

# Optimal Control Applied to Elliptic and Parabolic PDEs with Biological Applications

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- ▶ 1. Optimal Control Applied to Native-Invasive Species Competition via a PDE Model
- ▶ 2. Optimal Control of the Growth Coefficient on a Steady State Population Model

Adjust control(s) in a dynamic system to achieve a goal (objective functional)

System:

- ▶ Discrete/Difference Equations
- ▶ Ordinary Differential Equations
- ▶ Partial Differential Equations
- ▶ Stochastic Differential Equations

## 1. Optimal Control Applied to Native-Invasive Species Competition via a PDE Model

# Motivating Example: Native - Invasive Species Competition

## Cottonwoods (*Populus deltoides*)

- ▶ Native to Southwest
- ▶ Large trees, 60-100 feet tall
- ▶ Dominate naturally only along rivers or other areas with surface water
- ▶ Rely on natural flooding of rivers



Figure: Cottonwoods

# Salt Cedar (*Tamarix*)

One of the most significant threats to global biodiversity is the invasion of plants.

- ▶ Introduced into New Mexico in 19th century
- ▶ Spreading shrub or trees, long-lived (50-100 yrs), 6 to 26 feet (2-8m) tall
- ▶ More tolerant of drought and fire than cottonwoods
- ▶ Great reproductive capability, 600,000 seeds/yr



Figure: Salt Cedar

# Salt Cedar Distribution



Figure: Salt Cedar Distribution

<http://www.columbia.edu/itc/cerc/>

Number of young cottonwoods along southwest rivers is declining

Restriction of overbank flooding gives salt cedars competitive advantage over cottonwoods

Salt cedars occupy nearly every drainage system in arid areas west of the Great Plains and have been reported in most states

Suggest that salt cedars may exclude cottonwoods

D. Kern, S. Lenhart, R. Miller, and J. Yong, Optimal control applied to native-invasive population dynamics, Journal of Biological Dynamics, 1:4 (2007) 413-426.

Our Model:

$N_1(x, t)$ – Native species (cottonwood)

$N_2(x, t)$ – Invasive species (salt cedar)

$u(x, t)$ – Control variable (flooding)

$d_{ij}^k(x, t)$ – Diffusion coefficients  $k = 1, 2$

$r_i^k(x, t)$ – Convective coefficients  $k = 1, 2$

$a_{ij}$ –Interaction coefficients indicating how species  $j$  affects species  $i$

# The Model: State System

$$(N_1)_t = \sum_{i,j=1}^n \left( d_{ij}^1(x, t)(N_1)_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^1(x, t)(N_1)_{x_i} \\ + (\theta_1(t, u(x, t)) - a_{11}N_1)N_1 - a_{12}N_1N_2,$$

$$(N_2)_t = \sum_{i,j=1}^n \left( d_{ij}^2(x, t)(N_2)_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^2(x, t)(N_2)_{x_i} \\ + (\theta_2(t, u(x, t)) - a_{22}N_2)N_2 - a_{21}N_1N_2,$$

$$N_1(x, 0) = N_{10}(x), \quad x \in \Omega,$$

$$N_2(x, 0) = N_{20}(x), \quad x \in \Omega,$$

$$N_1(x, t) = N_2(x, t) = 0, \quad \partial\Omega \times (0, T).$$

Control set

$$U = \{u \in L^\infty(\Omega \times \Gamma) : 0 \leq u(x, t) \leq M\},$$

where

$$\Gamma = \bigcup_{i=1}^T [\sigma_i, \tau_i].$$

The intrinsic growth rates are

$$\begin{aligned}\theta_1(t, u(x, t)) &= (a_1 u^2(x, t) + b_1 u(x, t)) \chi_\Gamma + c_1, \\ \theta_2(t, u(x, t)) &= (a_2 u^2(x, t) + b_2 u(x, t)) \chi_\Gamma + c_2\end{aligned}$$

# Optimal Control Problem

Goal: To maximize the native species at final time subject to balancing the minimization of the invasive species and the cost to implement the control.

