Numerical approximation of some inverse problems arising in Elastography

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joint work with

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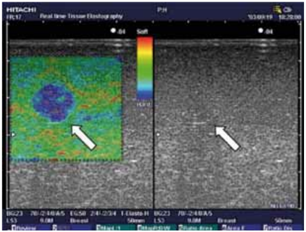
Outline

We consider: Geometric inverse problems for wave equation and Lamé system: **the unknown is the spatial domain**

- **Motivation: Elastrography** is noninvasive technique of imaging by ultrasound or MRI, allowing to detect elastic properties of a tissue in real time

- It is based on the fact that soft tissues are more deformable than stiff matter. When mechanical compression is applied, the stress in the tumor is less than into the surrounding tissue and the difference can be captured by images (a tumor tissue is 5–28 times stiffer than normal tissue, then the deformation after a mechanical action is smaller)



- It is used in various fields of Medicine (detection and description of bread, liver, prostate and other cancers, fibrosis, . . .)

(a) Direct problem:

Data: Ω, D, $T > 0$, $\varphi = \varphi(x,t)$ and $\gamma \subset \partial\Omega$
Result: the solution u

$$(1) \quad \begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad (\Omega \setminus \overline{D}) \times (0,T) \\ u = \varphi & \text{on} \quad \partial\Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial D \times (0,T) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{in} \quad \Omega \end{cases}$$

Information:

$$(2) \quad \frac{\partial u}{\partial n} := \widetilde{\alpha} \quad \text{on} \quad \gamma \times (0,T)$$

(b) Inverse problem:

(Partial) data: Ω, T, φ and $\gamma \subset \partial\Omega$
(Additional) information: $\widetilde{\alpha} = \widetilde{\alpha}(x,t)$
Goal: Find D such that the solution to (1) satisfies (2)

Lamé vibrations are much greater in one direction than the other, then neglecting small terms, one component of the displacement field approximately satisfies a wave equation...

(b) Inverse problem: given $\widetilde{\alpha} = \widetilde{\alpha}(x,t)$, $\varphi = \varphi(x,t)$, $\mu, \lambda > 0$, find D such that

$$\begin{cases} u_{tt} - \mu \Delta u + (\mu + \lambda)\nabla(\nabla \cdot u) = 0 & \text{in } \Omega \setminus \overline{D} \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

satisfies

$$\sigma(u) \cdot n = \Big(\mu(\nabla u + \nabla u^t) + \lambda(\nabla \cdot u)\mathbf{Id.}\Big) \cdot n := \widetilde{\alpha}(x,t) \quad \text{on} \quad \gamma \times (0,T)$$

Explanations:

- $u = (u_1, u_2, u_3)$ is the displacement vector
- $\sigma(u) \cdot n$ is normal stress
- Small displacements, hence linear elasticity
- The tissue is described by the Lamé coefficients λ and μ

Inverse problems: main questions

Uniqueness u^0 and u^1 solutions corresponding to D^0 and D^1 resp. and $\widetilde{\alpha}^0 \equiv \widetilde{\alpha}^1$ on $\gamma \times (0, T)$. Then, do we have $D^0 = D^1$?

- N-dimensional wave equation: OK
- N-dimensional isotropic Lamé system with constant coefficients: OK

Here we need only Unique Continuation, then no geometrical condition on γ

Stability Find an estimate of the "size" of $(D^0 \setminus D^1) \cup (D^1 \setminus D^0)$ in terms of the "size" of $\widetilde{\alpha}^0 - \widetilde{\alpha}^1$:

$$\text{Size}\Big((D^0 \setminus D^1) \cup (D^1 \setminus D^0)\Big) \leq CF\Big(\|\widetilde{\alpha}^0 - \widetilde{\alpha}^1\|_{A(\gamma\times(0,T))}\Big)$$

for all D^1 "close" to D^0, for some $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $F(s) \to 0$ as $s \to 0$, some suitable space $A(\gamma \times (0, T))$ and $C = C(D^0, \Omega, \gamma, T, \varphi_0)$

Reconstruction Devise iterative algorithms to compute D from $\widetilde{\alpha}$

1. Method 1: Optimization Problem, FEM, FreeFem++
2. Method 2: Mesh-less method, MFS, Optimization Problem, MATLAB

Assume: $N = 2$, $D = B(x_0, y_0; r)$

Inverse problem: given $\widetilde{\alpha} = \widetilde{\alpha}(x, t)$, find x_0, y_0, r such that $D \subset \Omega$ and the solution u to the Lamé system satisfies

$$\sigma[x_0, y_0; r] := \left(\mu(x)(\nabla u + \nabla u^t) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}\right) \cdot n = \widetilde{\alpha}(x, t) \quad \text{on } \gamma \times (0, T)$$

