Construction of bounded cochain projections and their role in the FE exterior calculus

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Outline Of Talk

- I. Motivation
- II. Review of Finite Element Exterior Calculus
- III. Canonical projection operators
- IV. Nonlocal bounded cochain projections
- V. Local bounded cochain projections
- VI. A double complex

Motivation

Elliptic equation, $-\operatorname{div}(a \operatorname{grad} u) = f$ in Ω, $u = 0$ on $\partial\Omega$.

Mixed formulation: Find $(\sigma, u) \in H(\operatorname{div}; \Omega) \times L^2(\Omega)$,
$\sigma = a \operatorname{grad} u$, such that

$$\begin{aligned} \langle a^{-1}\sigma, \tau\rangle + \langle u, \operatorname{div}\tau\rangle &= 0, \quad \tau \in H(\operatorname{div}; \Omega), \\ \langle \operatorname{div}\sigma, v\rangle &= \langle f, v\rangle, \quad v \in L^2(\Omega). \end{aligned}$$

$H(\operatorname{div}; \Omega) = \{\tau \in L^2(\Omega) : \operatorname{div}\tau \in L^2(\Omega)\}$.

Mixed finite element approximation

Choose finite dimensional spaces $\Sigma_h \times V_h \subset H(\mathrm{div};\Omega) \times L^2(\Omega)$.

Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$\begin{aligned} \langle a^{-1}\sigma_h, \tau\rangle + \langle u_h, \mathrm{div}\,\tau\rangle &= 0, \quad \tau \in \Sigma_h, \\ \langle \mathrm{div}\,\sigma_h, v\rangle &= \langle f, v\rangle, \quad v \in V_h. \end{aligned}$$

If $\mathrm{div}\,\Sigma_h \subset V_h$, stability follows from

$$\sup_{\tau \in \Sigma_h} \frac{\langle v, \mathrm{div}\,\tau\rangle}{\|\tau\|_{H(\mathrm{div})}} \geq \alpha \|v\|_{L^2}, \quad v \in V_h.$$

Stability and Fortin operators

To satisfy sup condition, let $\tau = \operatorname{grad}\phi$, where ϕ satisfies

$$\Delta\phi = v, \text{ in } \Omega, \qquad \phi = 0 \text{ on } \partial\Omega.$$

Then $\operatorname{div}\tau = \Delta\phi = v$ and $\|\tau\|_{H(\operatorname{div})} \le C\|v\|_{L^2}$.

In fact, $\exists W \subset H(\operatorname{div})$ such that $\|\tau\|_W \le C\|v\|_{L^2}$.
For example, if Ω is a convex polygon, $W = H^1(\Omega)$.

Assume there exists a (Fortin) operator $\pi_h : W \to \Sigma_h$ such that

$$\langle v, \operatorname{div}\pi_h\tau\rangle = \langle v, \operatorname{div}\tau\rangle, \quad v \in V_h, \qquad \|\pi_h\tau\|_{H(\operatorname{div})} \le C'\|\tau\|_W.$$

Then for $v \in V_h$,

$$\sup_{\tau\in\Sigma_h} \frac{\langle v, \operatorname{div}\tau\rangle}{\|\tau\|_{H(\operatorname{div})}} \ge \frac{\langle v, \operatorname{div}\pi_h\tau\rangle}{\|\pi_h\tau\|_{H(\operatorname{div})}} \ge \frac{\langle v, \operatorname{div}\tau\rangle}{C'\|\tau\|_W}$$

$$\ge \frac{\|v\|_{L^2}^2}{C'C\|v\|_{L^2}} \ge \alpha\|v\|_{L^2}.$$

Commuting diagram

Alternatively, if P_h (L^2 projection into V_h) and π_h satisfy commuting diagram:

$$\begin{array}{ccc} W & \xrightarrow{\text{div}} & L^2(\Omega) \\ \big\downarrow{\scriptstyle \pi_h} & & \big\downarrow{\scriptstyle P_h}, \\ \Sigma_h & \xrightarrow{\text{div}} & V_h \end{array}$$

then for $\tau \in W$ and $v \in V_h$,

$$\langle v, \operatorname{div} \tau \rangle = \langle v, P_h \operatorname{div} \tau \rangle = \langle v, \operatorname{div} \pi_h \tau \rangle.$$

Commuting projections have been a standard tool of stability analysis for FEM for a long time.