Our Objective Functional is

$$J(u) = \int_{\Omega} \left( AN_1(x, T) - BN_2(x, T) \right) dx \\ - \iint_{\Omega \times \Gamma} \left( B_1 u(x, t) + B_2 u^2(x, t) \right) dxdt,$$

## Theorem

*Given  $u \in U$ , there exists a unique  $(N_1, N_2)$  in  $(V \times V) \cap L^\infty(Q)$  solving the state system.*

# Existence of an Optimal Control

Needed since maximizing sequence  $u_n$  converge weakly in  $L^2$  but  $u_n^2$  do not converge weakly in  $L^2$ .

## Theorem

Suppose there exists  $\kappa \geq 0$  such that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \kappa \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad B_1\kappa - B_2 \leq 0.$$

Then there exists an optimal control  $u^*$  in  $U$  with corresponding states  $N_1^*, N_2^*$  that maximizes the objective functional  $J(u)$ .

# Adjoint Equations

Define the adjoint system as

$$\mathbf{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} A \\ -B \end{pmatrix},$$

where

$$\mathbf{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -p_t - \sum_{i,j=1}^n \left( d_{ij}^1(x, t) p_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^1(x, t) p_{x_i} \\ -q_t - \sum_{i,j=1}^n \left( d_{ij}^2(x, t) q_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^2(x, t) q_{x_i} \end{pmatrix} + M^T \begin{pmatrix} p \\ q \end{pmatrix},$$

$$M = \begin{pmatrix} -(a_1 u^2 + b_1 u) \chi_\Gamma - c_1 + 2a_{11} N_1 + a_{12} N_2 & a_{12} N_1 \\ a_{21} N_2 & -(a_2 u^2 + b_2 u) \chi_\Gamma - c_2 + 2a_{22} N_2 + a_{21} N_1 \end{pmatrix}$$

$$p = 0, \text{ on } \partial\Omega \times (0, T), \quad p(x, T) = A, \quad x \in \Omega,$$

$$q = 0, \text{ on } \partial\Omega \times (0, T), \quad q(x, T) = -B, \quad x \in \Omega.$$

## Theorem

If there exists  $\kappa \geq 0$ , such that  $B_1\kappa - B_2 < 0$ ,  $a_i = \kappa b_i$ ,  $i = 1, 2$ , then given an optimal control  $u^*$  and the corresponding states  $N_1^*$  and  $N_2^*$ , there exist solutions  $p$  and  $q$  to the adjoint system.

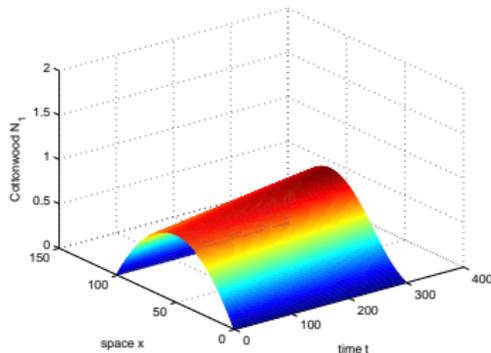
Moreover, let  $S = \{(x, t) \mid B_2 - a_1 N_1^* p - a_2 N_2^* q = 0\}$  and  $m(S)$  is the Lebesgue measure of  $S$ , then this optimal control  $u^*$  is characterized by the following:

- (1) if  $m(S) > 0$ , then  $u^* = M$  on  $S$ .
- (2) if  $m(S) = 0$ , then for  $(x, t) \notin S$ ,

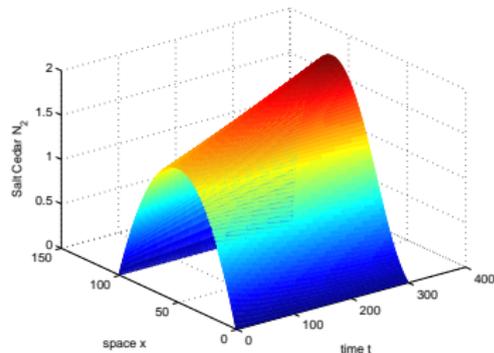
$$u^* = \min\left\{M, \max\left\{0, \frac{b_1 N_1^* p + b_2 N_2^* q - B_1}{2B_2 - 2a_1 N_1^* p - 2a_2 N_2^* q}\right\}\right\},$$

and it holds on  $\Omega \times \Gamma$  a.e.

