Multisymplecticity of hybridizable discontinuous Galerkin methods*

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Motivation

- Given $H = H(t, q, p)$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, **Hamilton's equations** are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad -\dot{p}_i = \frac{\partial H}{\partial q^i}.$$

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- What about finite element methods?

Approach

- The framework of **hybridizable discontinuous Galerkin (HDG) methods**[†] makes it particularly natural to talk about local, per-element conservation laws, like the multisymplectic conservation law, for finite element methods.

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- We establish multisymplecticity criteria for HDG methods and show that many popular finite element methods are actually multisymplectic.

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Canonical Hamiltonian PDEs and the multisymplectic conservation law

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- Define the canonical 2-forms $\omega^\mu := \mathrm{d}u^i \wedge \mathrm{d}\sigma_i^\mu$, where $\mu = 1, \ldots, m$. Solutions satisfy the **multisymplectic conservation law** $\boxed{\partial_\mu \omega^\mu = 0}$.

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- The $m = 1$ case corresponds to canonical Hamiltonian systems of ODEs, along with the usual symplectic conservation law.

Example: semilinear elliptic PDE

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Remark

We can also obtain hyperbolic PDEs (and more) by changing the signature of a.

What does the multisymplectic conservation law really say?

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- If (u, σ) is a solution and (v, τ) and (v', τ') are variations tangent to the space of solutions—i.e., solutions to the linearized problem at (u, σ)—then

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- This is deeply related to **reciprocity laws** in physical systems: Green's reciprocity in electrostatics, Betti reciprocity in elasticity, etc.
- Laplace's equation: symmetry of the Dirichlet-to-Neumann operator $v|_{\partial K} \mapsto \operatorname{grad} v \cdot \mathbf{n}|_{\partial K}$, related to Green's identity.

Flux formulation for canonical PDEs

- Consider a system of PDEs on U in the following canonical form:

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$$\int_{\partial K} u^i \tau^\mu_i \,\mathrm{d}^{m-1} x_\mu = \int_K \left(u^i \partial_\mu \tau^\mu_i + \phi^i_\mu \tau^\mu_i \right) \mathrm{d}^m x,$$
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- Numerical integration analogy: think of u, σ as internal stages or collocation polynomials, and $\widehat{u}$, $\widehat{\sigma}$ as the endpoint values.

HDG methods and numerical fluxes

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- To determine the extra unknown $\widehat{u}$, we add the **conservativity condition**,

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- For Galerkin variational integrators, the conservativity condition corresponds to the **discrete Euler–Lagrange equations**.
- An HDG method is defined by specifying the the global trace space in which $\widehat{u}$ lives, and for each $K \in \mathcal{T}_h$, the local spaces in which $u|_K$ and $\sigma|_K$ live, together with the numerical flux $\widehat{\sigma}|_{\partial K}$.

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Multisymplecticity and strong multisymplecticity of HDG methods

Definition

An HDG method is **multisymplectic** if, when applied to a Hamiltonian system of PDEs, solutions satisfy

$$\int_{\partial K} (\mathrm{d}\widehat{u}^i \wedge \mathrm{d}\widehat{\sigma}_i^\mu)\, \mathrm{d}^{m-1} x_\mu = 0,$$

for all $K \in \mathcal{T}_h$. It is **strongly multisymplectic** if

$$\int_{\partial(\overline{\bigcup \mathcal{K}})} (\mathrm{d}\widehat{u}^i \wedge \mathrm{d}\widehat{\sigma}_i^\mu)\, \mathrm{d}^{m-1} x_\mu = 0,$$

for any collection of elements $\mathcal{K} \subset \mathcal{T}_h$.

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Theorem (McLachlan–S.)

If a multisymplectic HDG method has a **strongly conservative** *numerical flux, i.e.,* $[\![\widehat{\sigma}]\!] = 0$*, then it is strongly multisymplectic.*

Multisymplecticity criteria

Lemma (McLachlan–S.)

