Principal eigenvalues for k-Hessian operators by maximum principle methods

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1. Introduction

Proposal: Study maximum principles and principal eigenvalues for

$$\text{(EVP)} \quad \begin{cases} F(x, D^2u) + \lambda u|u|^{k-1} = 0 & \text{in } \Omega \Subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $F = F(x, A)$ is continuous with $F(x, 0) = 0$ and

- homogeneous of degree k in $A \in \mathcal{S}(N)$;
- elliptic in the sense of Krylov [TAMS'95]; increasing in A along $\Theta(x) \subsetneq \mathcal{S}(N)$ an elliptic set for each $x \in \Omega$

Define $\lambda_1^-(F, \Theta)$ as the sup over $\lambda \in \mathbb{R}$ for which there is a negative subsolution of (EVP).

1. Do suitably defined supersolutions satisfy a minimum principle for $\lambda < \lambda_1^-(F, \Theta)$?
2. Exists $\psi_1 < 0$ in Ω corresponding to $\lambda = \lambda_1^-(F, \Theta)$?

Test case: k-Hessian operators

For $k = 1, 2, \ldots, N$ consider $F(D^2u) = S_k(D^2u)$ defined by

$$S_k(D^2u) := \sigma_k(\lambda(D^2u)) \quad \text{where}$$

$$\sigma_k(\lambda(A)) := \sum_{1 \leq i_1 < \cdots i_k \leq N} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{and}$$

$$\lambda(A) = (\lambda_1(A), \ldots, \lambda_N(A)) \text{ for } \ A \in \mathcal{S}(N).$$

- $S_1(D^2u) = \mathrm{tr}\,(D^2u)$: "know everything" about $\lambda_1^{\pm}(\Delta u)$ - [Berestycki-Nirenberg-Varadhan, CPAM'94]
- For each k there is a variational structure [Reilly, MMJ'73]
- $S_N(D^2u) = \det\,(D^2u)$: variational description of $\lambda_1^-(S_k(D^2u))$ simple w/ convex eigenfunction $\psi_1 < 0$ on Ω strictly convex w/ $\partial\Omega \in C^2$ - [P.L. Lions, AMPA'85]
- $k = 2, \ldots, N$: similar result for Ω strictly $(k-1)$-convex and with k-convex $\psi_1 < 0$ - [X.J. Wang, IUMJ'94]

Objectives:

For $k = 2, \ldots, N$ on $\Omega \Subset \mathbb{R}^N$ which is $(k-1)$-convex and $\partial\Omega \in C^2$

1. Characterize $\lambda_1^-(S_k(D^2 u))$ by the validity of a minimum principle.
2. Capture ψ_1 by an iterative viscosity method for $\lambda_n \nearrow \lambda_1^-$ a la [Birindelli-Demengel, CPAA'07]

In order to do this, we will:

- encode the needed notions of k-convexity into the language of elliptic sets $\Theta \subsetneq \mathcal{S}(N)$;
- define suitable notions of admissible viscosity supersolutions;
- exploit the boundary convexity for constructing suitable barriers;
- follow the usual path of [BD].

2. Notions of k-convexity

Consider the open convex cone (in $\mathbb{R}^N$) with vertex at the origin

$$\Gamma_k := \{\lambda \in \mathbb{R}^N : \ \sigma_j(\lambda) > 0, \ j = 1, \ldots, k\}$$

and define the closed cone in $\mathcal{S}(N)$ by

$$\Theta_k := \{A \in \mathcal{S}(N) : \ \lambda(A) \in \overline{\Gamma}_k\}$$

where $\lambda(A) = (\lambda_1(A), \ldots, \lambda_N(A)) \in \mathbb{R}^N$ are the evals of A

- Θ_k is an elliptic set; that is, $\Theta_k \subsetneq \mathcal{S}(N)$ is closed, non empty and

$$A \in \Theta_k, P \geq 0 \quad \Rightarrow \quad A + P \in \Theta_k$$

- S_k is increasing along Θ_k; that is, for each $A \in \Theta_k, P \geq 0$

$$S_k(A + P) := \sigma_k(\lambda(A + P)) \geq \sigma_k(\lambda(A)) := S_k(A)$$

where

$$\sigma_k(\lambda(A)) := \sum_{1 \leq i_1 < \cdots i_k \leq N} \lambda_{i_1} \cdots \lambda_{i_k}$$

k-convex functions on Ω

- For $u \in C^2(\Omega)$ one asks that for each $x \in \Omega$,

$$S_k(D^2 u(x)) \in \Theta_k \quad \left(\Leftrightarrow \quad \sigma_j(\lambda(D^2 u(x))) \geq 0, \ j = 1, \ldots, k \right)$$

N.B. For $k = 1, N$, u is subharmonic, convex respectively.

