Geometric regularity theory for fully nonlinear PDEs

Edgard A. Pimentel

(Jointly with Eduardo Teixeira and Ricardo Castillo)

PUC RIO

Banff Workshop – Mostly Maximum Principle

April, 2017

Outline of the Presentation

1. Introductory remarks

Outline of the Presentation

1. Introductory remarks;
2. An approximation method

Outline of the Presentation

1. Introductory remarks;
2. An approximation method;
3. The elliptic setting: Sobolev regularity

Outline of the Presentation

1. Introductory remarks;
2. An approximation method;
3. The elliptic setting: Sobolev regularity;
4. The parabolic case: Sobolev regularity and a universal modulus of continuity

Outline of the Presentation

1. Introductory remarks;
2. An approximation method;
3. The elliptic setting: Sobolev regularity;
4. The parabolic case: Sobolev regularity and a universal modulus of continuity;
5. Concluding remarks.

Introduction - model problems

Our model problems are:

Problem 1 – Elliptic setting

$$F(D^2u) = f(x) \quad \text{in} \quad B_1;$$

Introduction - model problems

Our model problems are:

Problem 1 – Elliptic setting

$$F(D^2u) = f(x) \quad \text{in} \quad B_1;$$

Problem 2 – Parabolic setting

$$u_t - F(D^2u) = g(x,t) \quad \text{in} \quad B_1 \times (-1,0) =: Q_1.$$

Introduction some assumptions

We work under the following assumptions:

1. The operator F is assumed to be $(\lambda, \Lambda)-$elliptic

Introduction some assumptions

We work under the following assumptions:

1. The operator F is assumed to be $(\lambda, \Lambda)-$elliptic, i.e.,

$$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|,$$

for any M, $N \in \mathcal{S}(d)$, $N \geq 0$

Introduction some assumptions

We work under the following assumptions:

1. The operator F is assumed to be $(\lambda, \Lambda)-$elliptic, i.e.,

$$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|,$$

   for any M, $N \in \mathcal{S}(d)$, $N \geq 0$;

2. The function $f : B_1 \to \mathbb{R}$ is continuous

Introduction some assumptions

We work under the following assumptions:

1. The operator F is assumed to be $(\lambda, \Lambda)-$elliptic, i.e.,

$$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|,$$

for any M, $N \in \mathcal{S}(d)$, $N \geq 0$;

2. The function $f : B_1 \to \mathbb{R}$ is continuous.
3. The function $g : B_1 \times (-1, 0) \to \mathbb{R}$ is continuous

Introduction some assumptions

We work under the following assumptions:

1. The operator F is assumed to be $(\lambda, \Lambda)-$elliptic, i.e.,

   $$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|,$$

   for any M, $N \in \mathcal{S}(d)$, $N \geq 0$;
2. The function $f : B_1 \to \mathbb{R}$ is continuous.
3. The function $g : B_1 \times (-1, 0) \to \mathbb{R}$ is continuous.

Continuity of the source term: in line with the theory of continuous viscosity solutions

Introduction some assumptions

We work under the following assumptions:

1. The operator F is assumed to be $(\lambda, \Lambda)-$elliptic, i.e.,

$$\lambda \|N\| \leq F(M+N) - F(M) \leq \Lambda \|N\|,$$

for any M, $N \in \mathcal{S}(d)$, $N \geq 0$;

2. The function $f : B_1 \to \mathbb{R}$ is continuous.
3. The function $g : B_1 \times (-1, 0) \to \mathbb{R}$ is continuous.

Continuity of the source term: in line with the theory of continuous viscosity solutions; however, our results depend on f or g through their norms in appropriate Lebesgue spaces.

Previous developments - elliptic setting

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form;

Harnack inequality and regularity $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0,1)$, unknown

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form;

Harnack inequality and regularity $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0,1)$, unknown;

Linearization argument: fully nonlinear case

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form;

- Harnack inequality and regularity $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0,1)$, unknown;
- Linearization argument: fully nonlinear case;
- Sharpness: best regularity in the absence of further structures.