Constrained optimization problem (case of a ball)

Find x_0, y_0 and r such that $(x_0, y_0, r) \in X_b$ and

$$J(x_0, y_0, r) \leq J(x_0', y_0', r') \quad \forall (x_0', y_0', r') \in X_b$$

the function $J : X_b \mapsto \mathbb{R}$ is defined by

$$J(x_0, y_0, r) := \frac{1}{2} \int_0^T \|\sigma[x_0, y_0, r] - \widetilde{\alpha}\|^2_{H^{-1/2}(\gamma)} \, dt$$

$$X_b := \{ (x_0, y_0, r) \in \mathbb{R}^3 \ : \ \overline{B}(x_0, y_0; r) \subset \Omega \}$$

The problem formulation contains inequality constraints

$$\begin{cases} \text{Minimize } f(x) \\ \text{Subject to } x \in X_0 = \{x \in \mathbb{R}^m : \underline{x}_j \leq x_j \leq \overline{x}_j, \quad 1 \leq j \leq m\} \\ c_i(x) \geq 0,\ 1 \leq i \leq I \end{cases}$$

We need numerical solution of PDE: **FreeFem++** (ff-NLopt- AUGLAG)

Slack variables s_i: $c_i(x) \geq 0$ rewired as $c_i(x) - s_i = 0$, $s_i \geq 0$, $1 \leq i \leq I$

Optimization problem: augmented Lagrangian

$$\begin{cases} \text{Minimize } \mathcal{L}_A(x, \lambda^k; \mu_k) := f(x) - \sum_{i=1}^{I} \lambda_i^k (c_i(x) - s_i) + \frac{1}{2\mu_k} \sum_{i=1}^{I} (c_i(x) - s_i)^2 \\ \text{Subject to } x \in X_0;\ s_i \geq 0,\ 1 \leq i \leq I \end{cases}$$

λ_i^k : multipliers, μ_k : penalty parameters

Subsidiary unconstrained optimization algorithms (among others):

- **CRS2** is a gradient-free algorithm a version of Controlled Random Search (CRS) for global optimization
- **DIRECTNoScal** is variant of the DIviding RECTangles algorithm for global optimization

Test 1: $N = 2$, $\Omega = B(0;10)$, $D = B(x0, y0; r)$, $T = 5$

$u_{01} = 10x$, $u_{02} = 10y$, $u_{11} = 0$, $u_{12} = 0$, $\varphi_1 = 10x$, $\varphi_2 = 10y$

x0ini = 0, y0ini = 0, rini = 0.6

x0des = −3, y0des = 0, rdes = 0.4

NLopt (AUGLAG + DIRECTNoScal), N° Iter = 1001, FreeFem++

x0cal	=	−3.000224338
y0cal	=	−0.0005268693985
rcal	=	0.4000228624

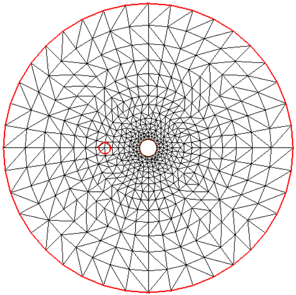


Figure: Test 1 – The initial geometrical configuration, the initial triangulation and the target *D*. Number of triangles: 992; number of vertices: 526.

