Randomized Model Order Reduction

Alessandro Alla
work in collaboration with J. Nathan Kutz

Data-Driven Methods for Reduced-Order Modeling and Stochastic Partial Differential Equations

Banff, January 30, 2016

Problem Settings

Dynamical System

$$\begin{cases} \mathbf{M}\,\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{f}(t, \mathbf{y}(t)), \quad t \in (0, T], \\ \mathbf{y}(0) = \mathbf{y_0}. \end{cases}$$

Assumptions

- $\mathbf{y_0} \in \mathbb{R}^n$ is a given initial data,
- $\mathbf{M}, \mathbf{A} \in \mathbb{R}^{n \times n}$ given matrices,
- $\mathbf{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function in both arguments and locally Lipschitz-type with respect to the second variable.

WARNING: High dimensional problems are computationally expensive.

Introduction

Low-rank approximation requires:

- Solutions of the original high-dimensional system (snapshots),
- Dimensionality-reduction produced by SVD,
- Galerkin projection of the dynamics on the low-rank subspace.

WARNING:

Offline stages are exceptionally expensive, but enable the (cheap) online stage to potentially run in real time.

GOAL: To improve the efficiency of the offline stage

Randomized techniques attempt to construct low-rank matrix decompositions, fast and accurate approximations of QR and SVD.

Outline

1. Model Order Reduction
   - Proper Orthogonal Decomposition
   - Discrete Empirical Interpolation Method
   - Dynamic Mode Decomposition
   - Coupling POD and DMD methods
2. Randomized Linear Algebra in Model Order Reduction
   - Compressed Model Order Reduction Techniques
   - Compressed POD
   - Compressive Sampling DMD
3. Numerical Tests

Outline

Proper Orthogonal Decomposition and SVD

Proper Orthogonal Decomposition (POD), [L. Sirovich '87]

- Simulate at different time instances,
- Take snapshots of the state,
- Perform POD (Proper Orthogonal Decomposition) using SVD (Singular Value Decomposition),
- Use the POD basis functions as (non local) FEM ansatz functions.

Proper Orthogonal Decomposition and SVD

Given **snapshots** $(y(t_0), \dots, y(t_n)) \in \mathbb{R}^m$

We look for an orthonormal basis $\{\psi_i\}_{i=1}^{\ell}$ in $\mathbb{R}^m$ with $\ell \ll \min\{n, m\}$ s.t.

$$J(\psi_1, \dots, \psi_\ell) = \sum_{j=1}^{n} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle \psi_i \right\|^2 = \sum_{i=\ell+1}^{d} \sigma_i^2$$

reaches a minimum where $\{\alpha_j\}_{j=1}^{n} \in \mathbb{R}^+$.

$$\min J(\psi_1, \dots, \psi_\ell) \quad s.t. \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

Singular Value Decomposition: $Y = \Psi \Sigma V^T$.

For $\ell \in \{1, \dots, d = rank(Y)\}$, $\{\psi_i\}_{i=1}^{\ell}$ are called **POD basis** of rank ℓ.

ERROR INDICATOR: $\mathcal{E}(\ell) = \dfrac{\sum_{i=1}^{\ell} \sigma_i}{\sum_{i=1}^{d} \sigma_i}$ with σ_i singular values of the SVD.

Reduced Order System

POD-Galerkin ansatz

$$\mathbf{y}(t) \approx \mathbf{\Psi}^{\text{POD}} \mathbf{y}^{\ell}(t), \quad \mathbf{\Psi}^{\text{POD}} \in \mathbb{R}^{n \times \ell}.$$

POD dynamical system

$$\begin{cases} \mathbf{M}^{\ell} \dot{\mathbf{y}}^{\ell}(t) = \mathbf{A}^{\ell} \mathbf{y}^{\ell}(t) + (\mathbf{\Psi}^{\text{POD}})^T f(t, \mathbf{\Psi}^{\text{POD}} \mathbf{y}^{\ell}(t)), \\ \mathbf{y}^{\ell}(0) = \mathbf{y}_{\mathbf{0}}^{\ell}. \end{cases}$$

Dimension of the entries

- $(\mathbf{M}^{\ell})_{ij} = \langle \mathbf{M}\psi_i, \psi_j \rangle \in \mathbb{R}^{\ell \times \ell}$,
- $(\mathbf{A}^{\ell})_{ij} = \langle \mathbf{A}\psi_i, \psi_j \rangle \in \mathbb{R}^{\ell \times \ell}$,
- $\mathbf{y}_{\mathbf{0}}^{\ell} = (\mathbf{\Psi}^{\text{POD}})^T \mathbf{y}_0 \in \mathbb{R}^{\ell}$.

