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Mostly maximum principle

Free boundary regularity in elliptic two phase problems

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Abstract.
In this talk I will deal with some recent results, obtained with Daniela De Silva and Sandro Salsa, about $C^{1,\gamma}$ regularity and higher regularity of free boundaries of solutions of some non-homogeneous elliptic two phase problems.

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The two phase problem

$$\begin{cases} \Delta u = f_+ & \text{in } B_1^+(u), \\ \Delta u = f_- & \text{in } B_1^-(u), \\ u_\nu^+ = G(u_\nu^-) & \text{on } F(u) := \partial B_1^+(u) \cap B_1. \end{cases} \tag{1}$$

Here B_1 is the unit ball in $\mathbb{R}^n$, centered at the origin, G is an increasing function such that $G(0) > 0, f_\pm \in C(B_1) \cap L^\infty(B_1)$,

$$B_1^+(u) := \{x \in B_1 : u(x) > 0\}, \quad B_1^-(u) := \{x \in B_1 : u(x) \leq 0\}^\circ.$$

u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $B_1^+(u)$ and $B_1^-(u)$ respectively.

Motivations

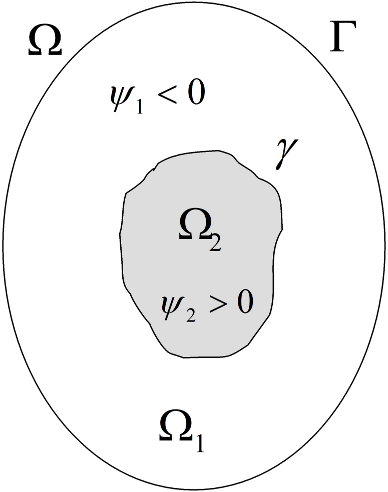
This type of problem arises in a number of applied contexts: the Prandtl-Bachelor model in fluid-dynamics (see e.g. [B1],[EM]), the eigenvalue problem in magnetohydrodynamics ([FL]), or in flame propagation models ([LW]).
B=Batchelor; EM= Elcrat-Miller; FL=Friedman-Liu; LW=Lederman-Wolanski

A bounded 2-dimensional domain is delimited by two simple closed curves γ, Γ.
For given constants $\mu < 0, \omega > 0$, consider functions ψ_1, ψ_2 satisfying

$$\Delta\psi_1 = 0 \text{ in } \Omega_1, \psi_1 = 0 \text{ on } \gamma, \psi_1 = \mu \text{ on } \Gamma,$$

$$\Delta\psi_2 = \omega \text{ in } \Omega_2, \psi_2 = 0 \text{ on } \gamma.$$

and $\Omega_1 := \{\psi_1 > 0\}, \Omega_2 := \{\psi_2 < 0\}$.



Prandtl-Batchelor flow configuration

ψ_1, ψ_2 represent respectively: the stream functions of an irrotational flow in Ω_1 and of a constant vorticity flow in Ω_2.
In the model proposed by Batchelor (coming from the limit of large Reynold number in the steady Navier-Stokes equation).
For the flow of this type is hypothesized that there is a jump in the tangential velocity along γ, namely

$$|\nabla \psi_2|^2 - |\nabla \psi_1|^2 = \sigma$$

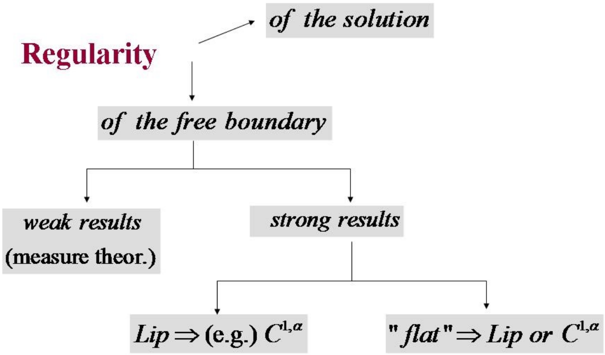
for some positive constant σ.
γ had to be determined = **Free boundary.**

History (recent)

Homogeneous case, i.e. $f_{\pm} = 0$: strong regularity properties of the f.b., Louis Caffarelli, [C1],[C2].
Existence of Lipschitz viscosity solutions, [C3] based on [ACF].
Inhomogeneous case: Lipschitz regularity was obtained by Caffarelli, Jerison and Kenig in [CJK].
Further results on homogeneous free boundary problem see for example: [F1,F2, CFS, FS1,Fe1, W1, W2, MT].
In the case of the non-homogeneous setting: [DFS, DFS2,DFS3, DFS4, DFS5(submitted)]
DFS=Daniela De Silva, F., Sandro Salsa

Results

- Existence of Lipschitz viscosity solutions and weak regularity properties of the free boundary.
- Strong regularity results.
- Higher regularity results for the free boundary.


Regularity
of the solution
of the free boundary
weak results
(measure theor.)
strong results
Lip ⇒ (e.g.) $C^{1,\alpha}$
"flat" ⇒ Lip or $C^{1,\alpha}$

Definitions

$x_0 \in F(u)$ is regular from the right (resp. left) if there is a ball $B \subset B_1^+(u)$ (resp. $B_1^-(u)$), such that $B \cap F(u) = \{x_0\}$.
$\nu = \nu(x_0)$ denotes the unit normal to ∂B at x_0, pointing towards $B_1^+(u)$.

Definition of viscosity solution of the f.b.p.

$u \in C(B_1)$ is a viscosity solution of f.b.p. (1) and for $G(\eta) = \sqrt{1+\eta^2}$ if:

i). $\Delta u = f_+$ in $B_1^+(u)$ and $\Delta u = f_-$ in $B_1^-(u)$;

ii). u satisfies the f. b. condition in the following sense:

1). If $x_0 \in F(u)$ is regular from the right with tangent ball B then

$$u^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B, \text{ with } \alpha \geq 0$$

and

$$u^-(x) \leq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{in } B^c, \text{ with } \beta \geq 0$$

with equality along every nontangential domain, and $\alpha^2 - \beta^2 \leq 1$.