# Numerical Simulation: 1-D

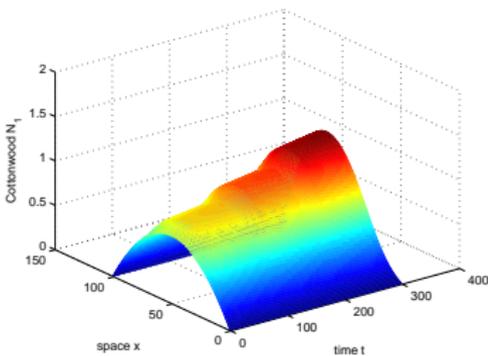


(a) Cottonwood - no control

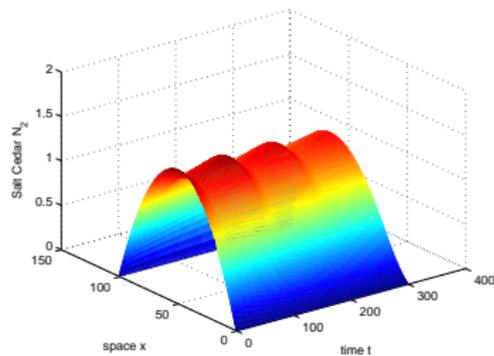


(b) Salt cedar - no control

Figure: Cottonwood and salt cedar without control,  $L=1$ ,  $T=3$

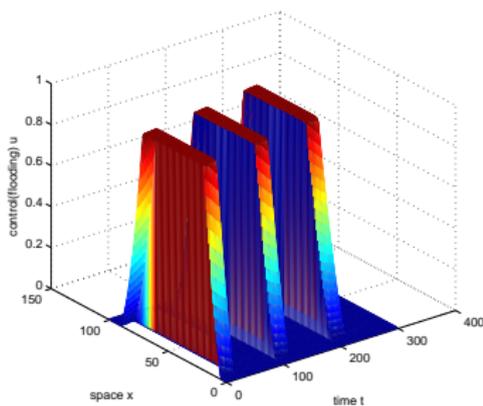


(a) cottonwood

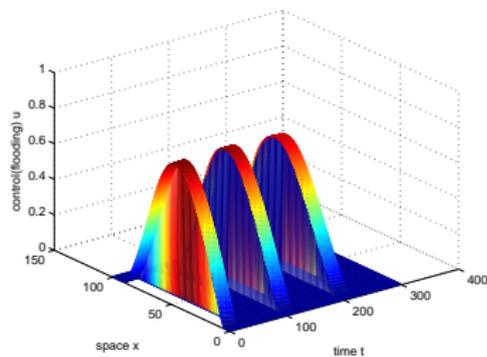


(b) salt cedar

Figure: Cottonwood - Salt cedar with  $B_2 = 5$ ,  $B_1 = 1$



(a) control:  $B_2 = 5$



(b) control:  $B_2 = 10$

Figure: Control - flood:  $B_2 = 5, 10$ , fix  $B_1 = 1$

# Conclusions

- ▶ Optimal control theory can be an appropriate tool for designing the intervention strategy of the invasive-native species interaction
- ▶ Proved existence of the optimal control when the control is quadratic in the growth function in the PDE system under certain conditions on the coefficients
- ▶ Gave numerical examples for different parameter values that can help natural resource managers to apply the most appropriate and cost-effective control methods to the invasive-native species scenario

## 2. Optimal Control of Growth Coefficient on a Steady State Population Model

- ▶ The Model
- ▶ Existence of an Optimal Control
- ▶ Necessary Conditions of Optimal Control
- ▶ Uniqueness of the Optimal Control
- ▶ Numerical Results

$$\begin{cases} -\lambda \Delta u = mu - u^2, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

$u(x)$  : density of the species at location  $x$

$\lambda$  : dispersal rate, a positive constant

$m(x)$  : intrinsic growth rate, measures the availability of the resources

Note: for every  $m \in U$ , (1) has a unique positive solution.