Discrete Empirical Interpolation Method

Problem:

Reduction of the nonlinearity is **NOT** independent from *n*:

$$\mathbf{F}(t, \mathbf{y}^{\ell}(t)) := (\mathbf{\Psi}^{\text{POD}})^T \mathbf{f}(t, \mathbf{\Psi}^{\text{POD}} \mathbf{y}^{\ell}(t)) = \langle \mathbf{f}(t, \mathbf{y}(t)), \mathbf{\Psi}^{\text{POD}} \rangle.$$

IDEA:

Do not evaluate the nonlinearity everywhere, but select the *most important* points via the **greedy** procedure.

References

- M. Barrault, Y. Maday, N.C. Nguyen, A.T. Patera, *An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations,* 2004.
- S. Chatarantabut, D. Sorensen, *Nonlinear Model Reduction via Discrete Empirical Interpolation,* 2010.

Discrete Empirical Interpolation Method

- Compute $\mathbf{y}(t_j)$ from the dynamical system,
- Evaluate $\mathbf{f}(t_j, \mathbf{y}(t_j))$,
- $\mathbf{U} \in \mathbb{R}^{n \times k}$ the POD basis function of rank k of the nonlinear part.
- The DEIM approximation of $\mathbf{f}(t, \mathbf{y}(t))$ is as follows

$$\mathbf{f}^{\text{DEIM}}(t, \mathbf{y}^{\text{DEIM}}(t)) := \mathbf{U}(\mathbf{S}^T\mathbf{U})^{-1}\mathbf{f}(t, \mathbf{y}^{\text{DEIM}}(t))$$

where $\mathbf{S} \in \mathbb{R}^{n \times k}$ and $\mathbf{y}^{\text{DEIM}}(t) = \mathbf{S}^T \mathbf{\Psi}^{\text{POD}} \mathbf{y}^{\ell}(t) \in \mathbb{R}^k$.

Interpolation Points: Matrix **S** tells where evaluate the nonlinearity.

- LU decomposition with pivoting (Chatarantabut, Sorensen, 2010),
- QR decomposition with pivoting (Drmac, Gugercin, 2015).

Discrete Empirical Interpolation Method

Let us define $\mathbf{\Psi}^{\text{DEIM}} := \mathbf{U}(\mathbf{S}^T\mathbf{U})^{-1} \in \mathbb{R}^{n \times k}$.
The reduced nonlinearity may be expressed as:

$$(\mathbf{\Psi}^{\text{POD}})^T \mathbf{f}^{\text{DEIM}}(t, \mathbf{y}^{\text{DEIM}}(t)) = (\mathbf{\Psi}^{\text{POD}})^T \mathbf{\Psi}^{\text{DEIM}} \mathbf{f}(t, \mathbf{y}^{\text{DEIM}})$$

where we only select a small (**sparse**) number of rows of $\mathbf{\Psi}^{\text{POD}}\mathbf{y}^{\ell}(t)$.

Computational expenses

$$\mathbf{S}^T\mathbf{\Psi}^{\text{POD}} \in \mathbb{R}^{k \times \ell},\ (\mathbf{S}^T\mathbf{U})^{-1} \in \mathbb{R}^{k \times k} \text{ and } (\mathbf{\Psi}^{\text{POD}})^T \mathbf{\Psi}^{\text{DEIM}} \in \mathbb{R}^{\ell \times k}$$

Remark

Precomputed quantities are **independent** of the full dimension n.

Discrete Empirical Interpolation Method

POD dynamical system

$$\begin{cases} \mathbf{M}^\ell \dot{\mathbf{y}}^\ell(t) = \mathbf{A}^\ell \mathbf{y}^\ell(t) + (\mathbf{\Psi}^{\text{POD}})^T f(t, \mathbf{\Psi}^{\text{POD}} \mathbf{y}^\ell(t)), \\ \mathbf{y}^\ell(0) = \mathbf{y}_0^\ell \end{cases}$$

POD-DEIM dynamical system

$$\begin{cases} \mathbf{M}^\ell \dot{\mathbf{y}}^\ell(t) = \mathbf{A}^\ell \mathbf{y}^\ell(t) + (\mathbf{\Psi}^{\text{POD}})^T \mathbf{\Psi}^{\text{DEIM}} \mathbf{f}(t, \mathbf{y}^{\text{DEIM}}) \\ \mathbf{y}^\ell(0) = \mathbf{y}_0^\ell. \end{cases}$$

low-rank approximation of the nonlinear term.