2). If $x_0 \in F(u)$ is regular from the left with tangent ball B, then

$$u^-(x) \geq \beta\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B, \text{ with } \beta \geq 0$$

and

$$u^+(x) \leq \alpha\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{in } B^c, \text{ with } \alpha \geq 0$$

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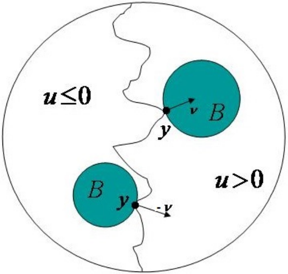


$B \subset \{u > 0\} \Rightarrow \alpha \leq G(\beta, y)$

(*y regular from the right,* subsolution condition)

$B \subset \{u < 0\} \Rightarrow \alpha \geq G(\beta, y)$

(*y regular form the left,* supersolution condition)

Definition of $\mathcal{F}$

A function $w \in \mathcal{F}$ if $w \in C(\overline{B}_1)$ and:
i) w is a solution to

$$\begin{cases} \Delta w \leq f_+ & \text{in } B_1^+(w), \\ \Delta w \leq f_- \chi_{\{w<0\}} & \text{in } B_1^-(w). \end{cases}$$

ii) If $x_0 \in F(u)$ is regular from the left, then, near x_0,

$$w^+ \leq \alpha \langle x - x_0, \nu(x_0) \rangle^+ + o(|x - x_0|), \quad \alpha \geq 0,$$

$$w^- \geq \beta \langle x - x_0, \nu(x_0) \rangle^- + o(|x - x_0|), \quad \beta \geq 0,$$

with

$$\alpha^2 - \beta^2 < 1.$$

iii) If $x_0 \in F(w)$ is not regular from the left, then near x_0,

$$w(x) = o(|x - x_0|).$$

Minorant subsolution

We say that a *locally Lipschitz* function $\underline{u}$, defined in B_1, is a *minorant* if:

a) $\underline{u}$ is a weak solution to

$$\begin{array}{ll} \Delta \underline{u} \geq f_+ & \text{in } B_1^+(\underline{u}) \\ \Delta \underline{u} \geq f_- \chi_{\{\underline{u}<0\}} & \text{in } B^-(\underline{u}). \end{array}$$

b) Every $x_0 \in F(u)$ is regular from the right and near x_0,

$$\underline{u}^- \leq \beta \langle x - x_0, \nu(x_0) \rangle^+ + o(| x - x_0 |),$$

$$\underline{u}^+ \geq \alpha \langle x - x_0, \nu(x_0) \rangle^- + o(| x - x_0 |),$$

with

$$\alpha^2 - \beta^2 > 1.$$

Consider the problem,

$$\begin{cases} \Delta u = f_+ & \text{in } B_1^+(u), \\ \Delta u = f_- \chi_{\{u<0\}} & \text{in } B_1^-(u), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } F(u) := \partial B_1^+. \end{cases} \tag{2}$$

Theorem ([DFS2])

Let ϕ be a continuous function on ∂B_1 and $\underline{u}$ be a minorant of our free boundary problem, with boundary data ϕ. Then

$$u = \inf\{w : w \in \mathcal{F},\ w \geq \underline{u} \text{ in } \overline{B_1}\}$$

is a locally Lipschitz viscosity solution to (2) such that $u = \phi$ on ∂B_1, as long as the set on the right is non-empty. The free boundary $F(u)$ has finite $(n-1)$-dimensional Hausdorff measure and there exist universal positive constants c, C, r_0 such that for every $r < r_0$ and every $x_0 \in F(u)$,

$$cr^{n-1} \leq \mathcal{H}^{n-1}(F(u) \cap B_r(x_0)) \leq Cr^{n-1}.$$

Moreover, if $F^*(u)$ denotes the reduced part of $F(u)$,

$$\mathcal{H}^{n-1}(F(u) \setminus F^*(u)) = 0.$$

Theorem (Flatness → $C^{1,\gamma}$, [DFS])

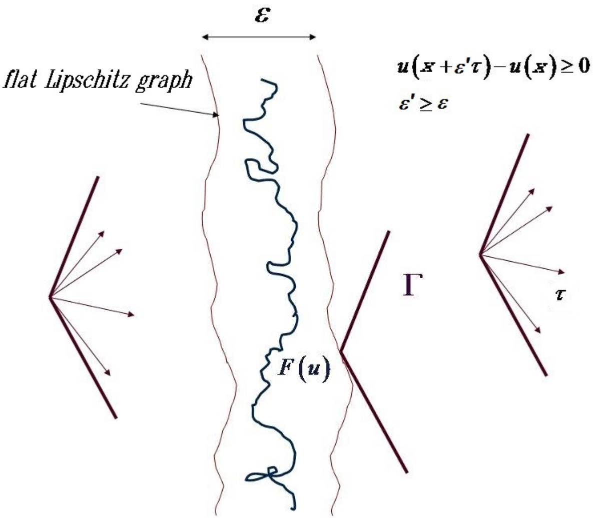
Let u be a solution of our n.h.f.b. problem. There exists a universal constant $\bar{\delta} > 0$ such that, if $0 \leq \delta \leq \bar{\delta}$ and

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \qquad (\delta - \textit{flatness}) \quad (3)$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$, with γ universal.

Theorem

Let u be a solution of our n.h.f.b. problem. If $F(u)$ is a Lipschitz graph in B_1, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$, with γ universal.


ε
flat Lipschitz graph
u(x+ε'τ)−u(x) ≥ 0
ε' ≥ ε
Γ
τ
F(u)

Then the basic step in the improvement of flatness.
Let

$$U_{\beta}(t) = \alpha t^{+} - \beta t^{-}, \ \ \beta \geq 0, \ \ \alpha = G(\beta) \equiv \sqrt{1+\beta^2}$$

and ν is a unit vector which plays the role of the normal vector at the origin. $U_{\beta}(x \cdot \nu)$ is a so-called *two plane solution*.

The strategy of flatness improvement works nicely in the one phase case ($\beta = 0$) or as long as the two phases u^+, u^- are, say, comparable (*nondegenerate case*).
The difficulties arise when the negative phase becomes very small but at the same time not negligible (*degenerate case.*) In this case the flatness assumption gives a control of the positive phase only, through the closeness to a *one plane solution* $U_0(x_n) = x_n^+$.
For simplicity we describe the nondegenerate situation.