The de Rham complex

In finite element exterior calculus, instead of studying discretizations of structure

$$H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

gain more insight by studying discretizations of complete de Rham complex

$$0 \to H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0,$$

where

$$H(\mathrm{curl};\Omega) = \{\, u : \Omega \to \mathbb{R}^3 \mid u \in L^2, \mathrm{curl}\, u \in L^2 \,\},$$
$$H(\mathrm{div};\Omega) = \{\, u : \Omega \to \mathbb{R}^3 \mid u \in L^2, \mathrm{div}\, u \in L^2 \,\}.$$

2-D de Rham sequences:

$$H^1(\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega),$$

$$H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{rot},\Omega) \xrightarrow{\mathrm{rot}} L^2(\Omega).$$

de Rham complex (continued)

3-D de Rham complex

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

is special case of general L^2 de Rham complex.

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d_0} H\Lambda^1(\Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} H\Lambda^n(\Omega) \to 0,$$

$$\text{where} \qquad H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) : d\omega \in L^2\Lambda^{k+1}(\Omega)\}$$

and $d_k : H\Lambda^k(\Omega) \to H\Lambda^{k+1}(\Omega)$ is exterior derivative.

Structure is called a *complex* since $d_{k+1} \circ d_k = 0$.

Complex called *exact* if $range(d_k) = ker(d_{k+1})$.

For 3-D de Rham complex, $d^0 = \text{grad}, \ d^1 = \text{curl}, \ d^2 = \text{div}$.

The Hodge Laplacian

Connected to this complex is operator $L = dd^* + d^*d$, called **Hodge Laplacian**, where d^* is adjoint of d. So

$$\langle du, v\rangle = \langle u, d^*v\rangle, u \in V^k \equiv H\Lambda^k(\Omega), v \in V^*_{k+1} \equiv \mathring{H}^*\Lambda^{k+1}(\Omega).$$

Domain of L is: $D_L = \{u \in V^k \cap V^*_k\}$. If u solves $Lu = f$, then

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, \quad v \in D_L.$$

Not a good formulation for FEM approximation: hard to construct useful subspaces of D_L.

In general: Harmonic forms $\mathfrak{H}^k = \{v \in D_L : dv = 0, d^*v = 0\}$.

Ignore for simplicity.

Mixed formulation of Hodge Laplacian

For $f \in L^2\Lambda^k(\Omega)$ given, find $(\sigma, u) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega)$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, & \tau &\in H\Lambda^{k-1}(\Omega), \\ \langle d\sigma, v \rangle + \langle du, dv \rangle &= \langle f, v \rangle, & v &\in H\Lambda^k(\Omega). \end{aligned}$$

First equation: u belongs to domain of d^* and $d^* u = \sigma$.

Second equation: du belongs to domain of d^* and $d^* du = f - d\sigma$.

Hence, $u \in D_L$ of L and solves Hodge Laplacian equation $Lu = f$.

Applications of the Hodge Laplacian

Let $\Omega \subset \mathbb{R}^3$. Mixed formulation gives:

$k = 0$: Neumann problem for Poisson's equation

$$-\operatorname{div}\operatorname{grad} u = f \text{ in } \Omega, \quad \int_\Omega u\,dx = 0, \quad \operatorname{grad} u \cdot n = 0 \text{ on } \partial\Omega.$$

$k = 1$: BVP for vector Laplacian

$$\sigma = -\operatorname{div} u, \qquad \operatorname{grad}\sigma + \operatorname{curl}\operatorname{curl} u = f \quad \text{ in } \Omega,$$
$$u \cdot n = 0, \ \operatorname{curl} u \times n = 0 \quad \text{on } \partial\Omega.$$

$$f = \operatorname{grad} F: \qquad -\operatorname{div} u = F, \qquad \operatorname{curl} u = 0.$$
$$\operatorname{div} f = 0: \qquad \operatorname{curl}\operatorname{curl} u = f, \qquad \operatorname{div} u = 0.$$

More applications of the Hodge Laplacian

$k = 2$: Another BVP for vector Laplacian

$$\sigma = \operatorname{curl} u, \ \operatorname{curl} \sigma - \operatorname{grad} \operatorname{div} u = f \quad \text{in } \Omega,$$
$$u \times n = 0, \ \operatorname{div} u = 0 \quad \text{on } \partial\Omega.$$

$$f = \operatorname{curl} F : \qquad \operatorname{curl} u = F, \qquad \operatorname{div} u = 0.$$
$$f = \operatorname{grad} F : \qquad \operatorname{div} u = F, \qquad \operatorname{curl} u = 0.$$