Error Estimation

$$\|\mathbf{f} - \mathbf{f}^{\text{DEIM}}\|_2 \leq c\|(\mathbf{I} - \mathbf{U}\mathbf{U}^T) f\|_2, \quad c = \|(\mathbf{S}^T \mathbf{U})^{-1}\|_2$$

with different performances depending on the matrix **S.**

Introduction to DMD

DMD is an equation-free, data-driven method capable of providing accurate model for complex system, and short-time future estimates of such a systems.

It traces its origins to pioneering work of **Bernard Koopman** in 1931. Koopman theory is a dynamical systems tool that provides information about a nonlinear dynamical system via an associated infinite-dimensional linear system.

DMD method was proposed as a data-driven algorithm for modeling complex flows as a special case of Koopman theory.
(Brunton, Kutz, Mezic, Noack, Rowley, Schmid, Tu, . . .)

Dynamic Mode Decomposition

Dynamic Mode Decomposition

Suppose we have a dynamical system and compute snapshots $\{(y(t_0), \ldots, y(t_m)\}$ and two sets of data

$$\mathbf{Y} = \begin{bmatrix} | & | & & | \\ \mathbf{y}(t_0) & \mathbf{y}(t_1) & \cdots & \mathbf{y}(t_{m-1}) \\ | & | & & | \end{bmatrix}, \quad \mathbf{Y}' = \begin{bmatrix} | & | & & | \\ \mathbf{y}(t_1) & \mathbf{y}(t_2) & \cdots & \mathbf{y}(t_m) \\ | & | & & | \end{bmatrix}$$

with $\mathbf{y}(t_j)$ an initial condition of the dynamical system and $\mathbf{y}(t_{j+1})$ its corresponding output $\Rightarrow \mathbf{Y}' = \mathbf{A_Y}\mathbf{Y}$ with $\mathbf{A_Y} \in \mathbb{R}^{n\times n}$ unknown.

The DMD modes are eigenvectors of

$$\mathbf{A_y} = \mathbf{Y}'\mathbf{Y}^\dagger$$

where $\dagger$ denotes the Moore-Penrose pseudoinverse.

Dynamic Mode Decomposition

- $\mathbf{A_y}$ is a finite dimensional approximation of the Koopman operator for a linear observable.
- The definition of DMD produces a regression procedure whereby the data snapshots in time are used to produce the **best-fit linear dynamical system** for the data $\mathbf{Y}$.

The DMD procedure constructs the proxy, approximate linear evolution

$$\frac{d\tilde{\mathbf{y}}}{dt} = \mathbf{A_y}\tilde{\mathbf{y}}$$

with $\tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0$ and whose solution is: $\tilde{\mathbf{y}}(t) = \sum_{i=1}^{n} b_i \psi_i \exp(\omega_i t)$, ψ_i and ω_i are the eigenfunctions and eigenvalues of the matrix $\mathbf{A_y}$.

SOLUTION

DMD circumvents the eigendecomposition of $\mathbf{A_y}$ by considering a rank-reduced representation in terms of a POD-projected matrix $\tilde{\mathbf{A}}_\mathbf{y}$.

Dynamic Mode Decomposition

DMD algorithm

Require: Snapshots $\{\mathbf{y}(t_0), \dots, \mathbf{y}(t_m)\}$,

1: Set $\mathbf{Y} = [\mathbf{y}(t_0), \dots, \mathbf{y}(t_{m-1})]$ and $\mathbf{Y}' = [\mathbf{y}(t_1), \dots, \mathbf{y}(t_m)]$,

2: Compute the **(reduced)** SVD of **Y**, $\mathbf{Y} = \mathbf{U\Sigma V}^T$

3: Define $\tilde{\mathbf{A}}_\mathbf{y} := \mathbf{U}^*\mathbf{Y}'\mathbf{V}\mathbf{\Sigma}^{-1}$

4: Compute eigenvalues and eigenvectors of $\tilde{\mathbf{A}}_\mathbf{y}\mathbf{W} = \mathbf{W\Lambda}$.