# The Goal

Given  $0 < \delta < |\Omega|$ , define the control set

$$U = \{m \in L^\infty(\Omega) \mid 0 \leq m(x) \leq 1, \int_{\Omega} m(x) dx = \delta\}.$$

We seek to find  $m^* \in U$ , such that  $J(m^*) = \max_m J(m)$ , where

$$J(m) = \int_{\Omega} [u - (Bm^2)] dx, \quad (2)$$

# Existence of an Optimal Control

## Theorem

There exists an optimal control  $m^* \in U$  maximizing the objective functional  $J(m)$ .

## Idea:

- ▶ Use maximizing sequence argument
- ▶ Need *a priori* estimates for  $u \in H^1(\Omega)$  and  $m \in L^2(\Omega)$
- ▶ Show weak convergence of

$$\lambda \int_{\Omega} \nabla u^n \cdot \nabla v \, dx = \int_{\Omega} m^n u^n v - (u^n)^2 v \, dx, \quad \forall v \in H^1(\Omega). \quad (3)$$

## Lemma

Assume for  $m \in U$ , the mapping  $m \in U \rightarrow u(m, \lambda)$  is differentiable at  $m$  in the following sense: there exists  $\psi \in H^1(\Omega)$ , such that

$$\frac{u(m + \epsilon l, \lambda) - u(m, \lambda)}{\epsilon} \rightharpoonup \psi \text{ weakly in } H^1(\Omega) \text{ as } \epsilon \rightarrow 0,$$

where  $m + \epsilon l \in U$ ,  $l \in L^\infty(\Omega)$ . And the sensitivity  $\psi = \psi(m, l, \lambda)$  satisfies

$$\begin{aligned} -\lambda \Delta \psi &= m\psi - 2u\psi + lu, \\ \frac{\partial \psi}{\partial n} &= 0. \end{aligned} \tag{4}$$

To handle

$$\int_{\Omega} m \, dx = \delta,$$

we introduce an extra state variable  $w$ , denoted by  $w(m)$ ,

$$\begin{cases} \Delta w = m, & x \in \Omega, \\ \frac{\partial w}{\partial n} = \frac{\delta}{|\partial\Omega|}, & x \in \partial\Omega. \end{cases} \quad (5)$$

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**Sensitivity 2:**

$$\begin{cases} \Delta \psi_2 = l, & x \in \Omega \\ \frac{\partial \psi_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \quad (6)$$

## Theorem

Given an optimal control  $m$  and corresponding states,  $u, w$ , there exists a solution  $p_1, p_2$  to the adjoint system, with  $p_1 \in H^2(\Omega)$  and  $p_2$  constant, satisfying

$$\begin{cases} -\lambda \Delta p_1 - (m - 2u)p_1 = 1, & x \in \Omega, \\ \frac{\partial p_1}{\partial n} = 0, & x \in \partial\Omega, \\ \Delta p_2 = 0, & x \in \Omega, \\ \frac{\partial p_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Furthermore, we have

$$m^* = \min\left\{\max\left\{0, \frac{up_1 + p_2}{2B}\right\}, 1\right\}.$$

## Theorem

If  $B > \frac{|\Omega|}{2\delta}$ , then  $m^* = u^* = \frac{\delta}{|\Omega|}$  is an optimal control and corresponding state.

But, if  $B$  is small enough, the constant solution  $m = u = \frac{\delta}{|\Omega|}$  is no longer an optimal control.

