

Optimal Mass transport as a natural extension of classical mechanics to Wasserstein Space

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Given a cost functional $c(y, x)$ on some product measure space $X_0 \times X_1$, and two probability measures μ on X_0 and ν on X_1 , we consider the optimal transport problem and its dual:

$$\begin{aligned} \inf \left\{ \int_{X_0 \times X_1} c(y, x) d\pi; \pi \in \mathcal{K}(\mu, \nu) \right\} \\ = \sup \left\{ \int_{X_1} \phi_1(x) d\nu(x) - \int_{X_0} \phi_0(y) d\mu(y); \phi_1, \phi_0 \in \mathcal{K}(c) \right\}, \end{aligned}$$

- ▶ $\mathcal{K}(\mu, \nu)$ is the set of probability measures π on $X_0 \times X_1$ whose marginal on X_0 (resp. on X_1) is μ (resp., ν) (*the transport plans*).
- ▶ $\mathcal{K}(c)$ is the set of functions $\phi_1 \in L^1(X_1, \nu)$ and $\phi_0 \in L^1(X_0, \mu)$ such that

$$\phi_1(x) - \phi_0(y) \leq c(y, x) \quad \text{for all } (y, x) \in X_0 \times X_1.$$

- ▶ Pairs of functions in $\mathcal{K}(c)$ can be assumed to satisfy

$$\phi_1(x) = \inf_{y \in X_0} c(y, x) + \phi_0(y) \quad \text{and} \quad \phi_0(y) = \sup_{x \in X_1} \phi_1(x) - c(y, x).$$

Say that ϕ_0 (resp., ϕ_1) is an initial (resp., final) Kantorovich potential.

Minimizing for $c(y, x) = \frac{1}{2}|y - x|^2$. Same as maximizing for $b(y, x) = \langle y, x \rangle$.

$$\begin{aligned} \sup \left\{ \int_{M \times M^*} \langle y, x \rangle d\pi; \pi \in \mathcal{K}(\mu, \nu) \right\} \\ = \inf \left\{ \int_{M^*} \phi_1(x) d\nu(x) + \int_M \phi_0(y) d\mu(y); \phi_1, \phi_0 \in \mathcal{K}(c) \right\}, \end{aligned}$$

that is ,over all ϕ_1, ϕ_0 such that

$$\phi_1(x) + \phi_0(y) \geq \langle y, x \rangle \quad \text{for all } (y, x) \in M \times M^*.$$

$\phi_1(x) = \sup \{ \langle y, x \rangle - \phi_0(y); y \in M \} = \phi_0^*(x)$ (Legendre transform of ϕ_0).

An optimal π_0 is supported on the set

$$\{(y, x) \in M \times M^*; \phi_0^*(x) + \phi_0(y) = \langle x, y \rangle\},$$

hence on the graph of the subdifferential of a convex function $\partial\phi_0$.

Two dynamic costs

Given a Lagrangian $L : [0, T] \times M \times M^* \rightarrow \mathbf{R}$,

The fixed-end cost on $M \times M$

$$c_T(y, x) := \inf \left\{ \int_0^T L(t, \gamma(t), \dot{\gamma}(t)) dt; \gamma \in C^1([0, T], M); \gamma(0) = x, \gamma(T) = y \right\}$$

and the ballistic cost on $M^* \times M$

$$b_T(v, x) := \inf \left\{ \langle v, \gamma(0) \rangle + \int_0^T L(t, \gamma(t), \dot{\gamma}(t)) dt; \gamma \in C^1([0, T], M); \gamma(T) = x \right\},$$

are both formally solutions to (HJ).