$k = 3$: Dirichlet problem for Poisson's equation

$$\sigma = -\operatorname{grad} u, \ \operatorname{div} \sigma = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Well-posedness of Mixed BVP for Hodge Laplacian

Let

$$\mathfrak{B}^k = dH\Lambda^{k-1}(\Omega), \qquad \mathfrak{Z}^k = \{w \in H\Lambda^k(\Omega) : d\omega = 0\},$$
$$\mathfrak{Z}^{k\perp} = \text{ orthogonal complement of } \mathfrak{Z}^k \text{ in } H\Lambda^k(\Omega).$$

Proof of well-posedness uses:

(i) Hodge decomposition of $u \in H\Lambda^k(\Omega)$:

$$u = P_{\mathfrak{B}^k} u \oplus P_{\mathfrak{Z}^{k\perp}} u.$$

(ii) Poincaré inequality:

$$\|v\|_{L^2\Lambda^k} \le c_P \|dv\|_{L^2\Lambda^{k+1}}, \quad v \in \mathfrak{Z}^{k\perp}.$$

to verify inf-sup condition (technical condition guaranteeing well-posedness).

Approximation of de Rham complexes

To approximate Hodge Laplacian, begin with approximation of de Rham complex.

Seek spaces $\Lambda_h^k \subset H\Lambda^k(\Omega)$ with $d\Lambda_h^k \subset \Lambda_h^{k+1}$, so that (Λ_h, d) is a subcomplex of $(H\Lambda, d)$.

Differential for subcomplex is restriction of d, but $d_h^* : \Lambda_h^{k+1} \to \Lambda_h^k$, defined by

$$\langle d_h^* u, v \rangle = \langle u, dv \rangle, \quad u \in \Lambda_h^{k+1}, v \in \Lambda_h^k,$$

not restriction of d^*. (Major technical difficulty.)

Then have discrete Hodge decomposition

$$\Lambda_h^k = \mathfrak{B}_h^k \oplus \mathfrak{Z}_h^{k\perp}.$$

Approximation of de Rham complex (continued)

Assume $\inf_{v \in \Lambda_h^k} \|u - v\|_{H\Lambda} \to 0$ as $h \to 0$ for some (or all) $u \in H\Lambda^k(\Omega)$,

where $\qquad \|v\|_{H\Lambda}^2 = \|v\|_{L^2}^2 + \|dv\|_{L^2}^2.$

Further assume: there exist bounded cochain projections $\pi_h^k : H\Lambda^k(\Omega) \mapsto \Lambda_h^k$, i.e., π_h^k leaves subspace invariant and satisfies

$$d^k \pi_h^k = \pi_h^{k+1} d^k, \qquad \|\pi_h^k v\|_{H\Lambda} \le c\|v\|_{H\Lambda}, \quad v \in H\Lambda^k(\Omega).$$

Have following commuting diagram relating complex $(H\Lambda(\Omega), d)$ to subcomplex (Λ_h, d):

$$\begin{array}{ccccccccc} 0 \to H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) \to 0 \\ \downarrow \pi_h & & \downarrow \pi_h & & & & \downarrow \pi_h \\ 0 \to \quad \Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda_h^n \quad \to 0. \end{array}$$

Galerkin approximation of Mixed Hodge Laplacian

Find $\sigma_h \in \Lambda_h^{k-1}$, $u_h \in \Lambda_h^k$, such that

$$\langle \sigma_h, \tau \rangle - \langle d\tau, u_h \rangle = 0, \qquad \tau \in \Lambda_h^{k-1},$$
$$\langle d\sigma_h, v \rangle + \langle du_h, dv \rangle = \langle f, v \rangle, \quad v \in \Lambda_h^k.$$

Under previous assumptions, (π_h^k a bounded cochain projection), get discrete Poincaré inequality

$$\|v\|_{L^2\Lambda^k} \le c_P \|\pi_h^k\|_{\mathcal{L}(H\Lambda, H\Lambda)} \|dv\|_{L^2\Lambda^k}, \qquad v \in \mathfrak{Z}_h^{k\perp}.$$