- For $u \in USC(\Omega)$ one uses a viscosity definition: for each $x_0 \in \Omega$ and for each $\varphi \in C^2(\Omega)$

$$u - \varphi \text{ has a local maximum in } x_0 \quad \Rightarrow \quad S_k(D^2 \varphi(x_0)) \geq 0;$$

or equivalently, if $(p, A) \in J^{2,+}u(x_0)$ then $A \in \Theta_k$.

Lemma (Trudinger-Wang AM'99)

$u \in USC(\Omega)$ is k-convex in Ω if and only if for each $\Omega' \Subset \Omega$ and for each $v \in C^2(\Omega') \cap C(\overline{\Omega})$ such that $S_k(D^2 v) \leq 0$ in Ω'

$$u \leq v \text{ on } \partial\Omega' \quad \Rightarrow \quad u \leq v \text{ in } \Omega'.$$

$(k-1)$-convex domains Ω

For $\Omega \Subset \mathbb{R}^N$ with $\partial\Omega \in C^2$ denote by $(\kappa_1(y), \ldots, \kappa_{N-1}(y))$ the principal curvatures at $y \in \partial\Omega$; i.e. the eigenvalues of $D^2\varphi(y')$ where $\varphi : B_r(y') \subset \mathbb{R}^{N-1} \to \mathbb{R}$ locally defines $\partial\Omega$ as a graph.

- Ω is strictly $(k-1)$-convex if

$$\sigma_{k-1}(\kappa_1(y), \ldots, \kappa_{N-1}(y)) > 0, \quad \text{for each} \quad y \in \partial\Omega$$

N.B. $(N-1)$-strict convexity is ordinary strict convexity.

- Since $\partial\Omega$ is compact, there exists $R > 0$ such that

$$\sigma_k(\kappa_1(y), \ldots, \kappa_{N-1}(y), R) > 0, \quad \text{for each} \quad y \in \partial\Omega;$$

i.e. $(\kappa_1(y), \ldots, \kappa_{N-1}(y), R) \in \Gamma_k$ for each $y \in \partial\Omega$.

- Equivalently $\partial\Omega$ is strictly $\overrightarrow{\Theta}_k$-convex in the sense of [Harvey-Lawson, CPAM'09]; expressed in terms of a local defining function $\rho : B_r(y) \subset \mathbb{R}^N \to \mathbb{R}$ for the boundary.

3. Minimum principle characterization of λ_1^-

With $\Phi_k^-(\Omega) := \{\psi \in USC(\Omega) : \psi$ is k-convex and $\psi < 0$ in $\Omega\}$ define the generalized principle eigenvalue $\lambda_1^-(S_k, \Theta_k)$ as

$$\sup\{\lambda \in \mathbb{R} : \ \exists\, \psi \in \Phi_k^-(\Omega) \text{ with } S_k(D^2\psi) + \lambda\psi|\psi|^{k-1} \geq 0 \text{ in } \Omega\},$$

where the inequality is in the viscosity sense: $\forall \ x \in \Omega, \varphi \in C^2(\Omega)$:

$$\psi - \varphi \text{ w/ local max in } x \ \Rightarrow \ S_k(D^2\varphi(x)) + \lambda\psi(x)|\psi(x)|^{k-1} \geq 0.$$

Theorem (Birindelli-P.'17)

Let Ω be a strictly $(k-1)$-convex domain in $\mathbb{R}^N$ with $k \in \{2, \ldots, N\}$. For every $\lambda < \lambda_1^-(S_k, \Theta_k)$ and for every $u \in LSC(\overline{\Omega})$ admissible viscosity supersolution of

$$S_k(D^2u) + \lambda u|u|^{k-1} = 0 \quad \text{in } \Omega \tag{1}$$

one has the following minimum principle

$$u \geq 0 \ \text{ on } \partial\Omega \quad \Rightarrow \quad u \geq 0 \ \text{ in } \Omega$$