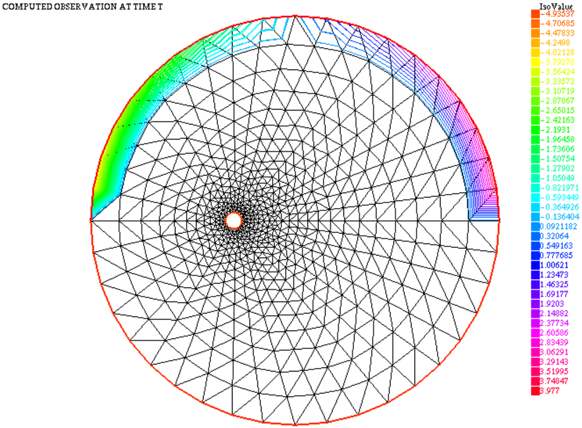


Figure: Computed center and radius

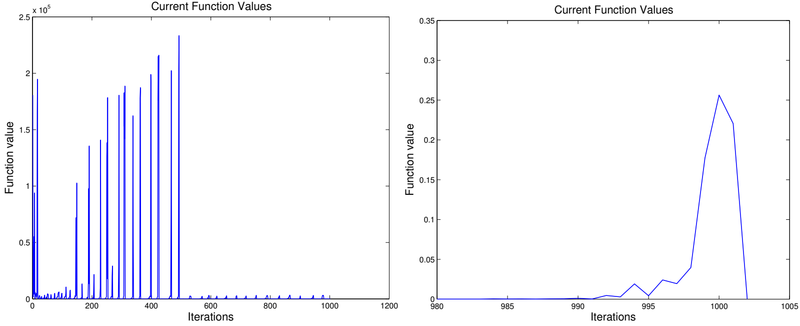


Figure: Test 1 – The evolution of the cost along the first 1001 iterations of `DIRECTNoScal` (Left) and a detail (Right).

Assume: $N = 2$, $D = E(x_0, y_0, \theta, a, b)$

Inverse problem: given $\tilde{\alpha} = \tilde{\alpha}(x, t)$, find x_0, y_0, θ, a, b such that $D \subset \Omega$ and the solution u to the Lamé system satisfies

$$\sigma[x_0, y_0, \theta, a, b] := \Big(\mu(x)(\nabla u + \nabla u^t) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}\Big) \cdot n = \tilde{\alpha}(x, t) \quad \text{on } \gamma \times (0, T)$$

Optimization problem: case of an ellipse

Find x_0, y_0 and θ and a, b such that $(x_0, y_0, \theta, a, b) \in X_e$ and

$$K(x_0, y_0, \theta, a, b) \leq K(x_0', y_0', \theta', a', b') \quad \forall\, (x_0', y_0', \theta', a', b') \in X_e,$$

the function $K : X_e \mapsto \mathbb{R}$ is defined by

$$K(x_0, y_0, \theta, a, b) := \frac{1}{2} \int_0^T \|\sigma[x_0, y_0, \theta, a, b] - \tilde{\alpha}\|^2_{H^{-1/2}(\gamma)}\, dt$$

$$X_e := \{\, (x_0, y_0, \theta, a, b) \in \mathbb{R}^5 \ :\ a, b > 0, \theta \in [0, \pi],\ \overline{E}(x_0, y_0, \theta, a, b) \subset \Omega \,\}$$

Test 2: $\Omega = B(0;10)$, $T = 5$, $u_{01} = 10x$, $u_{02} = 10y$, $u_{11} = 0$, $u_{12} = 0$, $\varphi_1 = 10x$, $\varphi_2 = 10y$

x0des=-3, y0des=0, sin(thetades)=0, ades=0.8, bdes=0.4

x0ini=-1, y0ini=-1, sin(thetaini)=0, aini=0.5, bini=0.5

NLopt (AUGLAG + DIRECTNoScal), N° Iter = 2002, FreeFem++:

x0cal	=-3.002591068
y0cal	=-3.001574963
sin(thetacal)	=0.00548696845
acal	=0.8036351166
bcal	=0.400617284

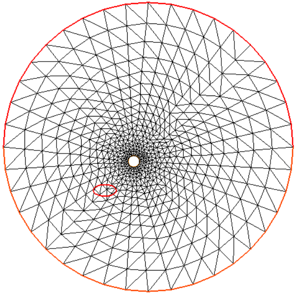


Figure: Test 2 – The initial geometrical configuration, the initial triangulation and the target *D*. Number of triangles: 1206; number of vertices: 633.

Test 3: $N = 3$, Ω is a sphere centered at $(0, 0, 0)$ and radius $R = 10$, $T = 5$,

$$u_{01} = 10x, \quad u_{02} = 10y \quad u_{03} = 10z, \quad u_{11} = 0, \quad u_{12} = 0 \quad u_{13} = 0$$
$$\varphi_1 = 10x, \quad \varphi_2 = 10y \quad \varphi_3 = 10z$$

```
x0des = -2,   y0des = -2,   z0des = -2,   rdes = 1
x0ini = 0,    y0ini = 0,    z0ini = 0,    rini = 0.6
```

NLopt (AUGLAG + DIRECTNoScal), FreeFem++:

```
x0cal  =  -1.981405274
y0cal  =  -2.225232904
z0cal  =  -2.148084171
 rcal  =  0.9504115226
```

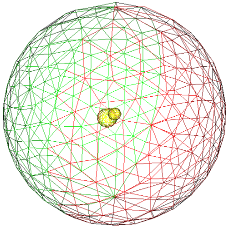


Figure: Test 3 – The initial mesh and the target D. Number of tetrahedra: 4023; number of vertices: 829; number of faces: 8406.