Actually, for a given function g , **the value function**

$$V_g(t, x) = \inf \left\{ g(\gamma(0)) + \int_0^t L(s, \gamma(s), \dot{\gamma}(s)) ds; \gamma \in C^1([0, T], M); \gamma(t) = x \right\},$$
$$V(0, x) = g(x),$$

is formally a solution of the Hamilton-Jacobi equation,

$$(HJ) \quad \partial_t V + H(t, x, \nabla_x V) = 0 \quad \text{on } [0, T] \times M,$$

where H is the associated Hamiltonian on $[0, T] \times T^*M$, i.e.,

$$H(t, y, x) = \sup_{v \in TM} \{ \langle v, x \rangle - L(t, y, v) \}.$$

Hopf-Lax and Dual Hopf-Lax formula

Both costs can be seen as "Kernels" that can be used to generate general solutions for (HJ).

- ▶ **General Hopf-Lax–Lower kernel:**

$$V_g(t, x) = \inf\{g(y) + c(t, y, x); y \in M\}$$

- ▶ **General Dual Hopf-Lax formula–Upper kernel:**

$$V_g(t, x) = \sup\{b(t, v, x) - g^*(v); v \in M^*\}$$

provided the Lagrangian L is jointly convex and the initial function g is convex.

- ▶ **Classical Hopf-Lax and Dual Hopf-Lax formulae**

If $L(x, v) = L_0(v)$ and L_0 convex, then

$$c_t(y, x) = tL_0\left(\frac{1}{t}|x - y|\right) \quad \text{and} \quad b_t(v, x) = \langle v, x \rangle - tH_0(v).$$

and

$$V_g(t, x) = \inf\{g(y) + tL_0\left(\frac{1}{t}|x - y|\right); y \in M\} = (g^* + tH_0)^*.$$

When defined, the upper kernel is much better than the lower kernel.

- ▶ One consequence is that the Legendre transform of the value functional $x \rightarrow V_g(t, x) := \inf\{g(y) + c(t, y, x); y \in M\}$ is another value functional

$$\tilde{V}_{g^*}(t, w) = \inf\{g^*(v) + \tilde{c}(t, v, w); v \in M^*\},$$

which yields that

$$b(t, v, x) = \inf\{\langle v, y \rangle + c(t, y, x); y \in M\} = \sup\{\langle w, x \rangle - \tilde{c}(t, v, w); w \in M^*\}.$$

- ▶ So, $x \rightarrow b(t, v, x)$ was a "solution" of the HJ equation

$$\begin{aligned}\partial_t b + H(t, x, \nabla_x b) &= 0 \quad \text{on } [0, T] \times M, \\ b_0(x) &= \langle v, x \rangle.\end{aligned}$$

- ▶ Now $v \rightarrow b(t, v, x)$ is also a solution for another H-J equation:

$$\begin{aligned}\partial_t b - H(t, \nabla_v b, v) &= 0 \quad \text{on } [0, T] \times M, \\ b_T(v) &= \langle v, x \rangle.\end{aligned}$$

The associated transport problem is

$$C_T(\nu_0, \nu_T) := \inf \left\{ \int_{M \times M} c_T(v, x) d\pi; \pi \in \mathcal{K}(\nu_0, \nu_T) \right\}.$$

Theorem (Bernard-Buffoni & Fathi-Figalli): Under suitable assumptions on L .

1. The following duality holds:

$$C_T(\nu_0, \nu_T) = \sup \left\{ \int_M \phi_T(x) d\nu_T(x) - \int_M \phi_0(y) d\nu_0(y); \phi_t \text{ solution of } (HJ) \right\}.$$

$$\begin{cases} \partial_t \phi + H(t, x, \nabla_x \phi) & = 0 \text{ on } [0, T] \times M, \\ \phi(0, x) & = \phi_0(x), \end{cases}$$

2. There exists a function $\psi : M \rightarrow \mathbf{R}$ such that

$$C_T(\nu_0, \nu_T) = \int_M c_T(y, \pi^* \Phi_T^H(y, \nabla \psi(y))) d\nu_0(y),$$

where Φ_t^H is the Hamiltonian flow from 0 to T .