Use to satisfy discrete version of inf-sup condition, so get stability with constant depending only on c_P and $\|\pi_h^k\|_{\mathcal{L}(V^k, V^k)}$. Also get quasi-optimal error estimate ($V^k = H\Lambda^k(\Omega)$):

$$\|\sigma - \sigma_h\|_{V^{k-1}} + \|u - u_h\|_{V^k} \le C \left(\inf_{\tau \in \Lambda_h^{k-1}} \|\sigma - \tau\|_{V^{k-1}} + \inf_{v \in \Lambda_h^k} \|u - v\|_{V^k} \right).$$

Finite element approximation of de Rham complex

To apply abstract approximation results for Hodge Laplacian, construct finite dimensional subspaces Λ_h^k of $H\Lambda^k(\Omega)$ satisfying:

(i) $d\Lambda_h^k \subset \Lambda_h^{k+1}$ so they form subcomplex (Λ_h, d) of de Rham complex.

(ii) There exist uniformly bounded cochain projections π_h from $H\Lambda^k$ to Λ_h^k.

(iii) Λ_h^k have good approximation properties.

Get two families of spaces of finite element differential forms.

$$\mathcal{P}_r\Lambda^k(\mathcal{T}_h) = \{\omega \in H\Lambda^k(\Omega) : \omega|_T \in \mathcal{P}_r\Lambda^k(T), \forall T \in \mathcal{T}_h\},$$
$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) = \{\omega \in H\Lambda^k(\Omega) : \omega|_T \in \mathcal{P}_r^-\Lambda^k(T), \forall T \in \mathcal{T}_h\}.$$

Generalize Raviart-Thomas and Brezzi-Douglas-Marini $H(\text{div})$ elements in 2-D and Nédélec 1st and 2nd kind $H(\text{div})$ and $H(\text{curl})$ elements in 3-D.

Simplest approximation of de Rham complex in 3-D

$$\begin{array}{ccccccccc}
0 \to H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl};\Omega) & \xrightarrow{\text{curl}} & H(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \to 0 \\
\downarrow \pi_h & & \downarrow \pi_h & & \downarrow \pi_h & & \downarrow \pi_h \\
0 \to H_h^1 & \xrightarrow{\text{grad}} & H_h(\text{curl}) & \xrightarrow{\text{curl}} & H_h(\text{div}) & \xrightarrow{\text{div}} & L_h^2 \to 0.
\end{array}$$

Simplest choice of finite element spaces:

- H_h^1 = piecewise linear scalar fields
- $H_h(\text{curl})$ = Nédélec edge element
- $H_h(\text{div})$ = Nédélec face element (or 3d Raviart–Thomas)
- L_h^2 = piecewise constants

all with respect to same simplicial mesh $\mathcal{T}_h$.

Degrees of freedom and canonical projections

For these spaces, commuting projections $\mathcal{I}_h$ can be constructed from degrees of freedom as follows:

- H_h^1 = piecewise linears, $\mathcal{I}_h^1 u(x) = u(x)$ at each vertex
- $H_h(\mathrm{curl})$ = edge element, $\int_e \mathcal{I}_h^c u \cdot t = \int_e u \cdot t$ at each edge
- $H_h(\mathrm{div})$ = face element, $\int_f \mathcal{I}_h^d u \cdot n = \int_f u \cdot n$ for each face
- L_h^2 = piecewise constants, $\int_T \mathcal{I}_h^0 u = \int_T u$ for each tetrahedron

These projections commute with differential operators:

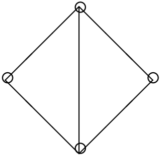
$$\mathrm{grad} \circ \mathcal{I}_h^1 = \mathcal{I}_h^c \circ \mathrm{grad}, \quad \mathrm{curl} \circ \mathcal{I}_h^c = \mathcal{I}_h^d \, \mathrm{curl}, \quad \mathrm{div} \circ \mathcal{I}_h^d = \mathcal{I}_h^0 \circ \mathrm{div} \, .$$

However, $\mathcal{I}_h^1$, $\mathcal{I}_h^c$, $\mathcal{I}_h^d$ are *not bounded* on spaces H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$, respectively.