Admissible supersolutions of (1)

The admissibility is in the sense of [Krylov, TAMS'95]; that is, $u \in LSC(\Omega)$ is an admissible viscosity supersolution of

$$S_k(D^2 u) + \lambda u|u|^{k-1} = 0 \text{ in } \Omega$$

if for each $x_0 \in \Omega$ and for each $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum (say zero) in x_0 then

$$D^2\varphi(x_0) \notin \Theta_k^\circ \quad \text{or} \quad S_k(D^2\varphi(x_0)) + \lambda\varphi(x_0)|\varphi(x_0)|^{k-1} \leq 0. \qquad (2)$$

hence

$$S_k(D^2\varphi(x_0)) + \lambda\varphi(x_0)|\varphi(x_0)|^{k-1} \leq 0 \quad (\text{if } D^2\varphi(x_0) \in \Theta_k^\circ). \qquad (3)$$

- Θ_k° corresponds to strict k-convexity.
- [Ishii-Lions, JDE'90] use the analog of (3) with $D^2\varphi(x_0) \in \Theta_k$ for supersolutions to Monge-Ampère equations.
- (2) also reflects *duality* of [Harvey-Lawson, CPAM'09].

- If k is odd, then so is S_k and one has a maximum principle characterization for $\lambda_1^+(S_k, -\Theta_k)$ via k-concave functions.
- The minimum principle shows that $\lambda_1^-(S_k, \Theta_k)$ agrees w/ the principal eigenvalue λ_1 of Wang'94 (Lions'85 for $k = N$)

$$\lambda_1^k := \inf_{u \in \Phi_0^k(\Omega)} \left\{ - \int_\Omega u S_k(D^2 u)\, dx : \quad ||u||_{L^{k=1}(\Omega)} = 1 \right\}$$

– $\Phi_0^k(\Omega)$ the set of strictly k-convex $u \in C^2(\Omega)$ w/ $u_{|\partial\Omega} = 0$.

Proof Since λ_1 has a k-convex principal eigenfunction ψ_1 with

$$\psi_1 < 0 \text{ in } \Omega \quad \text{and} \quad \psi_1 = 0 \text{ on } \partial\Omega,$$

one has $\lambda_1 \leq \lambda_1^-(S_k, \Theta_k)$ by definition. If $\lambda_1 < \lambda_1^-(S_k, \Theta_k)$, then ψ_1 would be an admissible supersolution of (1) with $\lambda = \lambda_1$ and hence $\psi_1 \geq 0$ in Ω by the minimum principle, which is absurd.

Idea of proof for the minimum principle

Need $u \geq 0$ on Ω for supersoln. with $0 < \lambda < \lambda_1(S_k, \Theta_k)$ of

$$S_k(D^2u) + \lambda u|u|^{k-1} = 0 \text{ in } \Omega. \qquad (4)$$

Argue by contradiction and compare u with $\gamma\psi$ where

$$\psi < 0 \text{ subsoln. of (4)} \leftrightarrow \tilde{\lambda} \in (\lambda, \lambda_1^-) \quad \text{and} \quad \gamma \in \left(0, \gamma' := \sup_{\Omega} \frac{u}{\psi}\right)$$

- ψ exists: the set of λ competing for $\lambda_1^-(S_k, \Theta_k)$ is an interval.
- $\gamma' < \infty$: use semicontinuity of u, ψ away from $\partial\Omega$ and construct barriers near $\partial\Omega$.

Find $\tilde{x} \in \Omega$ such that $u(\tilde{x}) < 0$ and

$$\lambda|u(\tilde{x})|^k \geq \gamma^k \tilde{\lambda}|\psi(\tilde{x})|^k; \text{ i.e. } \frac{\lambda}{\tilde{\lambda}}\gamma^k \leq \left(\frac{u(\tilde{x})}{\psi(\tilde{x})}\right)^k \leq (\gamma')^k. \qquad (5)$$

Pick $\gamma > \gamma'(\lambda/\tilde{\lambda})^{1/k}$ to contradict (5).