Lemma (Main[DFS])

Let u satisfy (1) and

$$U_\beta(x_n-\varepsilon)\le u(x)\le U_\beta(x_n+\varepsilon),\quad in \quad B_1,\quad 0\in F(u),$$

with $0<\beta\le L$ *and*

$$||f||_{L^\infty(B_1)}\le\varepsilon^2\beta.$$

If $0<r\le r_0$ *for* r_0 *universal, and* $0<\varepsilon\le\varepsilon_0$ *for some* ε_0 *depending on* r*, then*

$$U_{\beta'}(x\cdot\nu_1-r\frac{\varepsilon}{2})\le u(x)\le U_{\beta'}(x\cdot\nu_1+r\frac{\varepsilon}{2})\quad in\ B_r, \tag{4}$$

with $|\nu_1|=1$, $|\nu_1-e_n|\le\tilde{C}\varepsilon$*, and* $|\beta-\beta'|\le\tilde{C}\beta\varepsilon$ *for a universal constant* $\tilde{C}$.

Consequence

Assume the lemma above holds. To prove the Theorem "Flatness $\rightarrow C^{1,\gamma}$" in hypotheses of flatness conditions.
We rescale considering a blow up sequence

$$u_k(x) = \frac{u(\rho_k x)}{\rho_k} \quad \rho_k = \bar{r}^k, \quad x \in B_1 \tag{5}$$

for suitable $\bar{r} \leq \min\left\{r_0, \frac{1}{16}\right\}, \tilde{\varepsilon} \leq \varepsilon_0(\bar{r})$, as required

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We iterate to get, at the kth step,

$$U_{\beta_k}(x\cdot\nu_k-\rho_k\varepsilon_k)\leq u_k(x)\leq U_{\beta_k}(x\cdot\nu_k+\rho_k\varepsilon_k)\quad\text{in }\ B_{\rho_k},$$

with $\varepsilon_k=2^{-k}\tilde{\varepsilon}$, $|\nu_k|=1$, $|\nu_k-\nu_{k-1}|\leq\tilde{C}\varepsilon_{k-1}$,

$$|\beta_k-\beta_{k-1}|\leq\tilde{C}\beta_{k-1}\varepsilon_{k-1},\ \ \varepsilon_k\leq\beta_k\leq L.$$

Note that **in the non-degenerate case**, $\beta \geq \tilde{\varepsilon}$, at each step we have the correct inductive hypotheses.
Starting with $\beta = \beta_0 \geq \varepsilon_0 = \tilde{\varepsilon}$, if $k \geq 1$ and $\beta_{k-1} \geq \varepsilon_{k-1}$, then

$$\begin{aligned} \beta_k &\geq \beta_{k-1}(1 - \tilde{C}\varepsilon_{k-1}) \geq 2^{-k+1}\tilde{\varepsilon}\left(1 - \tilde{C}2^{-k+1}\tilde{\varepsilon}\right) \\ &\geq 2^{-k}\tilde{\varepsilon} = \varepsilon_k. \end{aligned}$$

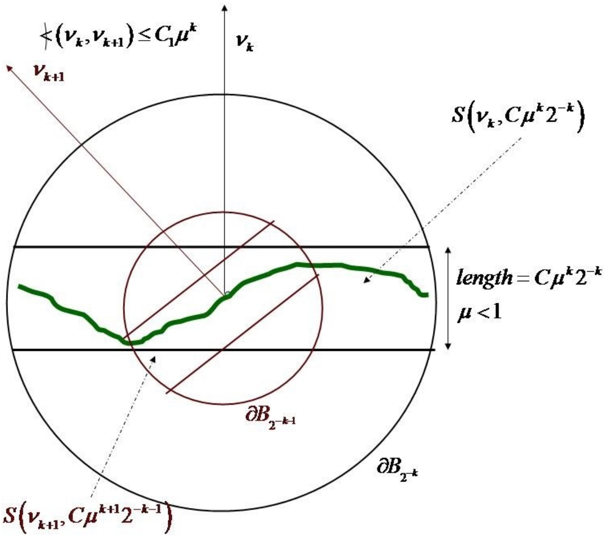
Thus, since

$$f_k(x) = \rho_k f(\rho_k x), \quad x \in B_1$$

(recall that $\bar{\eta} = \tilde{\varepsilon}^3$)

$$\|f_k\|_{L^\infty(B_1)} \leq \rho_k \tilde{\varepsilon}^3 \leq \tilde{\varepsilon}_k^2 \beta_k = \tilde{\varepsilon}_k^2 \min\{\alpha_k, \beta_k\}.$$

The figure below describes the step from k to $k+1$.


$\sphericalangle(\nu_k,\nu_{k+1})\le C_1\mu^k$
ν_k
ν_{k+1}
$S(\nu_k, C\mu^k 2^{-k})$
$length = C\mu^k 2^{-k}$
$\mu<1$
$\partial B_{2^{-k-1}}$
$\partial B_{2^{-k}}$
$S(\nu_{k+1}, C\mu^{k+1}2^{-k-1})$

This implies that $F(u)$ is $C^{1,\gamma}$ at the origin. Repeating the procedure for points in a neighborhood of $x = 0$, (all estimates are universal), we conclude that there exists a unit vector $\nu_\infty = \lim \nu_k$ and $C > 0$, $\gamma \in (0,1]$, both universal, such that, in the coordinate system $e_1, \ldots, e_{n-1}, \nu_\infty, \nu_\infty \perp e_j, e_j \cdot e_k = \delta_{jk}$, $F(u)$ is $C^{1,\gamma}$ graph, say $x_n = g(x')$, with $g(0') = 0$ and

$$\left| g(x') - \nu_\infty \cdot x' \right| \leq C \left| x' \right|^{1+\gamma}$$

in a neighborhood of $x = 0$.

Proof of Lemma (Main)[DFS]

We argue by contradiction.

Step 1. Fix $r \leq r_0$, to be chosen suitably. Assume that for a sequence $\varepsilon_k \to 0$ there is a sequence u_k of solutions of our free boundary problem in B_1, with right hand side f_k such that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2 \min\{\alpha_k, \beta_k\}$, and

$$U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{in } B_1, \;\; 0 \in F(u_k), \tag{6}$$

with $0 \leq \beta_k \leq L$, $\alpha_k = \sqrt{1 + \beta_k^2}$, but the conclusion of Lemma (Main) does not hold for every $k \geq 1$.