If an HDG method is applied to a Hamiltonian system of PDEs, then

$$\int_{\partial K} (\mathrm{d}\widehat{u}^i \wedge \mathrm{d}\widehat{\sigma}_i^\mu)\, \mathrm{d}^{m-1}x_\mu = \int_{\partial K} \big[\mathrm{d}(\widehat{u}^i - u^i) \wedge \mathrm{d}(\widehat{\sigma}_i^\mu - \sigma_i^\mu)\big]\, \mathrm{d}^{m-1}x_\mu .$$

Consequently, the method is multisymplectic if and only if

$$\int_{\partial K} \big[\mathrm{d}(\widehat{u}^i - u^i) \wedge \mathrm{d}(\widehat{\sigma}_i^\mu - \sigma_i^\mu)\big]\, \mathrm{d}^{m-1}x_\mu = 0.$$

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If an HDG method is applied to a Hamiltonian system of PDEs, then

$$\int_{\partial K} (\mathrm{d}\widehat{u}^i \wedge \mathrm{d}\widehat{\sigma}_i^\mu)\, \mathrm{d}^{m-1}x_\mu = \int_{\partial K} \left[\mathrm{d}(\widehat{u}^i - u^i) \wedge \mathrm{d}(\widehat{\sigma}_i^\mu - \sigma_i^\mu)\right] \mathrm{d}^{m-1}x_\mu .$$

Consequently, the method is multisymplectic if and only if

$$\int_{\partial K} \left[\mathrm{d}(\widehat{u}^i - u^i) \wedge \mathrm{d}(\widehat{\sigma}_i^\mu - \sigma_i^\mu)\right] \mathrm{d}^{m-1}x_\mu = 0.$$

Remark

This condition is usually straightforward to check from the numerical flux $\widehat{\sigma}$, since it only depends on the jump between the actual and approximate traces.

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  5. *IP-H (hybridized interior penalty)*
- *The CG-H (hybridized continuous Galerkin) method is multisymplectic but* **not strongly multisymplectic** *except when* $m = 1$.

Proof ideas

- Show that the methods satisfy

$$\int_{\partial K} \left[\mathrm{d}(\widehat{u}^i - u^i) \wedge \mathrm{d}(\widehat{\sigma}_i^\mu - \sigma_i^\mu) \right] \mathrm{d}^{m-1} x_\mu = 0.$$

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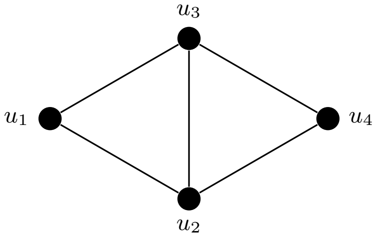
- The LDG-H method takes $\widehat{\sigma} - \sigma = \lambda(\widehat{u} - u)\mathbf{n}$, where λ is a penalty parameter and $\mathbf{n}$ is the outer unit normal to ∂K. Substituting this above, the antisymmetry of the wedge product implies

$$\lambda \delta_{ij} \mathrm{d}(\widehat{u}^i - u^i) \wedge \mathrm{d}(\widehat{u}^j - u^j) = 0,$$

so the integral vanishes. The IP-H method is similar.

The CG-H method is not strongly multisymplectic



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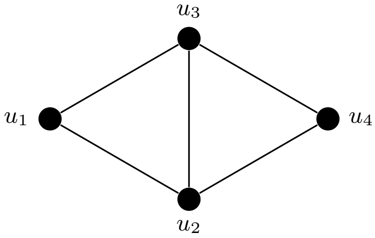

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- Assembling the matrix for the Dirichlet-to-Neumann map $\widehat{v} \mapsto \widehat{\tau} \cdot \mathbf{n}$ on ∂U yields (up to a scaling factor)

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix},$$

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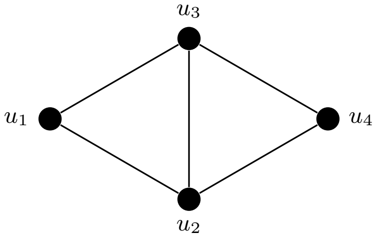

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  which is not symmetric.
- This is due to the fact that $\widehat{\sigma}$ is only weakly conservative for CG-H.

Concluding thoughts

- The multisymplectic conservation law is a local condition, describing a physically important “perturbation-response” symmetry for boundary conditions (i.e., symmetry of a Poincaré–Steklov operator).

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- Multisymplecticity has some important physical manifestations: Betti reciprocity in elasticity, symmetry of the Dirichlet-to-Neumann map used in electrical impedance tomography, etc.
- Ongoing work: other multisymplectic methods (Reich-type collocation methods, Marsden–Patrick–Shkoller-type variational methods) can be seen as HDG with "variational crimes" in the local solvers, e.g., quadrature.