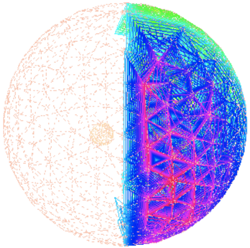


Figure: Test 3 – The computed first component of the observation at final time and the final mesh.

Similar results for the wave equation...

AD, E. Fernández-Cara, *Some geometric inverse problems for the linear wave equation*, Inverse Problems and Imaging, **9** (2015), no. 2, 371–393

AD, E. Fernández-Cara, *Some geometric inverse problems for the Lamé system with application in elastography*, submitted

MFS is meshless method developed for solving N dimensional wave equations (direct problem), based on:

1. Wave equation is considered as Poisson equation with time-dependent source term: $-\Delta u = -u_{tt}$
2. Houbolt finite difference, then Poisson problem
3. Method of particular solutions (MPS)- fundamental solutions (MFS):

$$\boxed{u(x) = u_P(x) + u_H(x) = \sum_{j=1}^{Nf} \beta_j F(|x - \eta_j|) + \sum_{k=1}^{Nb} \alpha_k G(|x - \xi_k|),} \quad \text{where}$$

- u_P is particular solution of nonnhomogeneous equation
- u_H is homogeneous solution of Laplace equation
- F is integrated radial basis function: $\Delta F(r) = f(r)$, $f(r)$ is radial basis func.
- β_j coefficients of the basis function, α_k intensity of the source points
- G is fundamental solution of the Laplace equation
- Nf is number of the field points
- Nb is number of the source points

4. PDE+BC+IC $\Rightarrow$ resolution of linear system: $\boxed{M \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = Z}$ for β_j, α_k

For Inverse Problem: for simplicity, we consider Poisson equation. . .

Inverse problem: (Partial) data: Ω, T, φ, f and $\gamma \subset \partial\Omega$
(Additional) information: $\widetilde{\alpha} = \widetilde{\alpha}(x)$
Goal: Find D such that the solution u to (3) satisfies (4)

$$(3) \qquad \begin{cases} -\Delta u + au = f & \text{in} \quad \Omega \setminus \overline{D} \qquad \text{(PDE)} \\ u = \varphi & \text{on} \quad \partial\Omega \qquad \text{(BC on } \partial\Omega \text{)} \\ u = 0 & \text{on} \quad \partial D \qquad \text{(BC on } \partial D \text{)} \end{cases}$$

$$(4) \qquad \frac{\partial u}{\partial n} = \widetilde{\alpha} \quad \text{on} \quad \gamma \quad \text{(BC on } \gamma \text{)}$$

We take

$$u(x) = u_P(x) + u_H(x) = \sum_{j=1}^{Nf} \beta_j F(|x - \eta_j|) + \sum_{k=1}^{Nb} \alpha_k G(|x - \xi_k|)$$

Assume: $\Omega = B(0, 10)$, D is a ball: $D = B(x_0, y_0; rho)$

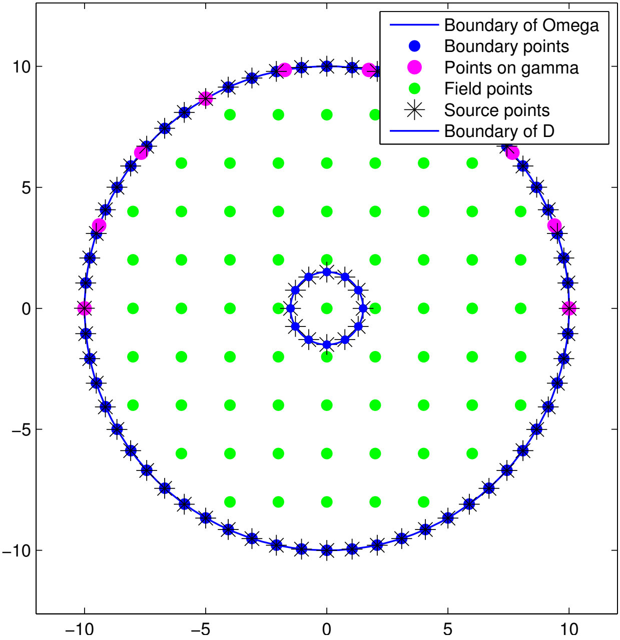
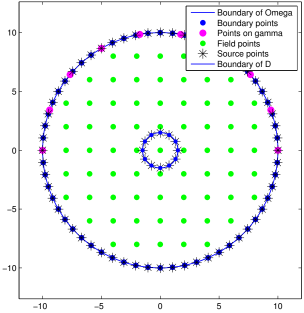