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form;

- Harnack inequality and regularity $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0,1)$, unknown;
- Linearization argument: fully nonlinear case;
- Sharpness: best regularity in the absence of further structures.

Evans (82) and Krylov (83, 84): convexity assumptions on F

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form;

- Harnack inequality and regularity $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0,1)$, unknown;
- Linearization argument: fully nonlinear case;
- Sharpness: best regularity in the absence of further structures.

Evans (82) and Krylov (83, 84): convexity assumptions on F;

- Solutions to $F(D^2u) = 0$ are locally a priori $\mathcal{C}^{2,\alpha}$, for some $\alpha \in (0,1)$, unknown

Previous developments - elliptic setting

Krylov and Safonov (79, 80): linear elliptic equations in nondivergence form;

- Harnack inequality and regularity $\mathcal{C}^{1,\alpha}$, for some $\alpha \in (0,1)$, unknown;
- Linearization argument: fully nonlinear case;
- Sharpness: best regularity in the absence of further structures.

Evans (82) and Krylov (83, 84): convexity assumptions on F;

- Solutions to $F(D^2u) = 0$ are locally a priori $\mathcal{C}^{2,\alpha}$, for some $\alpha \in (0,1)$, unknown;
- Theory of classical solutions;

Elliptic setting - $W^{2,p}$-estimates

Caffarelli (89): launches the foundations of the $W^{2,p}$ regularity theory for

$$F(x, D^2u) = f(x) \quad \text{in} \;\; B_1$$

Elliptic setting - $W^{2,p}$-estimates

Caffarelli (89): launches the foundations of the $W^{2,p}$ regularity theory for

$$F(x, D^2u) = f(x) \quad \text{in} \;\; B_1;$$

1. $F(x_0, M)$ is convex (concave) with respect to M, for every x_0

Elliptic setting - $W^{2,p}$-estimates

Caffarelli (89): launches the foundations of the $W^{2,p}$ regularity theory for

$$F(x, D^2u) = f(x) \quad \text{in } B_1;$$

1. $F(x_0, M)$ is convex (concave) with respect to M, for every x_0;
2. For $x_0 \in B_1$ fixed, $F(x_0, M)$ and $F(x, M)$ are *close*

Elliptic setting - $W^{2,p}$-estimates

Caffarelli (89): launches the foundations of the $W^{2,p}$ regularity theory for

$$F(x, D^2u) = f(x) \quad \text{in} \; B_1;$$

1. $F(x_0, M)$ is convex (concave) with respect to M, for every x_0;
2. For $x_0 \in B_1$ fixed, $F(x_0, M)$ and $F(x, M)$ are *close*:

$$\beta_F(x_0, x) := \sup_{M \in B_1^{\mathcal{S}(d)}} \frac{|F(x_0, M) - F(x, M)|}{\|M\|}$$

is such that

$$\|\beta_F(x_0, \cdot)\|_{L^p(B_1)} \ll 1.$$

Elliptic setting - $W^{2,p}$-estimates

Swiech (97): studies regularity of solutions in $W^{1,p}$

Elliptic setting - $W^{2,p}$-estimates

Swiech (97): studies regularity of solutions in $W^{1,p}$;
Builds upon $W^{2,p}$ regularity for

$$F(x, D^2u) = f(x) \quad \text{in} \quad B_1$$

to produce $W^{2,p}$ estimates for the solutions of

$$F(x, u, Du, D^2u) = f(x) \quad \text{in} \quad B_1$$

Elliptic setting - $W^{2,p}$-estimates

Swiech (97): studies regularity of solutions in $W^{1,p}$;

Builds upon $W^{2,p}$ regularity for

$$F(x, D^2u) = f(x) \quad \text{in} \quad B_1$$

to produce $W^{2,p}$ estimates for the solutions of

$$F(x, u, Du, D^2u) = f(x) \quad \text{in} \quad B_1;$$

Uniqueness of solutions

Elliptic setting - $W^{2,p}$-estimates

Swiech (97): studies regularity of solutions in $W^{1,p}$;

Builds upon $W^{2,p}$ regularity for

$$F(x, D^2u) = f(x) \quad \text{in} \quad B_1$$

to produce $W^{2,p}$ estimates for the solutions of

$$F(x, u, Du, D^2u) = f(x) \quad \text{in} \quad B_1;$$

Uniqueness of solutions.