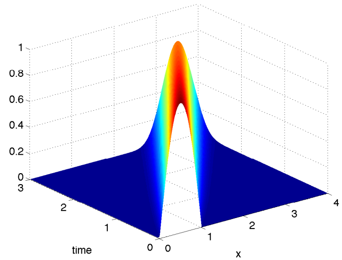
5: Set $\mathbf{\Psi}^{\text{DMD}} = \mathbf{Y}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{W}$

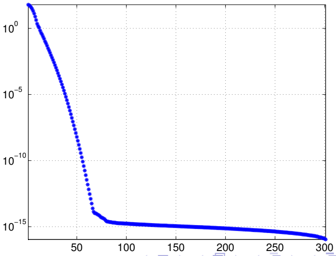
Example: DMD-Galerkin approximation

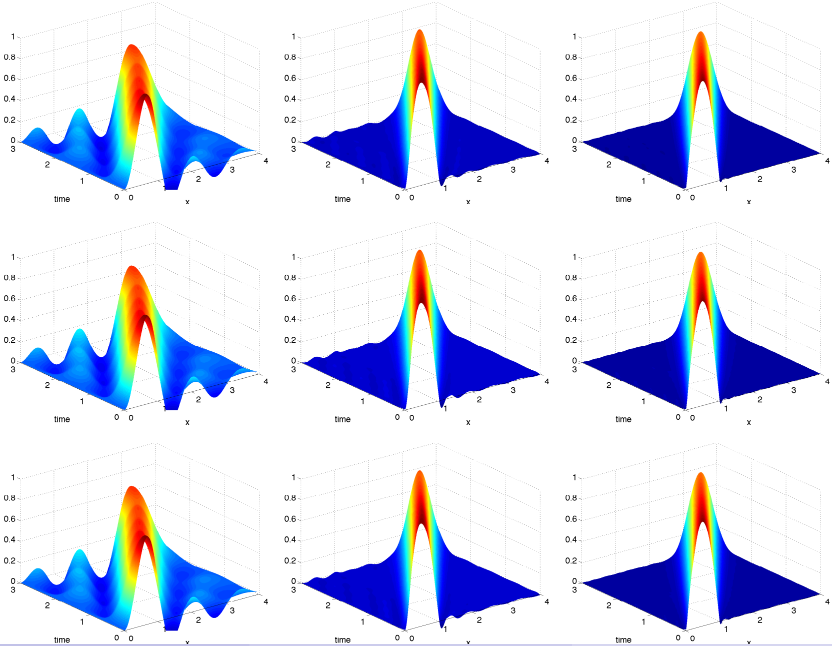
$$\begin{aligned} y_t(x,t) + y_x(x,t) &= 0 & (x,t) &\in [0,4]\times[0,3], \\ y(x,0) &= y_0(x) & x &\in [0,4], \\ y(0,t) = 0 &= y(4,t) & t &\in [0,T], \end{aligned}$$

where $y_0(x) = \sin(\pi x)$ if $0 \leq x \leq 1$ and 0 elsewhere.

DMD Ansatz: $\mathbf{y}(t) \approx \mathbf{\Psi}^{\text{DMD}}\mathbf{y}^{\text{DMD}}(t)$.






time
x

Example: DMD-Galerkin approximation

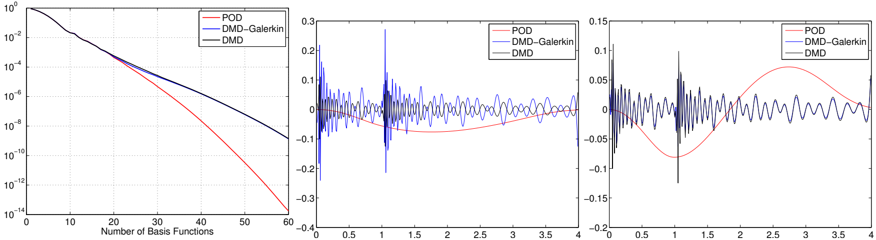


Figure: Error analysis with respect to the Frobenius norm (left), first mode (middle), second mode (right).

POD-DMD method

MAIN IDEA

The evaluation of the nonlinearity is the **most expensive** part in model order reduction. We aim faster approximation of the nonlinear term.

We need snapshots!

- $\{\mathbf{y}(t_0), \dots, \mathbf{y}(t_m)\}$, to compute the POD basis functions,
- $\{\mathbf{f}(t_0, \mathbf{y}(t_0)), \dots, \mathbf{f}(t_m, \mathbf{y}(t_m))\}$ to compute the DMD basis functions.