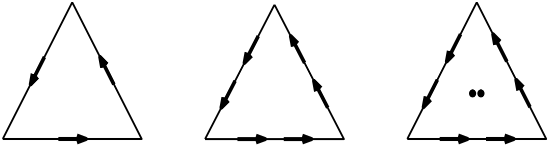
Example: $u(x, y) = \log \log(2/r)$, $r^2 = x^2 + y^2 \in H^1(\Omega)$, $\Omega =$ unit disk, but is unbounded at origin, so $\mathcal{I}_h^1$ not defined if origin is a vertex of triangulation.

Examples of DOF

C^0 piecewise P_1 on triangulation $\mathcal{T}_h$ of $\Omega \in \mathbb{R}^2$. Shape fcns are P_1 on each $T \in \mathcal{T}_h$. DOF are $\omega \mapsto \omega(v_i)$, v_i vertices of $\mathcal{T}_h$.



Shape fcns for $\mathcal{P}_1^-\Lambda^1(T) = \begin{pmatrix} a - by \\ c + bx \end{pmatrix}$. DOF: $\omega \mapsto \int_e \omega \cdot t_e$.



$\mathcal{P}_1^-\Lambda^1(T)$ $\mathcal{P}_1\Lambda^1(T)$ $\mathcal{P}_2^-\Lambda^1(T)$

DOF and canonical projections for more general subspaces

For a d-dimensional subsimplex f of T, DOF have form

$$\mathcal{P}_r\Lambda^k(T): \qquad \omega \mapsto \int_f \operatorname{tr}_{T,f} \omega \wedge \eta, \qquad \eta \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f)$$

$$\mathcal{P}^-_r\Lambda^k(T): \qquad \omega \mapsto \int_f \operatorname{tr}_{T,f} \omega \wedge \eta, \qquad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

Key idea: if subsimplex shared by more than one simplex in triangulation, DOF associated with subsimplex are single-valued.

Determines interelement continuity of finite element space – resulting finite element spaces are subspaces of $H\Lambda^k(\Omega)$.

Implicitly defines canonical projections: *not bounded* in $H\Lambda^k(\Omega)$,

$$\int_f \operatorname{tr}_{T,f} \mathcal{I}_h\omega \wedge \eta = \int_f \operatorname{tr}_{T,f} \omega \wedge \eta, \qquad \eta \text{ as above}$$

since all traces not defined for $\omega \in H\Lambda^k(\Omega)$.

Construction of bounded cochain projections

Consider operators of form

$$Q^k_{\epsilon,h} = \mathcal{I}^k_h \circ R^k_{\epsilon,h},$$

where $R^k_h = R^k_{\epsilon,h}$ is a smoothing operator which commutes with exterior derivative d and $\mathcal{I}^k_h$ are canonical projections.

Operator of form Q^k_h can be made bounded on $L^2\Lambda^k(\Omega)$ and will commute with d. However, in general it is *not a projection* onto finite element space Λ^k_h.

So called *smoothed projections* are of form:

$$\pi^k_h = (Q^k_{\epsilon,h}|_{\Lambda_h})^{-1} \circ Q^k_{\epsilon,h},$$

for ϵ sufficiently small, but not too small. (cf. Schöberl 2007, Christiansen 2007, A–F–W 2006).

This construction gives bounded, but *nonlocal* cochain projections.

Why do we care about having local projections?

A posteriori error estimation and adaptive FEM.

Goal: Estimate local errors using only quantities known from the computation and use this information to modify mesh to introduce smaller elements where local error is big.

Need: localized a posteriori error estimates

L. Chen and Y. Wu, Convergence of adaptive mixed finite element methods for Hodge Laplacian equation: without harmonic forms

A. Demlow, Convergence and quasi-optimality of adaptive finite element methods for harmonic forms

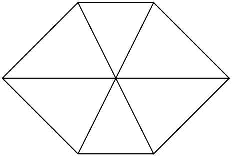
Macroelements and the Clément interpolant

Problem of defining interpolants on non-smooth functions (only in $L^2(\Omega)$) solved by Clément.

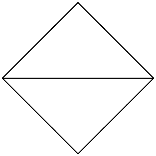
Consider subspaces $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ of H^1. To define Clément interpolant for each $f \in \Delta(\mathcal{T}_h)$, introduce associated macroelement Ω_f by

$$\Omega_f = \bigcup \{\, T \mid T \in \mathcal{T}_h,\, f \in \Delta(T)\,\}.$$

Vertex macroelement, $n = 2$.