# Uniqueness of the Optimal Control

## Theorem

*For  $B$  sufficiently large, the optimal control maximizing  $J(m)$  is unique.*

## Idea:

For  $m, l \in U$  and  $0 \leq \epsilon \leq 1$ , we will show that

$$g(\epsilon) = J(\epsilon l + (1 - \epsilon)m) = J(m + \epsilon(l - m))$$

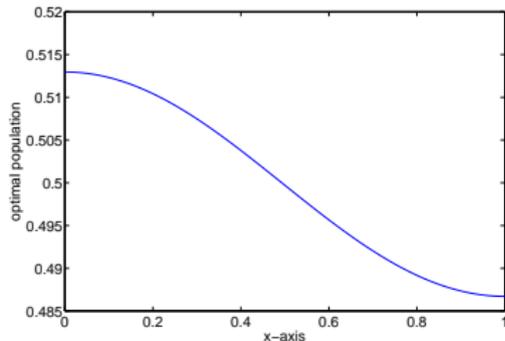
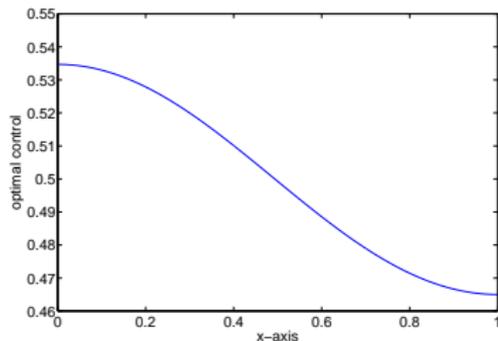
is strictly concave, which implies the uniqueness of the optimal control.

$$\begin{cases} -\lambda \Delta u = mu - u^2, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \Delta w = m, & x \in \Omega, \\ \frac{\partial w}{\partial n} = \frac{\delta}{|\partial\Omega|}, & x \in \partial\Omega. \end{cases} \quad (7)$$

$$\begin{cases} -\lambda \Delta p_1 - (m - 2u)p_1 = 1, & x \in \Omega, \\ \frac{\partial p_1}{\partial n} = 0, & x \in \partial\Omega, \\ \Delta p_2 = 0, & x \in \Omega, \\ \frac{\partial p_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

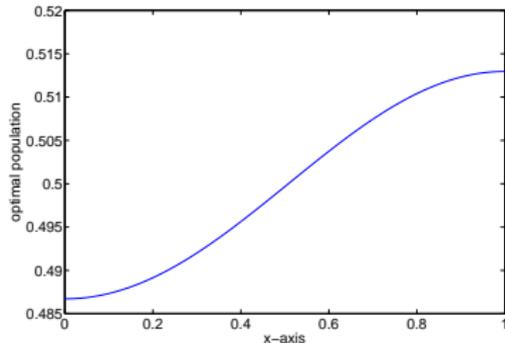
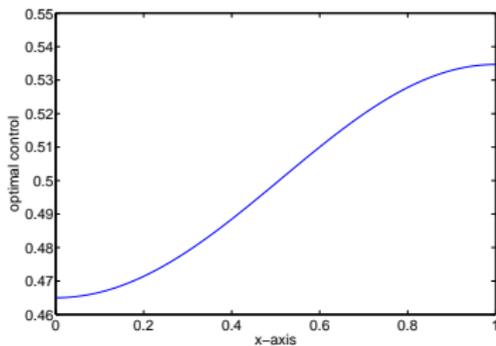
$$m^* = \min\left\{\max\left\{0, \frac{up_1 + p_2}{2B}\right\}, 1\right\}.$$

# Nonuniqueness for $B < \frac{|\Omega|}{2\delta} = 1$



**Figure:** An Optimal Control and Corresponding State in 1D for  $\lambda = 0.1$ ,  $B = 0.5$

# Nonuniqueness for $B < \frac{|\Omega|}{2\delta} = 1$



**Figure:** Another Optimal Control and Corresponding State in 1D for  $\lambda = 0.1$ ,  $B = 0.5$

$$B = 0.001$$

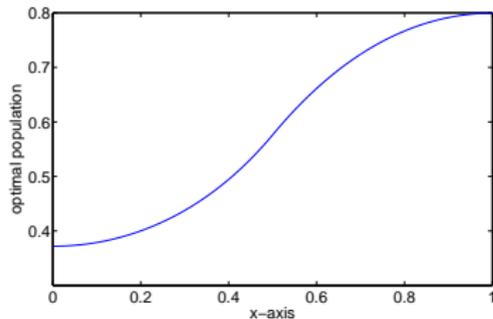
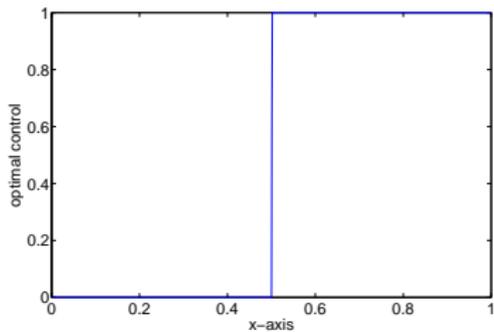