POD-DMD method **(NO EVALUATION OF THE NONLINEARITY)** (A., Kutz, 2016)

$$\begin{cases} \mathbf{M}^\ell \dot{\mathbf{y}}^\ell(t) = \mathbf{A}^\ell \mathbf{y}^\ell(t) + (\mathbf{\Psi}^{\text{POD}})^T \mathbf{\Psi}^{\text{DMD}} diag(e^{\omega^{\text{DMD}} t})\mathbf{b}, \\ \mathbf{y}^\ell(0) = \mathbf{y_0}^\ell. \end{cases}$$

Example: Semi-Linear Parabolic Equation

$$
\begin{aligned}
y_t - \theta \Delta y + \mu(y - y^3) &= 0 && (x,t) \in \Omega \times [0,T], \\
y(x,0) &= y_0(x) && x \in \Omega, \\
y(\cdot, t) &= 0 && x \in \partial\Omega, t \in [0,T],
\end{aligned}
$$

Parameters:

$\Omega = [0,1] \times [0,1]$, $T = 3$,
$y_0(x) = 0.1$ if $0.1 \le x_1 x_2 \le 0.6$ and 0 elsewhere.

Example: Semi-Linear Parabolic Equation

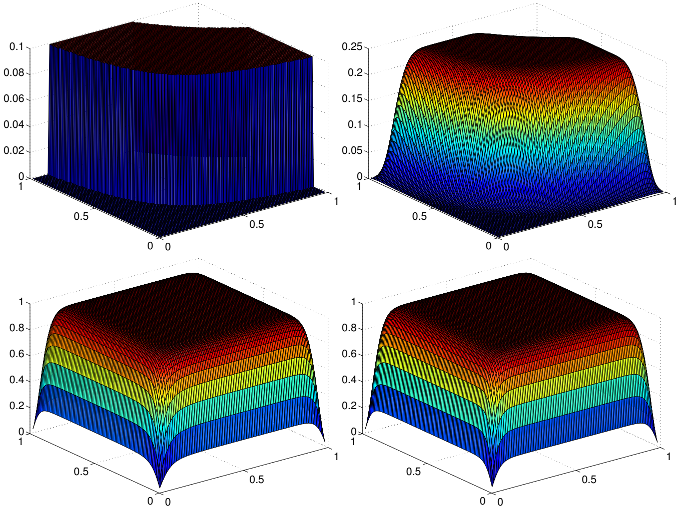


Figure: Solution at time $t = \{0, 0.1\}$ (top) and $t = \{1.5, 3\}$ (bottom)

Example: Semi-Linear Parabolic Equation

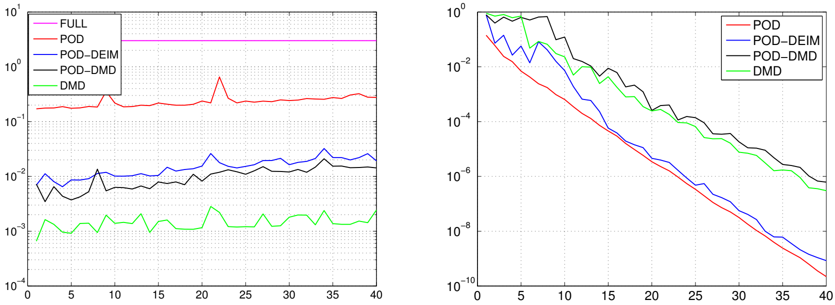


Figure: CPU-time online stage (left) and Relative Error wrt Frobenius norm. Number of POD modes and DEIM/DMD points are the same

Example: Semi-Linear Parabolic Equation

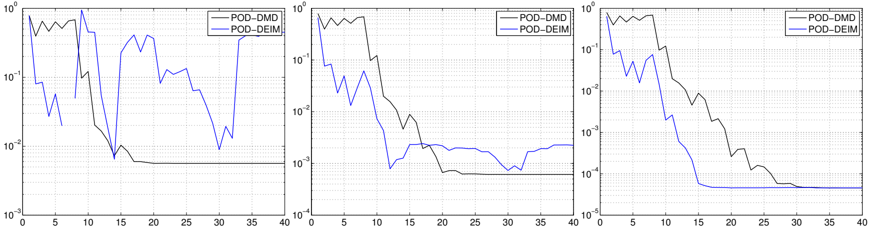
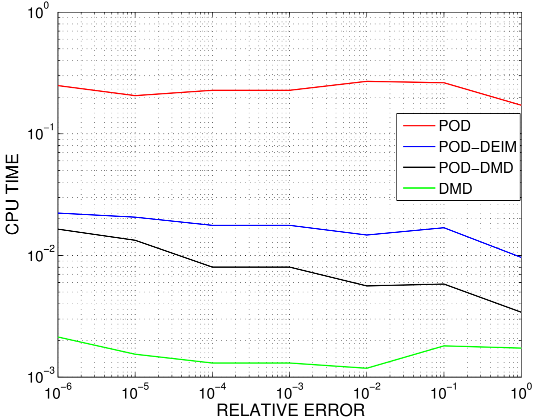