Barriers for S_k

For $\delta > 0$ small enough, there exist $C_1, C_2 > 0$ such that

$$\psi(x) \leq -C_1 d(x) \quad \text{and} \quad u(x) \geq -C_2 d(x),$$

in $\Omega_\delta := \{x \in \Omega : \ d(x) = \mathrm{dist}(x, \partial\Omega) < \delta\}$.

- Compare ψ to $w \in C^2(\Omega)$ standard radial function in an annular region touching $\partial\Omega$ (Hopf lemma).
- Easy to calculate S_k on radial functions $w(x) = h(|x - x_0|)$.
- Compare u to $v(x) = -M \log(1 + td(x))$ with $t \geq 2R$ where $R\ \sigma_{k-1}(\kappa_1(y), \ldots, \kappa_{N-1}(y), R) > 0$ all $y \in \partial\Omega$.
- Easy to calculate $S_k v$ for $v = g \circ d$ in a *principal coordinate system* based at $y_0 = y(x_0)$ with $x_0 \in \Omega_{d_0}$:

$$S_k(D^2 v(x_0)) = \sigma_k \left(\frac{-\kappa_1 g'(d)}{1 - \kappa_1 d}, \ldots, \frac{-\kappa_{N-1} g'(d)}{1 - \kappa_{N-1} d}, g''(d) \right),$$

where $\kappa_i = \kappa_i(y_0)$, $d = d(x_0)$ and $1 - \kappa_i d > 0$ for δ small.

Ishii's Lemma and admissibility

In order to find $\tilde{x} \in \Omega$ such that (5) holds when comparing $u, \gamma\psi$, look at the maximum values of

$$\Psi_j(x,y) := \gamma\psi(x) - u(y) - \frac{j}{2}|x-y|^2, \quad (x,y) \in \overline{\Omega} \times \overline{\Omega}, j \in \mathbb{N}.$$

- $\Psi_j \leq 0$ on the complement of $\Omega \times \Omega$.
- $\Psi_j(\bar{x}, \bar{x}) \geq (\gamma - \gamma')\psi(\bar{x}) > 0$ where $\min_\Omega u = u(\bar{x}) < 0$, so Ψ_j has a positive maximum in $(x_j, y_j) \in \Omega \times \Omega$.
- By Ishii's lemma, $\exists X_j, Y_j \in \mathcal{S}(N)$ such that

$$(j(x_j - y_j), X_j) \in \overline{J}^{2,+}\gamma\psi(x_j)) \text{ and } (j(x_j - y_j), Y_j) \in \overline{J}^{2,-}\gamma u(y_j))$$

$$(x_j, y_j) \to (\tilde{x}, \tilde{x}) \quad \text{and} \quad X_j \leq Y_j.$$

- $X_j \in \Theta_k$ since ψ is k-convex and hence $Y_k \in \Theta_k$, so

$$\tilde{\lambda}\gamma^k |\psi(x_j)|^k \leq S_k(X_j) \leq S_k(Y_j) \leq \lambda |u(y_j)|^k$$

  and pass to the limit to get (5).

Construction of a principal eigenfunction

We know that a negative k-convex eigenfunction ψ_1 exists associated to $\lambda_1 = \lambda_1^-(S_k, \Theta_k)$, but to prepare for non variational perturbations of S_k seek a maximum principle approach.

Idea: [Birindelli-Demengel] Pick $\{\lambda_n\}_{n\in\mathbb{N}}$ with $0 < \lambda_n \nearrow \lambda_1^-$.

- Start with $u_0 = 0$ and solve inductively

$$\begin{cases} S_k(D^2 u_n) = 1 - \lambda_n u_{n-1}|u_{n-1}|^{k-1} := f_n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

  for $\{u_n\}_{n\in\mathbb{N}} \subset C(\overline{\Omega})$ a decreasing sequence of k-convex solns.
- The PDE in (6) is proper as u_n does not appear explicitly.
- A strong comparison principle shows that $u_n < 0$ in Ω.
- Pass to the limit (along a subsequence) as $n \to +\infty$ using a uniform Hölder bound on $||u_n||_{C^{0,\gamma}(\overline{\Omega})}$ for each $n \in \mathbb{N}$ and some $\gamma \in (0, 1]$.