Construct the corresponding sequence of renormalized functions

$$\tilde{u}_k(x) = \begin{cases} \frac{u_k(x)-\alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\ \frac{u_k(x)-\beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

Up to a subsequence $\beta_k \to \tilde{\beta}$ so that $\alpha_k \to \tilde{\alpha} = \sqrt{1+\tilde{\beta}^2}$. At this point we need compactness to show that the graphs of $\tilde{u}_k$ converge in the Hausdorff distance to a Hölder continuous $\tilde{u}$ in $B_{1/2}$. The compactness is provided by the Harnack inequality stated in the following Theorem (Harnack)

Theorem (Harnack type, [DFS])

Let u be a solution of our f.b.p. in B_1 with Lipschitz constant L. There exists a universal $\tilde{\varepsilon} > 0$ such that, if $x_0 \in B_1$ and u satisfies the following condition:

$$U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_1 \tag{7}$$

with $\|f\|_{L^\infty(B_2)} \leq \varepsilon^2 \min\{\alpha, \beta\}, \quad 0 < \beta \leq L$, and $0 < b_0 - a_0 \leq \varepsilon r$ for some $0 < \varepsilon \leq \tilde{\varepsilon}$, then

$$U_\beta(x_n + a_1) \leq u(x) \leq U_\beta(x_n + b_1) \quad \text{in } B_{r/20}(x_0)$$

with $a_0 \leq a_1 \leq b_1 \leq b_0$ and $b_1 - a_1 \leq (1 - c)\varepsilon r$ and $0 < c < 1$ universal.

Corollary (Harnack type, [DFS])

Let u satisfies at some point $x_0 \in B_2$

$$U_\beta(x_n + a_0) \le u(x) \le U_\beta(x_n + b_0) \quad in \;\; B_1(x_0) \subset B_2, \tag{8}$$

for some $0 < \beta \le L$*, with* $b_0 - a_0 \le \varepsilon$*, and let*
$||f||_{L^\infty(B_2)} \le \varepsilon^2 \min\{\alpha, \beta\}, \quad 0 < \beta \le L$ *holds, for* $\varepsilon \le \bar{\varepsilon}$*,* $\bar{\varepsilon}$ *universal.*
Let us define in $B_1(x_0)$*,*

$$\tilde{u}_\varepsilon(x) = \begin{cases} \frac{u(x) - \alpha x_n}{\alpha\varepsilon}, & in \;\; B_2^+(u) \cup F(u) \\ \frac{u(x) - \beta x_n}{\beta\varepsilon}, & in \;\; B_2^-(u) \end{cases}$$

Then for all $x \in B_1(x_0)$*, with* $|x - x_0| \ge \varepsilon/\bar{\varepsilon}$

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \le C|x - x_0|^\gamma.$$

Indeed, if u satisfies (7) with, say $r = 1$, then we can apply Harnack inequality repeatedly and obtain

$$U_\beta(x_n + a_m) \leq u(x) \leq U_\beta(x_n + b_m) \quad \text{in } \ B_{20^{-m}}(x_0),$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon$$

for all m's such that

$$(1-c)^m 20^m \varepsilon \leq \bar{\varepsilon}.$$

This implies that for all such m's, the oscillation of the renormalized functions $\tilde{u}_k$ in $B_r(x_0)$, $r = 20^{-m}$, is less than $(1-c)^m = 20^{-\gamma m} = r^\gamma$.
Since in the proof of Lemma (Harnack type),

$$-1 \leq \tilde{u}_k(x) \leq 1, \quad \text{for } x \in B_1,$$

we can implement previous corollary and use Ascoli-Arzela theorem to obtain that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2}$.
Thus the improvement of flatness process in the nondegenerate case can be concluded.

Step 2: Transmission problem.
$\tilde{u}$ solves the "linearized problem" ($\tilde{\alpha} \neq 0$)

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_1 \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{9}$$

Moreover according with the following result

Theorem (Regularity of the transmission problem)

Let $\tilde{u}$ be a viscosity solution to (9) in B_1 such that $\|\tilde{u}\|_\infty \leq 1$. Then $\tilde{u} \in C^\infty\left(\bar{B}_1^\pm\right)$ and in particular, there exists a universal constant $\bar{C}$ such that

$$|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'}\tilde{u}(0) \cdot x' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq \bar{C}r^2, \quad in \;\; B_r \qquad (10)$$

for all $r \leq 1/2$ and with $\tilde{\alpha}^2\tilde{p} - \tilde{\beta}^2\tilde{q} = 0$.

Step 3 (Contradiction). We can prove the last step.
We can show that (for k large and $r \le r_0$)

$$\widetilde{U}_{\beta'_k}(x\cdot\nu_k - \varepsilon_k \frac{r}{2}) \le \tilde{u}_k(x) \le \widetilde{U}_{\beta'_k}(x\cdot\nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

where again we are using the notation:

$$\widetilde{U}_{\beta'_k}(x) = \begin{cases} \frac{U_{\beta'_k}(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(U_{\beta'_k}) \cup F(U_{\beta'_k}) \\ \frac{U_{\beta'_k}(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(U_{\beta'_k}). \end{cases}$$

This will clearly imply that

$$U_{\beta'_k}(x\cdot\nu_k-\varepsilon_k\frac{r}{2})\leq u_k(x)\leq U_{\beta'_k}(x\cdot\nu_k+\varepsilon_k\frac{r}{2}),\quad \text{in } B_r$$

leading to a contradiction with the assumption that the thesis of the Lemma (Main) is false.

Indeed, recalling the Theorem (Regularity of the transmission problem), it is sufficient to show that in B_r :

$$\widetilde{U}_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2$$

and

$$\widetilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) \geq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2.$$

This can be shown after some elementary calculations as long as $r \leq r_0$, r_0 universal, and $\varepsilon \leq \varepsilon_0\,(r)$.