Figure: Relative Error for 5 POD basis functions (left), 10 POD basis (middle), 15 POD basis (right)

Example: Semi-Linear Parabolic Equation

A fair comparison



Outline

Randomized SVD (Haiko, Martinsson, Tropp, 2011)

Computational Costs for $\mathbf{Y} \in \mathbb{R}^{m \times n}$:

- SVD: $O(mn^2)$
- Randomized SVD: $O(mn\ell)$,

First Steps:

- Choose desired target rank $\ell \ll \{m, n\}$,
- Create a random (gaussian) sampling matrix $\mathbf{\Omega} \in \mathbb{R}^{n \times \ell}$,
- Sampled matrix $\mathbf{X} \in \mathbb{R}^{n \times \ell}$ is computed as: $\mathbf{X} = \mathbf{Y\Omega}$.

Remarks:

- If the matrix $\mathbf{Y}$ has exact rank ℓ, then the sampled matrix $\mathbf{X}$ spans, with high probability, a basis for the column space.
- In practice, it is favorable to *slightly oversample* $\ell = \ell + p$, were p denotes the number of additional samples.

Randomized SVD

Second Steps: Obtain low-rank SVD from a compressed matrix

- Compute $\mathbf{X} = \mathbf{QR}$, $\mathbf{Q} \in \mathbb{R}^{n\times\ell}$ orthonormal,
- $\mathbf{Y}$ is projected into this low-dimensional space $\mathbf{B} \in \mathbb{R}^{\ell\times m}$:

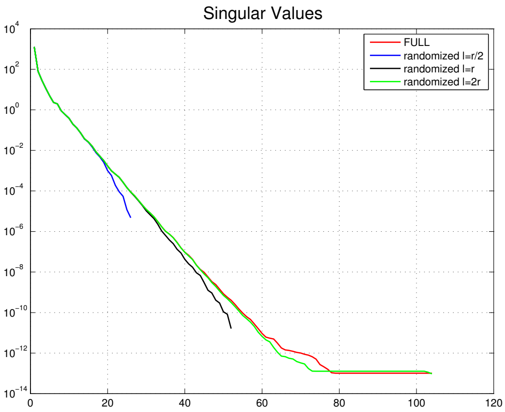
$$\mathbf{B} = \mathbf{Q}^{\intercal}\mathbf{Y},$$

- Compute the (cheap) SVD $\mathbf{B} = \hat{\mathbf{U}}\mathbf{\Sigma}\mathbf{V}^T$,
- Set $\mathbf{U} = \mathbf{Q}\hat{\mathbf{U}}$

Remark: if ℓ is large enough:

$$\begin{aligned}\mathbf{Y} &\approx \mathbf{Q}\mathbf{Q}^{\intercal}\mathbf{Y} \\ &\approx \mathbf{QB} \\ &\approx \mathbf{Q}\tilde{\boldsymbol{U}}\mathbf{\Sigma}\mathbf{V}^T \\ &\approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\end{aligned}$$

Randomized SVD



How to use RSVD in model reduction?

Model order reduction techniques:

- are based on snapshots of the dynamical system.
- SVD decomposition of the snapshots matrix provides a low-dimension projector operator that allows one to obtain surrogate models.
- WARNING: SVD may be computationally expensive to reduced the offline cost of the method.

IDEA:

Compute basis functions from a few spatially incoherent measurements (not from the full set of measurements)

Compressed POD

- we collect the snapshot set,
- we solve the POD optimization problem,
- optimality conditions provide eigenvalue problems,
- RSVD computes compressed POD basis functions in a significantly **faster** way.

Algorithm

Require: Snapshot Matrix $\mathbf{Y} \in \mathbb{R}^{n \times m}$, ℓ number of basis functions., p number of measurements.

1: Compute the Randomized SVD $[\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}] = rsvd(\mathbf{Y}, p)$

2: Set $\mathbf{\Psi}_i = \mathbf{U}_i$ for $i = 1, \ldots, \ell$.