Figure: Initial configuration

$\Omega = B(0,10), \quad \varphi(x) = 10x, \quad f(x) = 0, \quad a = 1$

- $Nb0 = 60$: Nb of boundary points on $\partial\Omega$
- $Nb00 = 10$: Nb. of boundary points on γ
- $Nd = 12$: Nb. of boundary points on ∂D
- $Nb = Nb0 + Nd$: Nb. of source points




```
x0ini = 0,   y0ini = 0,   rho0ini = 1.5
x0des = -6,  y0des = 0,   rhodes = 1.2
```

$$u(x) = u_P(x) + u_H(x) = \sum_{j=1}^{Nf} \beta_j F(|x-\eta_j|) + \sum_{k=1}^{Nb} \alpha_k G(|x-\xi_k|),$$

PDE + BC's on $\partial\Omega$, ∂D, γ yield to nonlinear system of equations

$$M(x0, y0, rho) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = Z \quad \Rightarrow$$ Last square formulation **fmincon**, MATLAB...

```
x0cal = -5.999991,  y0cal = 0,  rhocal = 1.199999
```

`x0cal = -5.999991,  y0cal = 0,  rhocal = 1.199999`

`x0des = -6,  y0des = 0,  rhodes = 1.2`

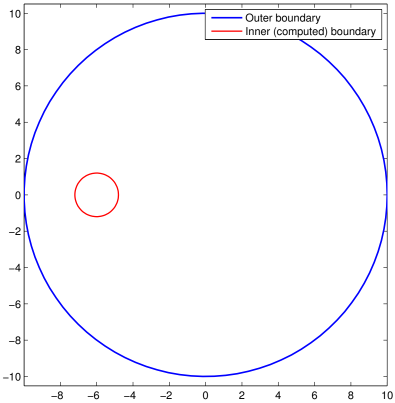


Figure: Desired configuration

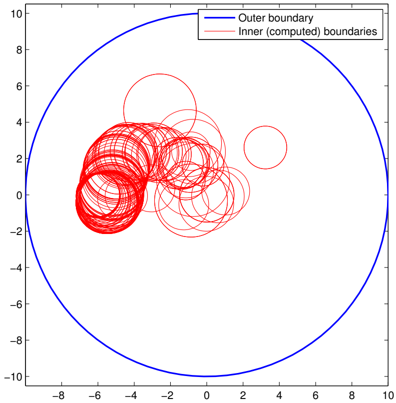


Figure: Computed domain and iterations

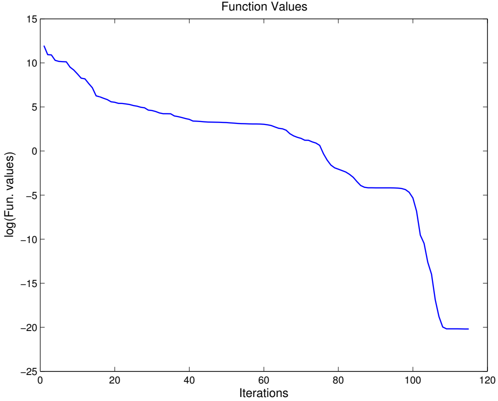


Figure: The evolution of the cost along 114 iterations of `fmincon`

Joint work with:

AD, E. Fernández-Cara, J. Rocha de Faria, P. de Carvalho

1. With the Method 2: Formulation + Numerical results for wave equation
2. With the Method 1: numerical results for general elasticity system:

$$\begin{cases} -u_{tt} - \nabla \cdot \sigma(u) = 0 & \text{in } \Omega \setminus \overline{D} \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

$$\sigma_{kl}(u) = \sum_{i,j,k,l=1}^{N} a_{ijkl}\varepsilon_{ij}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

$$a_{ijkl} = a_{klij} = a_{ijlk} \in L^\infty(\Omega) \quad 1 \leq i,j,k,l \leq 3$$

$$\sum_{i,j,k,l=1}^{N} a_{ijkl}\xi_{ij}\xi_{kl} \geq \alpha \sum_{i,j=1}^{N} |\xi_{ij}|^2, \quad \forall \{\xi_{ij}\} \in \mathbb{R}^{N\times N}_{sym}$$

3. Ellipsoids, other more complicated geometries ?
4. Internal observations ?

- AD, E. Fernández-Cara, *Some geometric inverse problems for the linear wave equation*, Inverse Problems and Imaging, **9** (2015), no. 2, 371–393
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