Winter (09): Sobolev regularity up to the boundary.

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity.

Krylov (84): regularity in $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}$

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity.

Krylov (84): regularity in $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}$;

Convexity assumption on the operator

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity.

Krylov (84): regularity in $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}$;

Convexity assumption on the operator.

Tso (90): produces an ABP estimate

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity.

Krylov (84): regularity in $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}$;

Convexity assumption on the operator.

Tso (90): produces an ABP estimate;

Wang (92, 92a, 92b): extends Caffarelli's perspective to the parabolic setting

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity.

Krylov (84): regularity in $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}$;

Convexity assumption on the operator.

Tso (90): produces an ABP estimate;

Wang (92, 92a, 92b): extends Caffarelli's perspective to the parabolic setting;

Harnack inequality, regularity in Hölder spaces

Parabolic case - former advances

Krylov and Safonov (79, 80): regularity in $\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}$

The exponent $\alpha \in (0,1)$ is universal;

Scaling (renormalization) affects the regularity.

Krylov (84): regularity in $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}$;

Convexity assumption on the operator.

Tso (90): produces an ABP estimate;

Wang (92, 92a, 92b): extends Caffarelli's perspective to the parabolic setting;

Harnack inequality, regularity in Hölder spaces;

Assumes $g \in L^p(B_1)$ and proves estimates in $W^{2,1;p}_{loc}(B_1)$.

Regularity of solutions - counterexamples

Nadirashivili & Vlăduț (07, 08, 11): solutions may exhibit unbounded Hessians, failing to be of class $\mathcal{C}^{1,1}$

Regularity of solutions - counterexamples

Nadirashivili & Vlăduț (07, 08, 11): solutions may exhibit unbounded Hessians, failing to be of class $\mathcal{C}^{1,1}$;

Even more surprising : fix $0 < \tau < 1$

Regularity of solutions - counterexamples

Nadirashivili & Vlăduț (07, 08, 11): solutions may exhibit unbounded Hessians, failing to be of class $\mathcal{C}^{1,1}$;

Even more surprising : fix $0 < \tau < 1$;

$$F_\tau, \quad (\lambda, \Lambda) - \text{elliptic}$$

Regularity of solutions - counterexamples

Nadirashivili & Vlăduț (07, 08, 11): solutions may exhibit unbounded Hessians, failing to be of class $\mathcal{C}^{1,1}$;

Even more surprising : fix $0 < \tau < 1$;

$$F_\tau, \quad (\lambda, \Lambda) - \text{elliptic},$$

whose solutions fail to be of class $\mathcal{C}^{1,\tau}$

Regularity of solutions - counterexamples

Nadirashivili & Vlăduț (07, 08, 11): solutions may exhibit unbounded Hessians, failing to be of class $\mathcal{C}^{1,1}$;

Even more surprising : fix $0 < \tau < 1$;

$$F_\tau, \quad (\lambda, \Lambda) - \text{elliptic},$$

whose solutions fail to be of class $\mathcal{C}^{1,\tau}$;

Dong & Kim (14): examples of linear elliptic operators, with piecewise constant coefficients, for which $W^{2,p}$ estimates are not available

Regularity of solutions - counterexamples

Nadirashivili & Vlăduț (07, 08, 11): solutions may exhibit unbounded Hessians, failing to be of class $\mathcal{C}^{1,1}$;

Even more surprising : fix $0 < \tau < 1$;

$$F_\tau, \quad (\lambda, \Lambda) - \text{elliptic},$$

whose solutions fail to be of class $\mathcal{C}^{1,\tau}$;

Dong & Kim (14): examples of linear elliptic operators, with piecewise constant coefficients, for which $W^{2,p}$ estimates are not available;

Caffarelli & Stefanelli (08): parabolic case – solutions may fail to be of class $\mathcal{C}^{2,1}$.