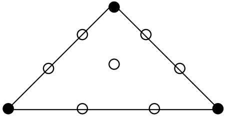


Edge macroelement, $n = 2$.



The Clément interpolant

Let $\mu_i : C(\bar{\Omega}) \to \mathbb{R}$ be usual DOfs for space $\mathcal{P}^r\Lambda^0(\mathcal{T}_h)$ and ϕ_i corresponding basis functions. Example: $P^3(T)$.



vertex v_i DOF: $u(v_i)$

edge e_i DOF: $\int_{e_i} u\,ds$, $\int_{e_i} u\,s\,ds$

triangle T DOF: $\int_T u\,dx$

Standard interpolant is $\mathcal{I}_h u = \sum_i \mu_i(u)\phi_i$.

Let S_i denote support of ϕ_i, i.e., macroelement where $\phi_i \neq 0$. Let $P_i : L^2(S_i) \to \mathcal{P}^r(S_i)$ be L^2 projection on S_i.

Clément operator $\tilde{\mathcal{I}}_h : L^2 \to \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ defined by

$$\tilde{\mathcal{I}}_h u = \sum_i \mu_i(P_i u)\phi_i.$$

$\tilde{\mathcal{I}}_h u$ bounded in L^2, but *not a projection*.

Degrees of freedom and geometric decompositions

Consequence of DOF that space $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ admits decomposition of form

$$P_r\Lambda^0(\mathcal{T}_h) = \bigoplus_{f\in\Delta(\mathcal{T}_h)} E_f(\mathring{\mathcal{P}}_r(f)),$$

where E_f is local extension operator mapping $\mathring{\mathcal{P}}_r\Lambda^0(f)$ into subspace of $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ with support in Ω_f.

Choose E_f to be discrete harmonic extension given by $\operatorname{tr}_f E_f\phi = \phi$,

$$\int_{\Omega_f} \operatorname{grad} E_f\phi \cdot \operatorname{grad} v = 0,$$

for all $v \in \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$, $\operatorname{supp} v \subset \Omega_f$, and $\operatorname{tr}_f v = 0$.

Geometric decomposition example: $p = 3$

Write $u \in \mathcal{P}_3\Lambda^0(\mathcal{T}_h) = u_2$, where

$$u_0 = \sum_{f_0 \in \Delta_0(\mathcal{T}_h)} E_{f_0} \operatorname{tr}_{f_0} u,$$

$$u_m = u_{m-1} + \sum_{f_m \in \Delta_m(\mathcal{T}_h)} E_{f_m} \operatorname{tr}_{f_m}(u - u_{m-1}), \quad m = 1, 2.$$

Show u and u_2 agree at degrees of freedom. Then $u = u_2$.

Since $\operatorname{tr}_{g_i \in \Delta_i} E_{f_i} \operatorname{tr}_{f_i} = 0$ unless $g_i = f_i$, $\operatorname{tr}_{g_0} u_0 = \operatorname{tr}_{g_0} u$ and $\operatorname{tr}_{g_1 \in \Delta_1} u_1 = \operatorname{tr}_{g_1} u_0 + \operatorname{tr}_{g_1}(u - u_0) = \operatorname{tr}_{g_1} u$. Similarly, $\operatorname{tr}_{g_2 \in \Delta_2} u_2 = \operatorname{tr}_{g_2 \in \Delta_2} u$.

Since for $j < i$, $\operatorname{tr}_{g_j \in \Delta_j} E_{f_i} \operatorname{tr}_{f_i} = 0$ if $g_j \notin f_i$ and $= \operatorname{tr}_{g_j \in \Delta_j}$ otherwise, also get:

$$\operatorname{tr}_{g_0 \in \Delta_0} u_2 = \operatorname{tr}_{g_0} u_1 = \operatorname{tr}_{g_0} u_0 = \operatorname{tr}_{g_0} u,$$

$$\operatorname{tr}_{g_1 \in \Delta_1} u_2 = \operatorname{tr}_{g_1} u_1 = \operatorname{tr}_{g_1} u.$$

The modified Clement operator π^0 onto $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$

Projection operator π^0 constructed by recursion wrt dimension of subsimplices $f \in \mathcal{T}_h$. Define $\mathcal{T}_{f,h}$ = restriction of $\mathcal{T}_h$ to Ω_f.