Theorem ([DFS5 submitted])

Let u be a (Lipschitz) viscosity solution to (1) in B_1. There exists a universal constant $\bar{\eta} > 0$ such that, if

$$\{x_n \leq -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\}, \quad \textit{for } 0 \leq \eta \leq \bar{\eta}, \tag{11}$$

then $F(u)$ is C^{2,γ^*} in $B_{1/2}$ for a small γ^* universal, with the C^{2,γ^*} norm bounded by a universal constant.

Theorem ([DFS5 submitted])

Let k be a nonnegative integer. Assume that $f_\pm \in C^{k,\gamma}(B_1)$. Then $F(u) \cap B_{1/2}$ is C^{k+2,γ^*}. If $f_\pm$ are C^∞ or real analytic in B_1, then $F(u) \cap B_{1/2}$ is C^∞ or real analytic, respectively.

We exploit an idea contained in a paper by Kinderlehrer, Nirenberg, Spruck ([KNS]).
For σ small, the partial hodograph map

$$y' = x', \quad y_n = u^+(x)$$

is $1-1$ from $\overline{B_1^+}(u) \cap B_\sigma(0)$ onto a neighborhood of the origin $U \subset \{y_n \geq 0\}$, and flattens $F(u)$ into a set $\Sigma \subset \{y_n = 0\}$.

The inverse mapping is the partial Legendre transformation

$$x' = y', \quad x_n = \psi(y),$$

where ψ satisfies $y_n = u^+\left(y', \psi\left(y\right)\right), y \in U$. The free boundary is the graph of $x_n = \psi\left(y', 0\right)$.

Differentiating we get

$$dy_n = \left(\nabla' u^+ + \partial_{x_n} u^+ \nabla' \psi\right) \cdot dy' + \partial_{x_n} u^+ \partial_{y_n} \psi dy_n$$

from which

$$\partial_{x_n} u^+ \left(y, \psi\left(y\right)\right) = \frac{1}{\partial_{y_n} \psi\left(y\right)}, \quad \nabla' u^+ \left(y, \psi\left(y\right)\right) = -\frac{\nabla' \psi\left(y\right)}{\partial_{y_n} \psi\left(y\right)}$$

in U.

Moreover $\Delta u^+ = f_+$ transforms into

$$\mathcal{F}_1(\psi) :\equiv -\frac{\partial_{y_n y_n}\psi}{(\partial_{y_n}\psi)^3} + \sum_{j=1}^{n-1}\left(-\partial_{y_j}\frac{\partial_{y_j}\psi}{\partial_{y_n}\psi} + \frac{\partial_{y_j}\psi}{\partial_{y_n}\psi}\partial_{y_n}\frac{\partial_{y_j}\psi}{\partial_{y_n}\psi}\right) = f_+(y', \psi(y))$$

in U.

Concerning the negative part, let C be a constant larger than

$$\partial_{y_n}\psi = \frac{1}{\partial_{x_n}u^{+}\left(y',\psi\left(y\right)\right)}$$

on Σ.
Introduce the reflection map

$$x' = y', \quad x_n = \psi(y) - Cy_n,$$

which is $1-1$ from a neighborhood of the origin $U_1 \subseteq U$ onto $\overline{B_1^-}\left(u\right) \cap B_{\sigma}\left(0\right)$ (choosing σ smaller, if necessary).

Define in U_1

$$\phi(y) = u^-(y', \psi(y) - Cy_n).$$

Differentiating we get

$$\nabla'\phi \cdot dy' + \partial_{y_n}\phi dy_n = (\nabla' u^- + \partial_{x_n} u^- \nabla'\psi) \cdot dy' + \partial_{x_n} u^- (\partial_{y_n}\psi - C) dy_n$$

from which

$$\partial_{x_n} u^- = \frac{\partial_{y_n}\phi}{\partial_{y_n}\psi - C}, \quad \nabla' u^- = \nabla'\phi - \frac{\partial_{y_n}\phi}{\partial_{y_n}\psi - C}\nabla'\psi.$$

The equation $\Delta u^- = f_-$ in $\overline{B_1^-}(u) \cap B_\sigma(0)$ transforms into the equation

$$
\begin{aligned}
\mathcal{F}_2(\phi,\psi) \equiv \frac{1}{\partial_{y_n}\psi - C}\partial_{y_n}\left(\frac{\partial_{y_n}\phi}{\partial_{y_n}\psi - C}\right) + \sum_{j=1}^{n-1}\partial_{y_j}\left(\partial_{y_j}\phi - \frac{\partial_{y_n}\phi}{\partial_{y_n}\psi - C}\partial_{y_j}\psi\right) \\
- \sum_{j=1}^{n-1}\frac{\partial_{y_j}\psi}{\partial_{y_n}\psi - C}\partial_{y_n}\left(\partial_{y_j}\phi - \frac{\partial_{y_n}\phi}{\partial_{y_n}\psi - C}\partial_{y_j}\psi\right) \\
= f_-(y', \psi(y) - Cy_n)
\end{aligned}
$$

in U_1.

Thus, in U_1 we have the following nonlinear system

$$\begin{cases} \mathcal{F}_1(\psi) = f_+(y', \psi(y)) \\ \mathcal{F}_2(\phi, \psi) = f_-(y', \psi(y) - Cy_n). \end{cases} \tag{12}$$

The free boundary conditions

$$u^+ = u^- \quad \text{and} \quad |\nabla u^+|^2 - |\nabla u^-|^2 = 1, \qquad \text{on } F(u)$$

become

$$
\begin{cases}
\phi(y',0) = 0 \\[2ex]
\dfrac{1+|\nabla'\psi(y',0)|^2}{(\partial_{y_n}\psi(y',0))^2} - \dfrac{(\partial_{y_n}\phi(y',0))^2}{(\partial_{y_n}\psi(y',0)-C)^2} \\
\qquad\qquad -||\nabla'\phi(y',0) - \frac{\partial_n\phi(y',0)}{\partial_{y_n}-C}\nabla'\psi(y',0)||^2_{\mathbb{R}^{n-1}} = 1.
\end{cases}
$$

That is, after a simple computation,

$$\begin{cases} \phi(y',0) = 0 \\ \\ \left(1 + |\nabla'\psi(y',0)|^2\right)\left(\dfrac{1}{(\partial_{y_n}\psi(y',0))^2} - \dfrac{(\partial_{y_n}\phi(y',0))^2}{(\partial_{y_n}\psi(y',0) - C)^2}\right) = 1. \end{cases}$$