Approximation argument

Approximation argument: the recession function

Aims at relating a given problem to an auxiliary one, through a genuinely geometric structure

Approximation argument: the recession function

Aims at relating a given problem to an auxiliary one, through a genuinely geometric structure;

Caffarelli's argument relating $F(x, D^2u)$ to $F(x_0, D^2u)$

Approximation argument: the recession function

Aims at relating a given problem to an auxiliary one, through a genuinely geometric structure;

Caffarelli's argument relating $F(x, D^2u)$ to $F(x_0, D^2u)$;

At the core: to build a path touching the original problem and connecting it to an auxiliary, model-problem

Approximation argument: the recession function

Aims at relating a given problem to an auxiliary one, through a genuinely geometric structure;

Caffarelli's argument relating $F(x, D^2u)$ to $F(x_0, D^2u)$;

At the core: to build a path touching the original problem and connecting it to an auxiliary, model-problem;

Various manners to design such a path: we focus on the idea of *recession function*.

The recession function - definition

Let

$$F : \mathcal{S}(d) \to \mathbb{R}$$

be a fully nonlinear elliptic operator

The recession function - definition

Let

$$F : \mathcal{S}(d) \to \mathbb{R}$$

be a fully nonlinear elliptic operator;

The *recession function* associated with F, denoted by F^*, is defined by the limit

$$F^*(M) := \lim_{\mu \to 0} F_\mu(M) = \lim_{\mu \to 0} \mu F\left(\mu^{-1} M\right)$$

The recession function - definition

Let

$$F : \mathcal{S}(d) \to \mathbb{R}$$

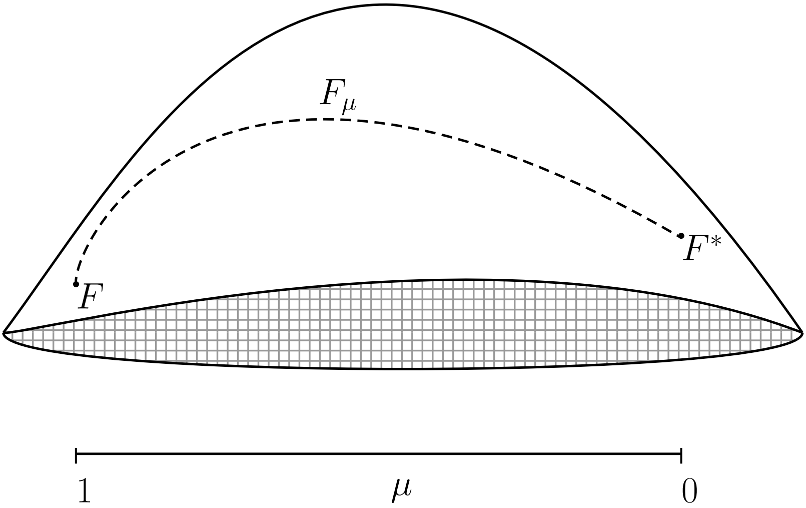
be a fully nonlinear elliptic operator;

The *recession function* associated with F, denoted by F^*, is defined by the limit

$$F^*(M) := \lim_{\mu \to 0} F_\mu(M) = \lim_{\mu \to 0} \mu F\left(\mu^{-1} M\right);$$

From the heuristic viewpoint, F^* encodes the behavior of F at the ends of $\mathcal{S}(d)$.