Compressed POD

Error (d rank of $\mathbf{Y}$, p number of samples)

$$\mathbb{E}\left(\sum_{j=1}^{m}\alpha_j\left\|\mathbf{y}(t_j)-\sum_{i=1}^{\ell}\langle\mathbf{y}(t_j),\psi_i\rangle\psi_i\right\|^2\right)=\left(1+\sqrt{\frac{\ell}{p-1}}\right)\sigma_{\ell+1}^2+\frac{\sqrt{\ell+p}}{p}\sum_{j=\ell+1}^{d}\sigma_j^2.$$

Remarks:

- we consider the expectation value of the error
- error depends on the computation of the set of snapshots and p,
- if the singular values of $\mathbf{Y}$ decay rapidly a few samples drives the error close to the theoretically minimum value.
- if the singular values do not decay rapidly we can lose accuracy.
- we suppose $\mathbf{Y}\in\mathbb{R}^{m\times n}$, such that $m\approx n$ (eigenvalue problem is also expensive)

Compressive DMD (Brunton, Proctor, Kutz, 2015)

GOAL:

Compute DMD and apply as Galerkin projection method.

Algorithm

Require: Snapshots $\{\mathbf{y}(t_0), \ldots, \mathbf{y}(t_m)\}$, $\mathbf{C} \in \mathbb{R}^{p\times m}$

1: Set $\mathbf{Y} = [\mathbf{y}(t_0), \ldots, \mathbf{y}(t_{m-1})]$ and $Y' = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_m)]$,

2: $\mathbf{X} = \mathbf{CY}$, $\mathbf{X}' = \mathbf{CY}'$

3: Compute the SVD of $\mathbf{X}$, $\mathbf{X} = \mathbf{U\Sigma V}^T$

4: Define $\tilde{\mathbf{A}}_\mathbf{x} := \mathbf{U}^*\mathbf{Y}'\mathbf{V}\mathbf{\Sigma}^{-1}$

5: Compute eigenvalues and eigenvectors of $\tilde{\mathbf{A}}_\mathbf{x}\mathbf{W} = \mathbf{W\Lambda}$.

6: Set $\mathbf{\Psi}^{\text{DMD}} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{W}$

Outline

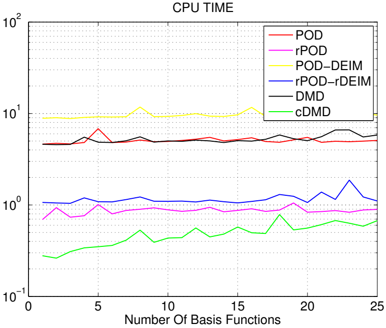
Test 1: Semi-Linear Parabolic Equation

$$
\begin{aligned}
y_t - \theta \Delta y + \mu(y - y^3) = 0 & \qquad (x,t) \in \Omega \times [0,T], \\
y(x,0) = y_0(x) & \quad x \in \Omega, \\
y(\cdot,t) = 0 & \qquad x \in \partial\Omega, t \in [0,T],
\end{aligned}
$$

Parameters:

$\Omega = [0,1] \times [0,1]$, $T = 3$,
$y_0(x) = 0.1$ if $0.1 \leq x_1 x_2 \leq 0.6$ and 0 elsewhere.

Test 1: Semi-Linear Parabolic Equation



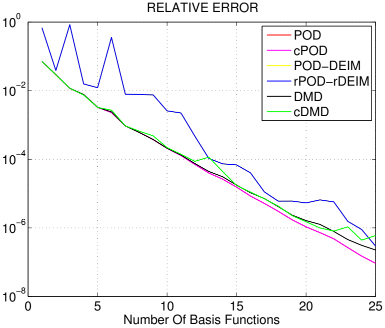


Figure: CPU-time online stage (left) and Relative Error wrt Frobenius norm. Number of POD modes and DEIM/DMD points are the same

Test 1: Semi-Linear Parabolic Equation

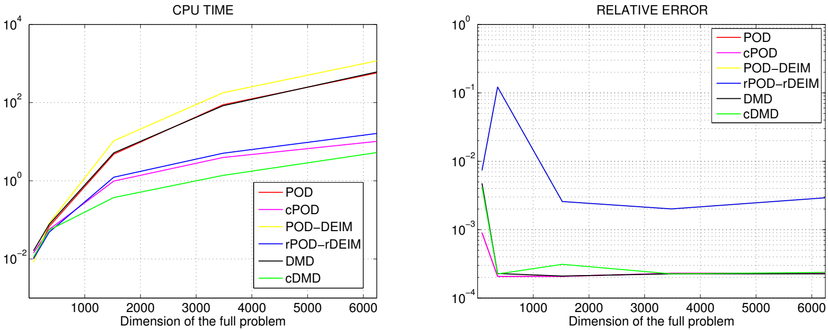


Figure: CPU-time online stage (left) and Relative Error wrt Frobenius norm. Number of POD modes and DEIM/DMD points are the same