An auxilliary existence and regularity result

Theorem (Birindelli-P.'17)

Let Ω be strictly $(k-1)$-convex of class C^2 and let $f \in C(\Omega)$ be nonnegative and bounded. There exists a unique k-convex solution $u \in C(\overline{\Omega})$ of the Dirichlet problem

$$S_k(D^2u) = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

Moreover, $\forall\, \gamma \in (0,1)$ *there exists* $C = C(\Omega, \gamma, ||u||_\infty, ||f||_\infty)$ *s.t.*

$$|u(x) - u(x_0)| \leq C|x - x_0|^\gamma, \quad \forall\, x, x_0 \in \overline{\Omega}. \tag{7}$$

- Existence for $\partial\Omega \in C^2$ follows from the main theorem of [Cirant-P., PM'17] since strict $(k-1)$-convexity implies the needed strict $\overrightarrow{\Theta}_k$ and $\overrightarrow{\widetilde{\Theta}}_k$ convexity since

$$\overrightarrow{\Theta}_k = \Theta_k \quad \text{and} \quad \Theta_k \subset \widetilde{\Theta}_k := -\Theta_k^\circ.$$

Hölder regularity via Ishii-Lions technique (JDE'90)

Interior estimate: Fix $d_0 > 0$ for the boundary estimate (nice tubular neighbourhood) and work in $\dot{B}_\delta(x_0) \Subset \Omega$ with $2\delta < d_0$.

- Compare $u(x)$ with $v_{x_0}(x) := u(x_0) + C|x - x_0|^\gamma$ where $C\delta^\gamma \geq 2||u||_\infty$).
- $u \leq v_{x_0}$ on $\partial\dot{B}_\delta(x_0)$.
- $S_k(D^2 v_{x_0}(x)) = C^k \gamma^k C_{N,k} |x - x_0|^{k(\gamma-2)}[(\gamma - 2)k + N]$ and u is k-convex s.t. $S_k(D^2 u) = f \geq 0$
- Use Trudinger-Wang if $(\gamma - 2)k + N \leq 0$ ($k > N/2$) and u (sub)solution with $f \geq 0$ otherwise.

Boundary estimate: In Ω_{d_0} compare $u(x)$ with $v(x) := -Cd(x)^\gamma$ with suitable $C = C(d_0(\Omega), \gamma, ||u||_\infty, ||f||_\infty)$ so that v is k-convex with

$$S_k(D^2 v) > ||f||_\infty \geq S_k(D^2 v) \quad \text{in } \Omega_{d_0}.$$

Apply comparison for k-convex functions ($u = 0 = v$ on $\partial\Omega$).

The existence theorem for ψ_1

Theorem (Birindelli-P.'17)

Let Ω be a strictly $(k-1)$-convex domain of class C^2. If $\{u_n\}_{n\in\mathbb{N}}$ is the sequence of k-convex solutions to the iteration scheme (6) with $0 < \lambda_n \nearrow \lambda_1^-$, then the normalized sequence defined by

$$w_n := u_n/||u_n||_\infty$$

admits a subsequence which converges uniformly to an eigenfunction $\psi_1 < 0$ of S_k associated to λ_1^-.

- Follow Birindelli-Demengel scheme.
- Monotonicity from comparison principle for k-convex functions.
- Hölder regularity above is the key.

Concluding remarks

Where do we go from here?

- Elements of a Fredholm theory and eigenvalue estimates.
- Anti-maximum principles.
- Symmetry of solutions.
- Non variational perturbations of $S_k(D^2u)$ like

$$S_k(D^2u + M(x)) + \lambda u|u|^{k-1} \quad \textit{with} \quad M \in UC(\Omega; \mathcal{S}(N))$$

  considered by [Cirant-P.] and [Y.Y. Li, CPAM'90].
- The "general case" $F(x, D^2u) + \lambda u|u|^{k-1} = 0$ with $F(x, A)$
  – continuous and $F(x, 0) = 0$;
  – homogeneous of degree k in $A \in \mathcal{S}(N)$;
  – $F(x, A)$ increasing in A along $\Theta : \Omega \to \mathcal{E} \subset \mathcal{S}(N)$ a uniformly continuous elliptic map;
  – uniformly continuous in $x \in \Omega$.

Thanks to one and all!!