A graphical representation



Example 1 - Eigenvalue q-momentum operator

Let $q \in 2\mathbb{N}+1$ and consider:

$$F_q(M) = F_q(\lambda_1, ..., \lambda_d) := \sum_{i=1}^{d} \left(1+\lambda_i^q\right)^{1/q} - d$$

Example 1 - Eigenvalue q-momentum operator

Let $q \in 2\mathbb{N}+1$ and consider:

$$F_q(M) = F_q(\lambda_1, ..., \lambda_d) := \sum_{i=1}^{d} \left(1 + \lambda_i^q\right)^{1/q} - d.$$

Easily one computes:

$$F_q^*(M) = \lim_{\mu \to 0} \sum_{i=1}^{d} \left(\mu^q + \lambda_i^q\right)^{1/q} - \mu d = \sum_{i=1}^{d} \lambda_i$$

Example 1 - Eigenvalue q-momentum operator

Let $q \in 2\mathbb{N} + 1$ and consider:

$$F_q(M) = F_q(\lambda_1, ..., \lambda_d) := \sum_{i=1}^{d} \left(1 + \lambda_i^q\right)^{1/q} - d.$$

Easily one computes:

$$F_q^*(M) = \lim_{\mu \to 0} \sum_{i=1}^{d} \left(\mu^q + \lambda_i^q\right)^{1/q} - \mu d = \sum_{i=1}^{d} \lambda_i;$$

Hence: F_q^* is the Laplacian operator.

Example 2 - Perturbation of the special Lagrangian equation

Let $0 < \alpha_1, ..., \alpha_d < +\infty$ and consider:

$$F(M) := \sum_{i=1}^{d} (\alpha_i \lambda_i + \arctan \lambda_i)$$

Example 2 - Perturbation of the special Lagrangian equation

Let $0 < \alpha_1, ..., \alpha_d < +\infty$ and consider:

$$F(M) := \sum_{i=1}^{d} (\alpha_i \lambda_i + \arctan \lambda_i) .$$

As before, one computes:

$$F^*(M) = \sum_{i=1}^{d} (1 + \alpha_i) \lambda_i$$

Example 2 - Perturbation of the special Lagrangian equation

Let $0 < \alpha_1, ..., \alpha_d < +\infty$ and consider:

$$F(M) := \sum_{i=1}^{d} (\alpha_i \lambda_i + \arctan \lambda_i) .$$

As before, one computes:

$$F^*(M) = \sum_{i=1}^{d} (1 + \alpha_i) \lambda_i ;$$

Therefore: F^* is a perturbation of the Laplacian operator.

Main results and a few consequences

Regularity in $W^{2,p}_{loc}(B_1)$ – elliptic setting

Theorem (P. & Teixeira, *J. Math. Pures Appl., 16*)

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F(D^2u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$ and F^* has $\mathcal{C}^{1,1}_{loc}(B_1)$ estimates.

Regularity in $W^{2,p}_{loc}(B_1)$ – elliptic setting

Theorem (P. & Teixeira, *J. Math. Pures Appl., 16*)

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F(D^2 u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$ and F^* has $\mathcal{C}^{1,1}_{loc}(B_1)$ estimates.

Then, $u \in W^{2,p}_{loc}(B_1)$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right),$$

where $C > 0$ is a universal constant.

A few remarks

1. The constraint $p > d$ can be slightly relaxed

A few remarks

1. The constraint $p > d$ can be slightly relaxed;

   There exists $\varepsilon > 0$ so that one may require $p > d - \varepsilon$
   Escauriaza's exponent

A few remarks

1. The constraint $p > d$ can be slightly relaxed;

   There exists $\varepsilon > 0$ so that one may require $p > d - \varepsilon$
   Escauriaza's exponent;

2. Variable coefficients, provided

$$\beta_{F^*}(x_0, x) := \sup_{M \in \mathcal{S}(d)} \frac{|F^*(M, x) - F^*(M, x_0)|}{1 + M}$$