Figure: An Optimal Control and Corresponding State in 1D for  $\lambda = 0.1$ ,  $B = 0.001$

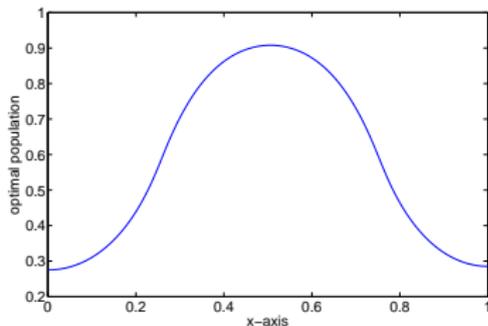
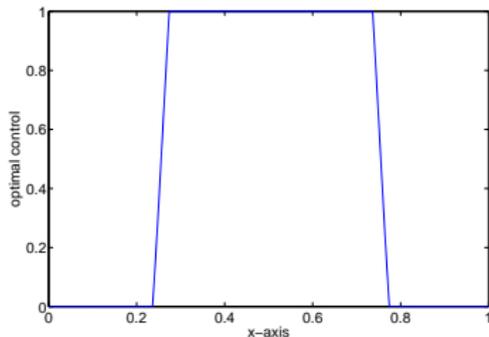


Figure: Optimal Control and State in 1D for  $\lambda = 0.011$ ,  $B = 0.1$

## 2-D, Nonuniqueness for $B$ small, $\delta = 0.5$

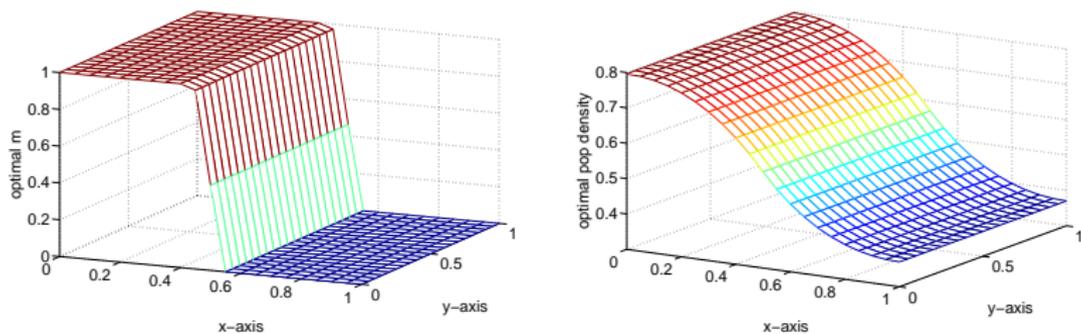


Figure: An Optimal Control and State in 2D for  $\lambda = 0.1$ ,  $B = 0.1$ ,  $\delta = 0.5$

$$\delta = 0.1$$

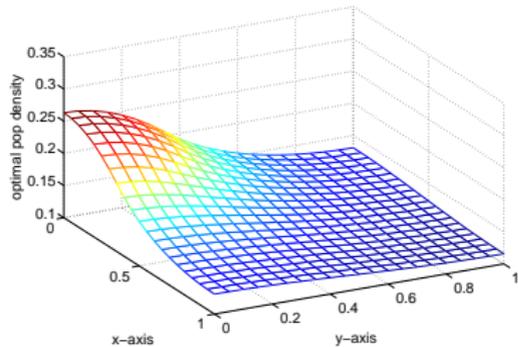
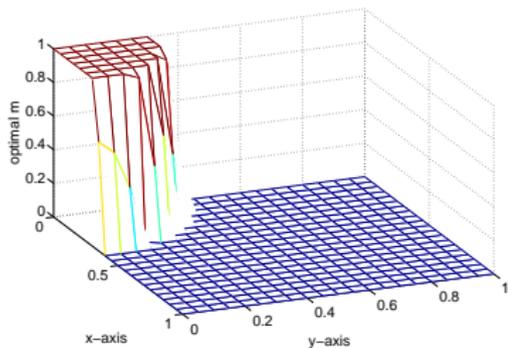


