}.$$

Non-local ABP Maximum Principle: Parabolic Case

Theorem (Parabolic Non-local Generalized ABP)

Let Ω be a bounded domain in $\mathbb{R}^n, T > 0$ and let $p_1 < p < n+1$, where p_1 is the exponent from the Classical Generalized ABP Maximum Principle. Suppose that $f \in L^p(Q)$. and R is a number such that $\mathrm{diam}(Q) \leq R$. Then there exists a constant $C = C(n, p, T, \lambda, \Lambda, \gamma, R, K(\cdot))$ such that if $u \in W^{1,2,p}_{\mathrm{loc}}(Q) \cap C_b([-T, 0] \times \mathbb{R}^n)$ is a strong subsolution of the extremal integro-PDE then

$$\sup_Q u \leq \sup_{\partial_{pn} Q} u + C(\mathrm{diam}(Q))^{2-\frac{n+2}{p}} \|f^+\|_{L^p(Q)}.$$