A few remarks

1. The constraint $p > d$ can be slightly relaxed;

   There exists $\varepsilon > 0$ so that one may require $p > d - \varepsilon$
   Escauriaza's exponent;

2. Variable coefficients, provided

$$\beta_{F^*}(x_0, x) := \sup_{M \in \mathcal{S}(d)} \frac{|F^*(M, x) - F^*(M, x_0)|}{1 + M}$$

   satisfies

$$\|\beta_{F^*}(x_0, \cdot)\|_{L^p(B_1)} \ll 1$$

Main ideas behind the proof

Standard results in harmonic analysis and measure theory

Main ideas behind the proof

Standard results in harmonic analysis and measure theory;

Control norms in $W^{2,p}$ by the decay rates of certain sets

Main ideas behind the proof

Standard results in harmonic analysis and measure theory;

Control norms in $W^{2,p}$ by the decay rates of certain sets;

Universal estimates in $W^{2,\delta}_{loc}(B_1)$

Main ideas behind the proof

Standard results in harmonic analysis and measure theory;

Control norms in $W^{2,p}$ by the decay rates of certain sets;

Universal estimates in $W^{2,\delta}_{loc}(B_1)$;

Finer decay rates: regime-switching from δ to p

Main ideas behind the proof

Standard results in harmonic analysis and measure theory;

Control norms in $W^{2,p}$ by the decay rates of certain sets;

Universal estimates in $W^{2,\delta}_{loc}(B_1)$;

Finer decay rates: regime-switching from δ to p;

1. $C^{1,1}$-estimates: competing inequality

Main ideas behind the proof

Standard results in harmonic analysis and measure theory;

Control norms in $W^{2,p}$ by the decay rates of certain sets;

Universal estimates in $W^{2,\delta}_{loc}(B_1)$;

Finer decay rates: regime-switching from δ to p;

1. $C^{1,1}$-estimates: competing inequality;

2. Made rigorous by means of an Approximation Lemma

Approximation Lemma

Proposition

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F_\mu(D^2u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$ and F^* has $\mathcal{C}^{1,1}_{loc}(B_1)$ estimates.

Approximation Lemma

Proposition

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F_\mu(D^2 u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$ and F^* has $\mathcal{C}^{1,1}_{loc}(B_1)$ estimates.
Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if

$$\mu + \|f\|_{L^p(B_1)} \leq \varepsilon$$

Approximation Lemma

Proposition

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F_\mu(D^2 u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$ and F^* has $\mathcal{C}^{1,1}_{loc}(B_1)$ estimates.
Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if

$$\mu + \|f\|_{L^p(B_1)} \leq \varepsilon,$$

there exists $h \in \mathcal{C}^{1,1}_{loc}(B_1)$, solution to

$$F^*(D^2 u) = 0 \quad \text{in} \quad B_{3/4},$$

satisfying

$$\|u - h\|_{L^\infty(B_{3/4})} \leq \delta.$$

Improved regularity

Corollary

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F(D^2u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in p - \mathrm{BMO}(B_1)$, for $p > d$ and F^* is convex.

Improved regularity

Corollary

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F(D^2u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in p - \mathrm{BMO}(B_1)$, for $p > d$ and F^* is convex.

Then, $u \in q - BMO(B_1)$ and there exists a universal constant $C > 0$, so that

$$\|u\|_{q-BMO(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{q-BMO(B_1)}\right),$$

for $q > 1$.

Density result

Corollary

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F(D^2u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$. Given $\delta > 0$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \in W^{2,p}_{loc}(B_1) \cap S(\lambda - \delta, \Lambda + \delta, f)$, converging locally uniformly to u.

Density result

Corollary

Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to

$$F(D^2u) = f(x) \quad \text{in} \quad B_1.$$

Suppose that $f \in L^p(B_1)$, for $p > d$. Given $\delta > 0$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \in W^{2,p}_{loc}(B_1) \cap S(\lambda - \delta, \Lambda + \delta, f)$, converging locally uniformly to u.