The Ballistic Optimal Transport Problem

The associated transport problems will be

$$\overline{B}_T(\mu_0, \nu_T) := \sup \left\{ \int_{M^* \times M} b_T(v, x) d\pi; \pi \in \mathcal{K}(\mu_0, \nu_T) \right\},$$

$$\underline{B}_T(\mu_0, \nu_T) := \inf \left\{ \int_{M^* \times M} b_T(v, x) d\pi; \pi \in \mathcal{K}(\mu_0, \nu_T) \right\},$$

where μ_0 (resp., ν_T) is a probability measure on M^* (resp., M), and $\mathcal{K}(\mu_0, \nu_T)$ is the set of probability measures π on $M^* \times M$ whose marginal on M^* (resp. on M) is μ_0 (resp., ν_T) (*the transport plans*).

Note that when $T = 0$, we have $b_0(x, v) = \langle v, x \rangle$, which is exactly the case considered by Brenier, that is

$$\overline{W}(\mu_0, \nu_0) := \sup \left\{ \int_{M^* \times M} \langle v, x \rangle d\pi; \pi \in \mathcal{K}(\mu_0, \nu_0) \right\},$$

$$\underline{W}(\mu_0, \nu_0) := \inf \left\{ \int_{M^* \times M} \langle v, x \rangle d\pi; \pi \in \mathcal{K}(\mu_0, \nu_0) \right\},$$

This is the dynamic version of the Wasserstein distance.

$$\begin{aligned}\underline{B}_T(\mu_0, \nu_0) : &= \inf \left\{ \int_{M^* \times M} b(v, x) d\pi; \pi \in \mathcal{K}(\mu_0, \nu_T) \right\} \\ &= \sup \left\{ \int_M \phi_1(x) d\nu_T(x) - \int_{M^*} \phi_0(v) d\mu_0(v); \phi_1, \phi_0 \in \mathcal{K}(b) \right\}.\end{aligned}$$

The Kantorovich functions in $\mathcal{K}(c)$ can be assumed to satisfy

$$\phi_1(x) = \inf_{v \in M^*} b(v, x) + \phi_0(v) \quad \text{and} \quad \phi_0(v) = \sup_{x \in M} \phi_1(x) - b(v, x).$$

- ▶ $b_T(v, x)$ is concave in v and convex in x . It is also Lipschitz continuous.
- ▶ In the case of $\underline{B}_T(\mu_0, \nu_T)$, **the initial Kantorovich potential ϕ_0 is convex**, though nothing can be said about ϕ_1 .
- ▶ For $\bar{B}_T(\mu_0, \nu_T)$, **the final potential is convex**. Nothing can be said for ϕ_1 .
- ▶ Even though $c(y, x)$ is jointly convex, nothing can be said about the Kantorovich potentials of $C_T(\mu_0, \mu_T)$ including the case where $L(x, v) = |v|^p$ ($p \geq 1$), that is when $c_1(y, x) = |x - y|^p$.

Theorem (A): Under suitable assumptions on the Lagrangian L . Let μ_0, ν_T be probabilities on M^* , M such that μ_0 is absolutely continuous with respect to Lebesgue measure. Then,

1. The following duality holds:

$$\underline{B}_T(\mu_0, \nu_T) = \sup \left\{ \int_M \phi_T(x) d\nu_T(x) + \int_M \tilde{\phi}_0(v) d\mu_0(v); \right. \\ \left. \phi_0 \text{ concave \& } \phi_t \text{ solution of (HJ)} \right\}.$$

$$\begin{cases} \partial_t \phi + H(t, x, \nabla_x \phi) & = 0 \text{ on } [0, T] \times M, \\ \phi(0, x) & = \phi_0(x), \end{cases}$$