Linearization at $y = 0$ gives (setting $A = C - \partial_{y_n}\psi(0)$,)

$$\begin{cases} \mathcal{L}_1(\psi) = |\nabla u^+(0)|^2 \partial_{y_n y_n}\psi + \sum_{k=1}^{n-1} \partial_{y_k y_k}\psi = 0, \\ \mathcal{L}_2(\psi,\phi) = \dfrac{1}{A^2}\partial_{y_n y_n}\phi + \sum_{k=1}^{n-1} \partial_{y_k y_k}\phi \\ \qquad -|\nabla u^-(0)| \left(\dfrac{1}{A^2}\partial_{y_n y_n}\psi + \sum_{k=1}^{n-1} \partial_{y_k y_k}\psi \right) = 0, \\ \mathcal{B}_1(\phi) = \phi = 0 \\ \mathcal{B}_2(\psi,\phi) = \left(|\nabla u^+(0)|^3 + \dfrac{1}{A}|\nabla u^-(0)|^2 \right) \partial_{y_n}\psi - \dfrac{1}{A}|\nabla u^-(0)|\partial_{y_n}\phi = 0. \end{cases}$$

This system is elliptic with coercive boundary conditions under the natural choices of weights $s_1 = s_2 = 0$ and $t_1 = t_2 = 2$ for $\mathcal{L}_1$ and $\mathcal{L}_2$ and $r_1 = -2$, $r_2 = -1$ for $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Indeed

$$\text{order}\mathcal{L}_j = s_j + t_j = 2 \qquad (j = 1, 2)$$

and

$$\text{order}\mathcal{B}_1 = t_1 + r_1 = 0, \text{order}\mathcal{B}_2 = t_2 + r_2 = 1.$$

The theorem follows from the results of [ADN] see [M].

Given $\omega \in \mathbb{R}^n$, with $|\omega| = 1$, and
let S_ω be an orthonormal basis containing ω.
Let $M \in S^{n\times n}$ satisfy

$$M\omega = 0$$

and define

$$P_{M,\omega}(x) = x \cdot \omega - \frac{1}{2}x^T Mx.$$

Let $\alpha > 0$, $\beta \geq 0$, $a, b \in \mathbb{R}^n$. We define

$$V^{\alpha,\beta}_{M,\omega,a,b}(x) = \alpha(1 + a \cdot x)P^{+}_{M,\omega}(x) - \beta(1 + b \cdot x)P^{-}_{M,\omega}(x).$$

These are our two-phase polynomials,
one-phase if $\beta = 0$.
In the particular case when $M = 0, a = b = 0, \omega = e_n$ we obtain the two-plane function:

$$U_\beta(x) = \alpha x_n^+ - \beta x_n^-.$$

The unit vector ω establishes the “direction of flatness”.

We shall need to work with a subclass, strictly related to problem (1), at least at the origin. We denote by $\mathcal{V}_{f_\pm}$ the class of functions of the form $V^{\alpha,\beta}_{M,\omega,a,b}$ for which

$$2\alpha a\cdot\omega - \alpha trM = f_+(0)$$

$$2\beta b\cdot\omega - \beta trM = f_-(0) \quad \text{if } \beta\neq 0,$$

$$\alpha^2-\beta^2 = 1, \quad \text{if } \beta\neq 0,$$

and

$$\alpha^2 a\cdot\omega^\perp = \beta^2 b\cdot\omega^\perp, \quad \forall\omega^\perp\in S_\omega.$$

The role of the last condition is to make $V^{\alpha,\beta}_{M,\omega,a,b}$ an “almost" viscosity subsolution.
When $\beta = 0$, then there is no dependence on b and $a \cdot \omega^{\perp} = 0$. Thus, we drop the dependence on β, b and f_{-} in our notation above and we indicate the dependence on $a_{\omega} := a \cdot \omega$.

We introduce the following definitions.

Definition ([DFS5])

Let $V = V^{\alpha,\beta}_{M,\omega,a,b}$. We say that u is (V,ϵ,δ) flat in B_1 if

$$V(x-\epsilon\omega) \leq u(x) \leq V(x+\epsilon\omega) \quad \text{in } B_1$$

and

$$|a|, |b'|, \|M\| \leq \delta\epsilon^{1/2}, \quad |b_n| \leq \delta^2, \quad |b_n|\|M\| \leq \delta^2\epsilon.$$

Definition ([DFS5])

Let $V = V_{M,\omega,a,b}^{\alpha,\beta}$. We say that u is (V, ϵ, δ) flat in B_r if the rescaling

$$u_r(x) := \frac{u(rx)}{r}$$

is $(V_r, \frac{\epsilon}{r}, \delta)$ flat in B_1.

Notice that if u is (V, ϵ, δ) flat in B_r then

$$V(x - \epsilon\omega) \leq u(x) \leq V(x + \epsilon\omega) \quad \text{in } B_r.$$

The parameter ϵ measures the level of polynomial approximation and δ is a flatness parameter (also controlling the $C^{0,\gamma}$ norms of f_+ and f_-).

To obtain uniform point wise C^{2,γ^*} regularity both for the solution and the free boundary in $B_{1/2}$ we have to show that u is $(V_k, \lambda_k^{2+\gamma^*}, \delta)$ flat in B_{λ_k} for $\lambda_k = \eta^k$ and all $k \geq 0$, for some δ, η small and a sequence of V_k converging to a final profile V_0.

The starting point in the proof of Theorem 8 is to show that the flatness condition (3) allows us to normalize our solution so that a rescaling $u_{\bar{r}}$ of u is close to a one or two-phase polynomial. This kind of dichotomy parallels in a sense what happens in the *flatness to* $C^{1,\gamma}$ case but at a quadratic order of approximation. Set

$$u_r(x) := \frac{u(rx)}{r}, \quad f_{\pm r}(x) = rf_{\pm}(rx), \quad x \in B_1.$$

Lemma

There exist universal constants $\bar{\epsilon}, \bar{\delta}, \bar{\lambda}$ such that if u satisfies (3) with $\bar{\eta} = \bar{\eta}(\bar{\epsilon})$ then either of these flatness conditions holds with $\bar{r} = \bar{r}(\bar{\epsilon})$.