Main idea of the proof: the sequence $(u_n)_{n\in\mathbb{N}}$ solve

$$F^n(D^2u_n) = f(x) \quad \text{in} \quad B_1,$$

where

$$F^n(M) := \max\{F(M),\, L_\delta(M) - C_n\}$$

with

$$L_\delta(M) := (\Lambda + \delta)\sum_{e_i>0} e_i + (\lambda - \delta)\sum_{e_i<0} e_i.$$

Sobolev regularity in the parabolic setting

Theorem (Castillo & P.)

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t - F(D^2 u) = g(x,t) \quad \text{in} \quad Q_1.$$

Suppose that $g \in L^p(Q_1)$, for $p > d+1$ and F^* has $\mathcal{C}^{1,0;1}_{loc}(Q_1)$ estimates.

Sobolev regularity in the parabolic setting

Theorem (Castillo & P.)

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t - F(D^2 u) = g(x,t) \quad \text{in} \quad Q_1.$$

Suppose that $g \in L^p(Q_1)$, for $p > d+1$ and F^* has $\mathcal{C}^{1,0;1}_{loc}(Q_1)$ estimates.

Then, u_t and $D^2 u$ are in $L^p_{loc}(Q_1)$ and

$$\|u_t\|_{L^p(Q_{1/2})} + \left\|D^2 u\right\|_{L^p(Q_{1/2})} \leq C\left(\|u\|_{L^\infty(Q_1)} + \|g\|_{L^p(Q_1)}\right),$$

where $C > 0$ is a universal constant.

Regularity in q-BMO spaces: the parabolic setting

Corollary (Castillo & P.)

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t - F(D^2u) = g(x,t) \quad \text{in} \quad Q_1.$$

Suppose that $g \in p - BMO(Q_1)$, for $p > d + 1$ and F^* has $\mathcal{C}^{1,0;1}_{loc}(Q_1)$ estimates.

Regularity in q-BMO spaces: the parabolic setting

Corollary (Castillo & P.)

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t - F(D^2 u) = g(x,t) \quad \text{in} \quad Q_1.$$

Suppose that $g \in p - BMO(Q_1)$, for $p > d+1$ and F^* has $\mathcal{C}^{1,0;1}_{loc}(Q_1)$ estimates.

Then, u_t and $D^2 u$ are in $q - BMO(Q_{1/2})$ and

$$\|u_t\|_{q-BMO(Q_{1/2})} + \left\|D^2 u\right\|_{q-BMO(Q_{1/2})} \leq C \left(\|u\|_{L^\infty(Q_1)} + \|g\|_{q-BMO(Q_1)} \right)$$

for $q > 1$.

Escauriaza's parabolic exponent

Proposition

Let $u \in \mathcal{C}(Q_1)$ be a nonnegative viscosity solution to

$$u_t - F(D^2 u) = g(x,t) \quad \text{in} \quad Q_r,$$

for $r > 0$. Then, there exists $C > 0$ and $\varepsilon_p > 0$ such that

Escauriaza's parabolic exponent

Proposition

Let $u \in \mathcal{C}(Q_1)$ be a nonnegative viscosity solution to

$$u_t - F(D^2 u) = g(x,t) \quad \text{in} \quad Q_r,$$

for $r > 0$. Then, there exists $C > 0$ and $\varepsilon_p > 0$ such that

$$\sup_{Q_{r/2}} u \le C \left[\inf_{Q_{r/2}} u + r^{2-\frac{d+1}{q}} \|f\|_{L^{d+1-\varepsilon_p}(Q_r)} \right].$$

Escauriaza's parabolic exponent

Proposition

Let $u \in \mathcal{C}(Q_1)$ be a nonnegative viscosity solution to

$$u_t - F(D^2u) = g(x,t) \quad \text{in} \quad Q_r,$$

for $r > 0$. Then, there exists $C > 0$ and $\varepsilon_p > 0$ such that

$$\sup_{Q_{r/2}} u \leq C \left[\inf_{Q_{r/2}} u + r^{2-\frac{d+1}{q}} \|f\|_{L^{d+1-\varepsilon_p}(Q_r)} \right].$$

Corollary

Sobolev regularity follows under the condition $p > d + 1 - \varepsilon_p$.