2. There exists a concave function $\phi_0 : M \rightarrow \mathbf{R}$ and a bounded locally Lipschitz vector field $X(x, t) : M \times]0, T[\rightarrow M$ such that, if $\Phi_s^t, (s, t) \in]0, T]^2$ is the flow of X from time s to time t , then

$$\underline{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \Phi_0^T \circ \nabla \tilde{\phi}_0(v)) d\mu_0(v).$$

$$\underline{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \pi^* \Phi_t^H(\nabla \tilde{\phi}_0(v), v)) d\mu_0(v).$$

- ▶ Introduce another cost functional

$$\tilde{c}_T(u, v) := \inf \left\{ \int_0^T \tilde{L}(t, \gamma(t), \dot{\gamma}(t)) dt; \gamma \in C^1([0, T], M); \gamma(0) = u, \gamma(T) = v \right\},$$

The new Lagrangian \tilde{L} is defined on $M \times M^*$ by

$$\tilde{L}(t, x, p) := L^*(t, p, x) = \sup \{ \langle p, y \rangle + \langle x, q \rangle - L(t, y, q); (y, q) \in M \times M^* \}.$$

The corresponding Hamiltonian is $H_{\tilde{L}}$ is then given by

$$H_{\tilde{L}}(x, y) = -H(y, x).$$

- ▶ Recall **Bolza's duality**: $(\mathcal{P}) = -(\tilde{\mathcal{P}})$, where

$$(\mathcal{P}) \quad \inf \left\{ \int_0^T L(\gamma(s), \dot{\gamma}(s)) ds + \ell(\gamma(0), \gamma(T)) \text{ over all } \gamma \in C^1([0, T], M) \right\}$$

and its dual

$$(\tilde{\mathcal{P}}) \quad \inf \left\{ \int_0^T \tilde{L}(\gamma(s), \dot{\gamma}(s)) ds + \ell^*(\gamma(0), -\gamma(T)) \text{ over all } \gamma \in C^1([0, T], M) \right\}$$

This has several consequences

Theorem (B): Under suitable assumptions on the Lagrangian L . Let μ_0, ν_T be probabilities on M^*, M such that ν_T is absolutely continuous with respect to Lebesgue measure. Then,

1. The following duality holds:

$$\bar{B}_T(\mu_0, \nu_T) = \inf \left\{ \int_M \psi_T^*(x) d\nu_T(x) + \int_M \psi_0(v) d\mu_0(v); \right. \\ \left. \psi_T \text{ convex \& } \psi_t \text{ solution of (dual-HJ)} \right\}.$$

$$\begin{cases} \partial_t \psi - H(\nabla_v \psi, v) & = 0 \text{ on } [0, T] \times M, \\ \psi(T, v) & = \psi_T(v), \end{cases}$$

2. There exists a convex function $\psi : M \rightarrow \mathbf{R}$ and a bounded locally Lipschitz vector field $Y(x, t) : M \times]0, T[\rightarrow M$ such that, if $\Psi_s^t, (s, t) \in]0, T[^2$ is the flow of Y from time s to time t , then

$$\bar{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \nabla \psi^* \circ \Psi_0^T(v)) d\mu_0(v).$$

Theorem (C): Assume $M = \mathbf{R}^d$ and that L satisfies hypothesis (A1), (A2) and (A3), and let μ_0 (resp. ν_T) be a probability measure on M^* (resp., M). If μ_0 is absolutely continuous with respect to Lebesgue measure, then

1. The following Hopf-Lax formula holds:

$$\underline{B}_T(\mu_0, \nu_T) = \inf\{\underline{W}(\mu_0, \nu) + C_T(\nu, \nu_T); \nu \in \mathcal{P}(M)\}.$$

2. The infimum is attained at some probability measure ν_0 on M .
 3. The initial Kantorovich potential for $C_T(\nu_0, \nu_T)$ is concave.
- **Worth noting:** If $L(x, v) = \frac{1}{2}|v|^2$ (i.e., $c(y, x) = \frac{1}{2}|x - y|^2$), the initial Kantorovich potential for $C_T(\nu_0, \nu_T)$ is then of the form

$$\phi_0(y) = g(y) - \frac{1}{2}|y|^2 \quad \text{where } g \text{ is a convex function.}$$

But ϕ_0 can still be concave if $0 \leq D^2g \leq I$, which is what occurs above in (3).