Figure: An Optimal Control and State in 2D for  $\lambda = 0.1$ ,  $B = 0.1$ ,  $\delta = 0.1$

$$\delta = 0.9$$

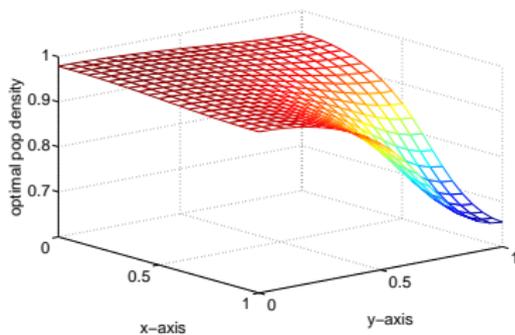
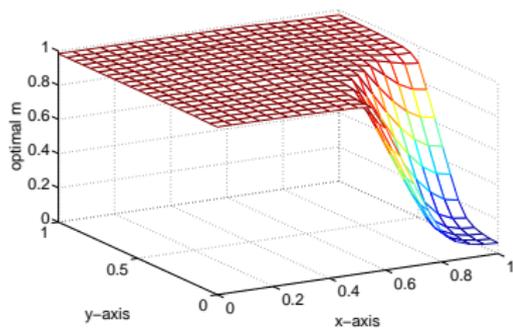


Figure: An Optimal Control and State in 2D for  $\lambda = 0.1$ ,  $B = 0.1$ ,  $\delta = 0.9$

- ▶ Studied the control problem of maximizing the total payoff in the conservation of a single species with a fixed amount of resource.
- ▶ The existence of an optimal control is established and uniqueness and characterization of the optimal control is investigated.
- ▶ Some necessary conditions are provided for the characterization of the optimal control. We introduced an extra state variable to handle the integral constraint for the control to get the characterization in the multi-dimensional space. For 1D case, we present a simpler version of this technique.

- ▶ For 1D habitat, the characterization of the optimal control depends on the choice of the diffusion rate  $\lambda$ . For small  $\lambda$  the optimal control seems to be symmetric, and so may be unique.
- ▶ When  $\lambda$  is suitably larger, where the optimal control is not unique and non-symmetric.
- ▶ For rectangular domains, the shape of the optimal control depends on the choice of the amount of total resources,  $\delta$ . When the amount is small, the optimal control is concentrated at one of the corners of the rectangle.
- ▶ When the amount of total resources is suitably large, for which the optimal control concentrates at a boundary edge of the rectangle.

- ▶ When  $B$  is small, numerical simulations indicate that the optimal control is close to “bang-bang.” Can one show that the optimal control is exactly “bang-bang” for  $B = 0$ ?
- ▶ It was shown that the total population size  $\int_{\Omega} u \, dx$  as a function of the diffusion rate  $\lambda$ , is not monotone. In fact,  $\int_{\Omega} u \, dx$  is exactly minimized at  $\lambda = 0$  and  $\lambda = \infty$  and maximized at some value of  $\lambda = \lambda^* \in (0, \infty)$ . From our numerical simulations we think that there exists some connection between  $\lambda^*$  and the symmetry of the optimal control for 1D habitat.

# Open Questions Cont'd

- ▶ For a high-dimensional habitat, we see that the profile of the optimal control may depend on the amount of total resources.
- ▶ Will the geometry of the boundary play some role in determining the optimal control? For example, when the total amount of resource is small, is it the best strategy to arrange resources near the most curved part of the boundary? Such questions seem to be rather challenging even for the simplest domains.

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