Universal modulus of continuity

Theorem (Castillo & P.)

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t - F(D^2u) = g(x,t) \quad \text{in} \quad Q_1.$$

Then, we have $u \in \mathcal{C}_{loc}^{\alpha^*, \frac{\alpha^*}{2}}(Q_1)$ and the following estimate is satisfied:

$$\|u\|_{\mathcal{C}^{\alpha^*, \frac{\alpha^*}{2}}(Q_{1/2})} \leq C\left[\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{d+1-\varepsilon_p}(Q_1)}\right],$$

where

$$\alpha^* = \alpha^*(d, \varepsilon_p) = \frac{d - 2\varepsilon_p}{d + 1 - \varepsilon_p}.$$

Main ingredients of the proof

Step 1 It suffices to verify the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\sup_{Q_{\rho^k}} |u - \xi_n| \leq \rho^{k\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}}$$

Main ingredients of the proof

Step 1 It suffices to verify the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\sup_{Q_{\rho^k}} |u - \xi_n| \leq \rho^{k\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}};$$

Step 2 The case $k = 1$: approximation lemma

Main ingredients of the proof

Step 1 It suffices to verify the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\sup_{Q_{\rho^k}} |u - \xi_n| \leq \rho^{k\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}};$$

Step 2 The case $k = 1$: approximation lemma;

Step 3 Suppose the case $k = m$ has been verified

Main ingredients of the proof

Step 1 It suffices to verify the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\sup_{Q_{\rho^k}} |u - \xi_n| \leq \rho^{k\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}};$$

Step 2 The case $k = 1$: approximation lemma;

Step 3 Suppose the case $k = m$ has been verified; introduce

$$v_m(x,t) := \frac{u(\rho^{m/2}x, \rho^m t) - \xi_m}{\rho^{m\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}}}$$

Main ingredients of the proof

Step 1 It suffices to verify the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\sup_{Q_{\rho^k}} |u - \xi_n| \leq \rho^{k\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}};$$

Step 2 The case $k = 1$: approximation lemma;

Step 3 Suppose the case $k = m$ has been verified; introduce

$$v_m(x,t) := \frac{u(\rho^{m/2}x, \rho^m t) - \xi_m}{\rho^{m\frac{d-2\varepsilon_P}{d+1-\varepsilon_P}}};$$

Step 4 Study the equation v_m satisfies and conclude the case $k = m + 1$.

Further comments

Concluding remarks

1. Recession strategy produces an approximation strategy

Concluding remarks

1. Recession strategy produces an approximation strategy;
2. Allows us to modify the operator outside of a large ball: density results

Concluding remarks

1. Recession strategy produces an approximation strategy;
2. Allows us to modify the operator outside of a large ball: density results;
3. Recession strategy preserves ellipticity

Concluding remarks

1. Recession strategy produces an approximation strategy;
2. Allows us to modify the operator outside of a large ball: density results;
3. Recession strategy preserves ellipticity;
4. Further insights into ellipticity-invariant objects

Concluding remarks

1. Recession strategy produces an approximation strategy;
2. Allows us to modify the operator outside of a large ball: density results;
3. Recession strategy preserves ellipticity;
4. Further insights into ellipticity-invariant objects;
5. Concrete example

Concluding remarks

1. Recession strategy produces an approximation strategy;
2. Allows us to modify the operator outside of a large ball: density results;
3. Recession strategy preserves ellipticity;
4. Further insights into ellipticity-invariant objects;
5. Concrete example: Escauriaza's exponent.

Thank you very much