However this formula doesn't lift:

$$c(t, y, x) = \sup\{b(t, v, x) - \langle v, y \rangle; v \in M^*\}.$$

Theorem (D): Assume ν_0 and ν_T are probability measures on M such that ν_0 is absolutely continuous with respect to Lebesgue measure. Then, TFAE:

1. The initial Kantorovich potential of $C_T(\nu_0, \nu_T)$ is concave.
2. The following holds:

$$C_T(\nu_0, \nu_T) = \sup\{\underline{B}_T(\mu, \nu_T) - \underline{W}(\nu_0, \mu); \mu \in \mathcal{P}(M^*)\}.$$

and the sup is attained at some probability measure μ_0 on M^* .

Corollary: Consider the cost $c(y, x) = c(x - y)$, where c is a convex function on M and let ν_0, ν_1 be probability measures on M such that the initial Kantorovich potential associated to $C_T(\nu_0, \nu_T)$ is concave. Then, there exist concave functions $\phi_0 : M \rightarrow \mathbf{R}$ and $\phi_1 : M^* \rightarrow \mathbf{R}$ such that

$$C_1(\nu_0, \nu_1) - K = \int_M c(\nabla\phi_1 \circ \nabla\phi_0(y) - y) d\nu_0(y) = \int_M \langle \nabla\tilde{\phi}_1(y) - \nabla\phi_0(y), y \rangle d\nu_0(y),$$

where $K = K(c)$ is a constant and $\tilde{\phi}$ is the concave Legendre transform of ϕ .

Theorem (E): For fixed probability measures μ_0 on M^* and ν_T on M ,

- **As a function of the end measure:**

$$\underline{B}_T(\mu_0, \nu_T) = \inf \left\{ \underline{W}(\mu_0, \rho_0) + \int_0^T \int_M L(x, w_t(x)) d\rho_t(x) dt; (\rho, w) \in P(0, T; \nu_T) \right\}$$

where $P(0, T; \nu_T)$ is the set of pairs (ρ, w) such that $t \rightarrow \rho_t \in \mathcal{P}(M)$, $t \rightarrow w_t \in \mathbf{R}^n$ are paths of Borel vector fields such that

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho w) & = 0 & \text{in } \mathcal{D}'((0, T) \times M) \\ \rho_T & = \nu_T. \end{cases}$$

- **As a function of the initial measure:**

$$\overline{B}_T(\mu_0, \nu_T) = \sup \left\{ \overline{W}(\nu_T, \rho_T) - \int_0^T \int_M \tilde{L}(x, w_t(x)) d\rho_t(x) dt; (\rho, w) \in P(0, T; \mu_0) \right\}$$

where $P(0, T; \mu_0)$ is the set of pairs (ρ, w) such that

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho w) & = 0 & \text{in } \mathcal{D}'((0, T) \times M) \\ \rho_0 & = \mu_0. \end{cases}$$

Started with $\phi_0(y) = \langle v, y \rangle$ and defined $b_v(t, x)$ as a **Value functional**

$$\begin{aligned} b_v(t, x) &= \inf \left\{ \phi_0(\gamma(0)) + \int_0^t L(s, \gamma(s), \dot{\gamma}(s)) ds; \gamma \in C^1([0, T], M); \gamma(t) = x \right\} \\ &= \inf \left\{ \phi_0(y) + c_t(y, x); y \in M \right\} \quad (\text{Hopf - Lax formula}). \end{aligned}$$

It satisfies the **Hamilton-Jacobi equation on M** .

$$\partial_t b + H(t, x, \nabla_x b) = 0 \quad \text{on } [0, T] \times M,$$

We then lifted b_v to Wasserstein space by defining $B_{\mu_0}(t, \nu) = \underline{B}_t(\mu_0, \nu)$.