1. *Non-degenerate case: $u_{\bar{r}}$ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat in B_1, with $V = V^{\alpha,\beta}_{0,e_n,a,b} \in \mathcal{V}_{f_\pm}$, $a' = b' = 0$, $\quad \beta \geq \frac{1}{2}\bar{\delta}^{1/2}\bar{\lambda}^{2+\gamma}$, and*

$$|f_{+\bar{r}}(x) - f_{+\bar{r}}(0)| \leq \bar{\delta}|x|^\gamma \quad |f_{-\bar{r}}(x) - f_{-\bar{r}}(0)| \leq \beta\bar{\delta}|x|^\gamma.$$

2. *Degenerate case: $u^+_{\bar{r}}$ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat in B_1, for $V = V^1_{0,e_n,a_n} \in \mathcal{V}_{f_+}$,*

$$|u^-_{\bar{r}} + \frac{1}{2}f_{-\bar{r}}(0)x_n^2| \leq \bar{\delta}^{1/2}\bar{\lambda}^{2+\gamma} \quad \text{in } B_1^-(u_{\bar{r}})$$

and

$$\|f_{-\bar{r}}\|_\infty \leq \bar{\delta}, \quad |f_{\pm\bar{r}}(x) - f_{\pm\bar{r}}(0)| \leq \bar{\delta}|x|^\gamma.$$

We describe the dichotomy as follows.
Case 1. (nondegenerate configuration). The two phases have comparable size and $u_{\bar{r}}$ is trapped between two translations of a genuine two-phase polynomials, with a positive slope β (not too small).

Case 2. (degenerate configuration). The negative phase that has either zero slope or a small one (but not negligible) with respect to $u_{\bar{r}}^{+}$, and $u_{\bar{r}}^{+}$ is trapped between two translations of a one-phase polynomial. Note that this situation cannot occur if $f_{-} \geq 0$ unless u^{-} is identically zero.

Next we examine how the initial flatness corresponding to cases 1 and 2 above improves successively at a smaller scale.
We construct the following two "subroutines", to be implemented in the course of the final iteration towards C^{2,γ^*} regularity.
The first one provides a *two-phase* $C^{2,\gamma}$ flatness improvement: if u is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat in B_λ then u is $(\bar{V}, \left(\eta\bar{\lambda}\right)^{2+\gamma}, \bar{\delta})$ flat in $B_{\bar{\lambda}\eta}$, with $\bar{V}$ close to V. This result applies to the non-degenerate case.

Theorem

Two-phase flatness improvement. There exist $\bar\eta, \bar\delta, \bar\lambda$ universal, such that, if for $\beta > 0$

$$u \text{ is } (V, \lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_\lambda, \lambda \leq \bar\lambda \tag{13}$$

with $V = V^{\alpha,\beta}_{M,e_n,a,b} \in V_{f_\pm},$,

$$|f_+(x) - f_+(0)| \leq \bar\delta|x|^\gamma, \quad |f_-(x) - f_-(0)| \leq \beta\bar\delta|x|^\gamma \tag{14}$$

and

$$\left|\nabla u^+\right|^2 - \left|\nabla u^-\right|^2 = 1 \quad \text{on } F(u) \cap B_{2/3\lambda}$$

then

$$u \text{ is } (\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta) \text{ in } B_{\bar\eta\lambda} \tag{15}$$

with $\bar V = V^{\bar\alpha,\bar\beta}_{\bar M,\bar\nu,\bar a,\bar b} \in V_{f_\pm}$ and $|\beta - \bar\beta| \leq C\lambda^{1+\gamma}$ for C universal.

The second one provides a *one-phase* flatness improvement.
It will be used with the degenerate case, i.e. when the flatness of the free boundary only guarantees closeness of the positive part u^+ to a quadratic profile. More precisely if u^+ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat in B_λ and $|\nabla u^+|$ is close to α on $F(u)$, then u^+ enjoys a $C^{2,\gamma}$ flatness improvement, with $\bar{V}$ close to V.

Theorem

There exist $\bar\eta, \bar\delta, \bar\lambda$ *such that if for* $\beta = 0$

$$u^+ \text{ is } (V, \lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_\lambda, \lambda \leq \bar\lambda \tag{16}$$

with $V = V^{\alpha}_{M,e_n,a_n} \in V_{f_+}$,

$$|f_+(x) - f_+(0)| \leq \bar\delta |x|^\gamma \tag{17}$$

and

$$|\,|\nabla u^+| - \alpha| \leq \bar\delta^{1/2} \lambda^{1+\gamma} \quad \text{on } F(u) \cap B_{2/3\lambda}, \tag{18}$$

in the viscosity sense, then

$$u^+ \text{ is } (\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta) \text{ flat in } B_{\bar\eta\lambda} \tag{19}$$

with $\bar V = V^{\alpha}_{\bar M,\bar\nu,\bar a_{\bar\nu}} \in V_{f_+}$.

The achievement of the improvements above relies on a higher order refinement of the Harnack inequalities. This gives the necessary compactness to pass to the limit in a sequence of renormalized functions of u of the type (e.g. in the genuine two-phase case)

$$\tilde{v}^{\epsilon}(x) = \begin{cases} \dfrac{v(x) - \alpha(1 + a \cdot x)P_{M,e_n}}{\alpha\epsilon}, & x \in B_1^+(u) \cup F(u) \\ \dfrac{v(x) - \beta(1 + b \cdot x)P_{M,e_n}}{\beta\epsilon}, & x \in B_1^-(u), \quad \beta > 0 \\ 0, & x \in B_1^-(u), \quad \beta = 0. \end{cases} \tag{20}$$

and obtain a limiting transmission or Neumann problem. From the regularity of the solution of this problem we get the information to improve the two-phase or one-phase approximation for u or u^+ respectively, and hence their flatness.

Now we can start iterating. As we have seen, according to Case 1 above, after a suitable rescaling, we face a first dichotomy "degenerate versus nondegenerate".
In the latter case the two-phase subroutine of Proposition 13 can be applied indefinitely to reach pointwise C^{2,γ^*} regularity for some universal γ^*.