Test 2: Parametric Elliptic Equation

$$
\begin{aligned}
-\Delta u(x,y) + s(u(x,y);\mu) &= f(x,y) \qquad && (x,y) \in \Omega \\
u(x,y) &= 0 && (x,y) \in \partial\Omega
\end{aligned}
$$

Parameters:

$$\Omega = [0,1] \times [0,1], \mu = (\mu_1, \mu_2) \in \mathcal{D} = [0.01, 10]^2$$

$$s(u,\mu) = \frac{\mu_1}{\mu_2}(e^{\mu_2 u} - 1), \quad f(x,y) = 100 \sin(2\pi x) \sin(2\pi y).$$

Test 2: Parametric Elliptic Equations

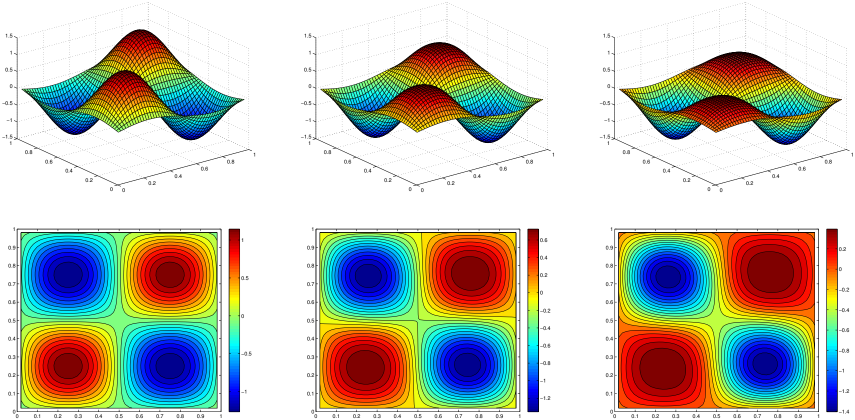


Figure: Test 2: Solution of problem for different parametric configurations (top) and countour lines (bottom)

Test 2: Parametric Elliptic Equations

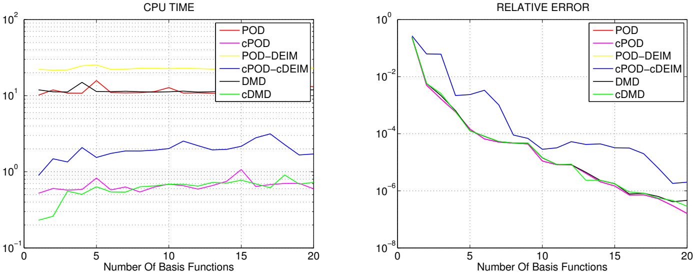


Figure: Test 2: CPU-time of the offline-online stages (left) and Relative Error in Frobenius norm (right). We compare the following methods: POD (red), cPOD (magenta), POD-DEIM (yellow), cPOD-cDEIM (blue, DMD (black), cDMD (green). Number of model are always the same for all the methods.

Conclusions

- Model order reduction is a successful technique that projects nonlinear high dimensional dynamical systems and PDEs into low dimensional surrogate models
- Compressed (randomized) techniques are a promising approach to circumventing expensive offline stages in model order reduction.
- DMD works successfully in a Galerkin projection framework

Thank you for your attention

- A. Alla, J.N. Kutz, *Randomized Model Order Reduction*, 2016.
- A. Alla, J.N. Kutz, *Nonlinear Model Reduction via DMD*, 2016.
- M. Barrault, Y. Maday, N.C. Nguyen, A.T. Patera, *An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations*, 2004.
- P. Benner, S. Gugercin and K. Willcox, *A Survey of Projection-Based Model Reduction Methods for Parametric Dynamical Systems,* 2015.
- S. L. Brunton, J. L. Proctor, and J. N. Kutz, *Compressive sampling and dynamic mode decomposition.* 2015.
- S. Chatarantabut, D. Sorensen, *Nonlinear Model Reduction via Discrete Empirical Interpolation.* 2010.
- Z. Drmac, S. Gugercin, *A new selection operator for the discrete empirical interpolation method - improved a priori error bound and extensions,* 2016.
- N. Halko, P.-G. Martinsson and J. Tropp, *Finding Structure with Randomness: Probabilistic Algorithms for Constructing Approximate Matrix Decompositions,* 2011.
- P. Schmid, *Dynamic mode decomposition of numerical and experimental data*, 2010.