$$\begin{aligned} B_{\mu_0}(t, \nu) &= \inf \{ \underline{W}(\mu_0, \tilde{\nu}) + C_t(\tilde{\nu}, \nu); \nu \in \mathcal{P}(M) \} \quad (\text{Hopf - Lax formula}) \\ &= \inf \{ \mathcal{U}_{\mu_0}(\varrho_0) + \int_0^t \mathcal{L}(\varrho, w) dt; (\varrho, w) \in P(0, t; \nu) \} \quad (\text{Value functional}) \end{aligned}$$

1. Do they satisfy a Hamilton-Jacobi equation on Wasserstein space?
2. Do they provide solutions to mean field games?

(Ambrosio-Feng) (at least in a particular case): Value functionals on Wasserstein space yield a unique metric viscosity solution for

$$\begin{cases} \partial_t B + \mathcal{H}(t, \nu, \nabla_\nu B(t, \mu)) = 0, \\ B(0, \nu) = \underline{W}(\mu_0, \nu) \end{cases}$$

Here the Hamiltonian on Wasserstein space is defined as

$$\mathcal{H}(\nu, \zeta) = \sup \left\{ \int \langle \zeta, \xi \rangle d\nu - \mathcal{L}(\nu, \xi); \xi \in T_\nu^*(\mathcal{P}(M)) \right\}$$

(Gangbo-Swiech) Value functions on Wasserstein space with suitable initial data yield solutions to the so-called Master equation for mean field games without diffusion and without potential term.

Theorem (Gangbo-Swiech) Assume $\mathcal{U}_0 : \mathcal{P}(M) \rightarrow \mathbb{R}$, $U_0 : M \times \mathcal{P}(M) \rightarrow \mathbb{R}$ are such that $\nabla_q U_0(q, \mu) \equiv \nabla_\mu \mathcal{U}_0(\mu)(q) \quad \forall q \in M \quad \mu \in \mathcal{P}(M)$, and consider the value functional,

$$\mathcal{U}(t, \nu) = \inf_{(\varrho, w) \in \mathcal{P}(0, t; \nu)} \int_0^t \mathcal{L}(\varrho, w) dt + \mathcal{U}_0(\varrho_0)$$

Then, there exists $U : [0, T] \times M \times \mathcal{P}(M) \rightarrow \mathbb{R}$ such that

$$\nabla_q U_t(q, \mu) \equiv \nabla_\mu \mathcal{U}_t(\mu)(q) \quad \forall q \in M \quad \mu \in \mathcal{P}(M).$$

and U satisfies the Master equation (but without diffusion)

This yields the existence for any probabilities μ_0, ν_T , a function $\beta : [0, T] \times M \times \mathcal{P}(M) \rightarrow \mathbb{R}$ such that

$$\nabla_x \beta(t, x, \mu) \equiv \nabla_\mu B_{\mu_0}(t, \mu)(x) \quad \forall x \in M \quad \mu \in \mathcal{P}(M).$$

There exists $\rho \in AC^2((0, T) \times \mathcal{P}(M))$ such that

$$\left\{ \begin{array}{l} \partial_t \beta + \int \langle \nabla_\mu \beta(t, x, \mu) \cdot \nabla H(x, \nabla_x \beta) \rangle d\mu + H(x, \nabla_x \beta(t, x, \mu)) = 0, \\ \partial_t \rho + \nabla(\rho \nabla H(x, \nabla_x \beta)) = 0, \\ \beta(0, \cdot, \cdot) = \beta_0, \quad \rho(T, \cdot) = \nu_T, \end{array} \right.$$

where $\beta_0(x, \rho) = \phi_\rho(x)$, where ϕ_ρ is the convex function such that $\nabla \phi_\rho$ pushes μ_0 into ρ .