When u falls into the degenerate case a *new kind of dichotomy appears.* First of all, to run the *one-phase* subroutine in Proposition 14 we need to make sure that the closeness of u^- to a purely quadratic profile makes u^+ to be a (viscosity) solution of a one-phase free boundary problem with $|\nabla u_\nu^+|$ close to an appropriate α on $F(u)$. At this point two alternatives occur at a smaller scale:

D1 : either u^- is closer to a purely quadratic profile at a proper $C^{2,\gamma}$ rate and u^+ enjoys a $C^{2,\gamma}$ flatness improvement;

D2 : or u^- is closer (at a $C^{2,\gamma}$ rate) to a one-phase polynomial profile with a small non-zero slope but u^+ only enjoys an "intermediate" C^2 flatness improvement.

To give a precise statement it is convenient to introduce a new class $\mathcal{Q}_{f_-}$ of functions, defined as

$$Q_{p,q,\omega,M} = (x\cdot\omega - \frac{1}{2}x^T Mx)(p + q\cdot x) - \frac{1}{2}(f_-(0) + ptrM)(x\cdot\omega)^2,$$

with $p \in \mathbb{R}, q \in \mathbb{R}^n, M \in S^{n\times n}$, such that

$$q\cdot\omega = 0, \quad M\omega = 0, \quad \|M\| \le 1.$$

In the degenerate case, we use these functions to approximate u^- in a $C^{2,\gamma}$ fashion at a smaller and smaller scale. We have the following facts.

There exist universal constants $\bar\lambda, \bar\delta, \bar\eta$ such that if

$$u^+ \text{ is } (V, \lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_\lambda, \lambda \leq \bar\lambda \tag{21}$$

with $V = V^1_{M,e_n,a_n} \in \mathcal{V}_{f_+}$,

$$|f_\pm(x) - f_\pm(0)| \leq \bar\delta |x|^\gamma, \quad \|f_-\|_\infty \leq \bar\delta \tag{22}$$

and

$$|u^- - Q_{0,0,e_n,0}| \leq \bar\delta^{1/2} \lambda^{2+\gamma}, \quad \text{in } B^-_\lambda(u) \tag{23}$$

then either one of the following holds:

$D1$. There exists $\bar V = V^1_{\bar M,\bar{\mathbf{e}},\bar a_{\bar{\mathbf{e}}}} \in \mathcal{V}_{f_+}$, such that

$$u^+ \text{ is } (\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta) \text{ flat in } B_{\bar\eta\lambda}, \tag{24}$$

and

$$|u^- - Q_{0,0,\bar{\mathbf{e}},0}| \leq \bar{\delta}^{1/2}(\bar{\eta}\lambda)^{2+\gamma}, \quad \text{in } B^-_{\bar{\eta}\lambda}(u); \tag{25}$$

*D*2. There exists $V^* = V^{\alpha^*}_{M^*,\mathbf{e}^*,a^*_{\mathbf{e}^*}} \in \mathcal{V}_{f_+}$, such that

$$u^+ \text{ is } (V^*, \bar{\eta}^2\lambda^{2+\gamma}, \bar{\delta}) \text{ flat in } B_{\bar{\eta}\lambda},$$

and

$$|u^- - Q_{p^*,q^*,\mathbf{e}^*,M^*}| \leq \bar{\delta}^{1/2}(\bar{\eta}\lambda)^{2+\gamma}, \quad \text{in } B^-_{\bar{\eta}\lambda}(u),$$

for $(\alpha^*)^2 - (p^*)^2 = 1$ and
$p^* < 0, |p^*| \sim (\bar{\delta}^{1/2}\lambda^{1+\gamma}), |q^*| = O(\bar{\delta}^{1/2}\lambda^{\gamma})$.

If D1 occurs indefinitely we are done. If it does not, we prove that the intermediate improvement in D2 is kept for a while, at smaller and smaller scale. The final and crucial step is to prove that, at a given universally small enough scale, the $C^{2,\gamma}$ one-phase approximation of u^-, together with the intermediate C^2 flatness improvement of u^+, is good enough to recover a full C^{2,γ^*} two-phase improvement of u with a universal $\gamma^* < \gamma$.

More precisely, at the beginning u^+ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat while u^- is $C^{2,\gamma}$ close to a pure quadratic profile. This closeness improves at a $C^{2,\gamma}$ rate until (possibly) the slope of the approximating polynomial Q is no longer zero, say at scale λ. However, to obtain the desired full flatness of u, we need to reach a scale $\rho = \lambda r$ for $r \sim \lambda^{1+1/\gamma}$.
It is necessary to exploit also the information that the flatness of u^+ is in fact improving at a C^2 rate for a little while, hence allowing us to continue the iteration on the negative side and to obtain that u^- is $C^{2,\gamma}$ close to a nondegenarate configuration at an even smaller scale.
We have seen that in the case of the $C^{1,\gamma}$ estimates this issue is not present. The key result is the following:

Theorem

There exist $\bar\lambda, \bar\delta, \gamma^*$ *universal such that if*

$$u^+ \text{ is } (V, r^2\lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_{r\lambda}, \lambda \le \bar\lambda$$

with $V = V^{\alpha}_{M,e_n,a_n} \in \mathcal{V}_{f_+}$*, for r such that* $\bar\delta^{1/2} r^\gamma \in [2\bar\eta^\gamma \lambda^{1+\gamma}, 2\lambda^{1+\gamma})$*, and*

$$|u^- - Q_{p,q,e_n,M}| \le \bar\delta^{1/2}(r\lambda)^{2+\gamma}, \quad \text{in } B^-_{r\lambda}(u),$$

for $\alpha^2 - p^2 = 1$ *and* $p < 0$, $|p| \sim \bar\delta^{1/2}\lambda^{1+\gamma}$, $|q| = O(\bar\delta^{1/2}\lambda^\gamma)$*, then*

$$u \text{ is } (\bar V, (r\lambda)^{2+\gamma^*}, \bar\delta) \text{ flat in } B_{r\lambda}$$

with $\bar V = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm}$, $\beta = |p|$.

From this point on we can go back to the two-phase subroutine to reach pointwise C^{2,γ^*} regularity.

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