What about solutions to mean field games that include diffusions?

$$\underline{B}_T^s(\mu_0, \nu_T) := \inf_{V \sim \mu_0} \inf_{X \in \mathcal{A}, X_T \sim \nu_T} \mathbf{E}_P \left\{ \langle V, X_0 \rangle + \int_0^T L(t, X(t), \beta(t, X)) dt \right\},$$

where \mathcal{A} is the class of all \mathbf{R}^d -valued continuous semimartingales $(X_t)_{0 \leq t \leq T}$ on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that there exists a Borel measurable $\beta_X : [0, T] \times C([0, T]) \rightarrow \mathbf{R}^d$ satisfying

1. $w \rightarrow \beta_X(t, w)$ is $\mathcal{B}(C([0, t]))_+$ -measurable for all t , where $\mathcal{B}(C([0, t]))$ denotes the Borel σ -field on $C([0, t])$.
2. $X(t) = X(0) + \int_0^t \beta_X(s, X) ds + W_X(t)$, where $W_X(t)$ is a $\sigma[X(s); 0 \leq s \leq t]$ -Brownian motion.

The fixed end measures cost has been studied by [Mikami](#), [Thieulin](#), [Leonard](#).

$$C_T^s(\nu_0, \nu_T) := \inf \mathbf{E}_P \left\{ \int_0^T L(t, X(t), \beta(t, X)) dt; X \in \mathcal{A}, X(0) \sim \nu_0, X(T) \sim \nu_T \right\},$$

Theorem (F): Under suitable conditions on L

1. Duality:

$$\underline{B}_T^S(\mu_0, \nu_T) = \sup \left\{ \int_M \phi_T(x) d\nu_T(x) + \int_M \tilde{\phi}_0(v) d\mu_0(v); \right. \\ \left. \phi_0 \text{ concave \& } \phi_t \text{ solution of (HJB)} \right\}.$$

$$\begin{cases} \partial_t \phi + \frac{1}{2} \Delta \phi + H(t, x, \nabla_x \phi) & = 0 \text{ on } [0, T] \times M, \\ \phi(0, x) & = \phi_0(x), \end{cases}$$

2. For any probability measure ν on M , we have

$$\underline{B}_T^S(\mu_0, \nu) = \inf \left\{ \underline{W}(\mu_0, \rho_0) + \int_0^T \int_M L(t, x, b_t(x)) d\rho_t(x) dt; (\rho, b) \in P(0, T; \nu) \right\}$$

where $P(0, T; \nu_T)$ is the set of pairs (ρ, b) such that $t \rightarrow \rho_t \in \mathcal{P}(M)$, $t \rightarrow b_t \in \mathbf{R}^n$ are paths of Borel vector fields such that

$$\begin{cases} \partial_t \rho - \frac{1}{2} \Delta \rho + \nabla \cdot (\rho b) & = 0 \text{ in } \mathcal{D}'((0, T) \times M) \\ \rho_T & = \nu. \end{cases}$$

There exists $\beta : [0, T] \times M \times \mathcal{P}(M) \rightarrow \mathbb{R}$ and $\rho \in AC^2((0, T) \times \mathcal{P}(M))$ such that

$$\left\{ \begin{array}{l} \partial_t \beta - \frac{1}{2} \Delta \beta + \int \langle \nabla_\mu \beta(t, x, \mu) \cdot \nabla H(x, \nabla_x \beta) \rangle d\mu + H(t, x, \nabla_x \beta(t, x, \mu)) = 0, \\ \partial_t \rho - \frac{1}{2} \Delta \rho + \nabla(\rho \nabla H(t, x, \nabla_x \beta)) = 0, \\ \beta(0, \cdot, \cdot) = \beta_0, \quad \rho(T, \cdot) = \nu_T, \end{array} \right.$$

where $\beta_0(x, \rho) = \phi_\rho(x)$, where ϕ_ρ is the convex function such that $\nabla \phi_\rho$ pushes μ_0 into ρ .