If $\dim f = 0$, i.e., f a vertex, first define $P_f^0 u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h})$ as H^1 projection of u, i.e., $P_f^0 u$ satisfies: $\int_{\Omega_f} P_f^0 u = \int_{\Omega_f} u$ and

$$\int_{\Omega_f} \operatorname{grad} P_f^0 u \cdot \operatorname{grad} v = \int_{\Omega_f} \operatorname{grad} u \cdot \operatorname{grad} v, \qquad v \in \mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h}).$$

$$\text{Define} \qquad \pi_0^0 u = \sum_{f \in \Delta_0(\mathcal{T}_h)} \mathcal{E}_f^0(\operatorname{tr}_f P_f^0 u),$$

where $\mathcal{E}_f^0(z)$ = piecewise linear function with value z at vertex f and zero at all other vertices.

The modified Clement operator π^0 continued

Use recursive approach based on geometric decomposition:

For $1 \le m \le n$, define π^0_m by

$$\pi^0_m u = \pi^0_{m-1} u + \sum_{f \in \Delta_m(\mathcal{T}_h)} E^0_f \operatorname{tr}_f P^0_f (u - \pi^0_{m-1} u).$$

For $\dim f \ge 1$, operators P_f are local H^1 projections onto space:

$$\check{\mathcal{P}}_r \Lambda^0(\mathcal{T}_{f,h}) = \{ u \in \mathcal{P}_r \Lambda^0(\mathcal{T}_{f,h}) \,|\, \operatorname{tr}_f u \in \mathring{\mathcal{P}}_r(f) \}.$$

Gives local projection $\pi^0 = \pi^0_n$, bounded in H^1.

Generalization to $\pi^k : H\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}_h)$

Construction of π^0 based on local projections P_f^0 defined with respect to associated macroelement Ω_f.

Let $f = [x_0, x_1]$. To get commuting projections ($\pi^1 du = d\pi^0 u$), need

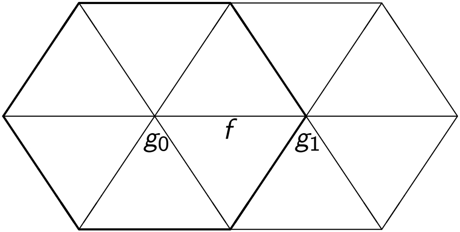
$$\int_f \mathrm{tr}_f \, \pi^1 du = \int_f \mathrm{tr}_f \, d\pi^0 u = \int_f \mathrm{grad}\, \pi^0 u \cdot t_f \, ds$$
$$= \int_{x_0}^{x_1} \frac{d}{ds} \pi^0 u \, ds = (\pi^0 u)(x_1) - (\pi^0 u)(x_0).$$

But RHS depends on u restricted to union of macroelements associated to vertices x_0 and x_1. So can't just take local projections on macroelement Ω_f to define π^1.

Extended macroelements

$$\Omega_f^e = \bigcup_{g \in \Delta_0(f)} \Omega_g, \quad f \in \Delta(\mathcal{T}_h).$$

If $g \in \Delta(f)$ then $\Omega_f \subset \Omega_g$ and $\Omega_f^e \supset \Omega_g^e$.



Extended macroelement Ω_f^e corresponding to union of two macroelements Ω_{g_0} (outlined by thick lines) and Ω_{g_1}, $n = 2$.

Construction of π^1 in simplest case

Consider (modified) Clement projection onto piecewise linear space $\mathcal{P}_1\Lambda^0(\mathcal{T}_h)$. Operator π^0 has form:

$$(\pi^0 u)(x) = \sum_{f \in \Delta_0(\mathcal{T}_h)} (P_f^0 u)(f)\lambda_f(x)$$

Projections P_f are local H^1 projections wrt to macroelement Ω_f and $\lambda_f(x)$ is barycentric coordinate associated to vertex f.

Define vol_{Ω_f} to be volume form on Ω_f, scaled so that $\int_{\Omega_f} \mathrm{vol}_{\Omega_f} = 1$. Rewrite $P_f u$ in form:

$$P_f u = \int_{\Omega} u \cdot \mathrm{vol}_{\Omega_f}\, dx + Q_f u,$$

where $Q_f u \in \mathcal{P}_1\Lambda^0(\mathcal{T}_{f,h})$ has mean value zero on Ω_f, and satisfies

$$\int_{\Omega_f} \mathrm{grad}\, Q_f u \cdot \mathrm{grad}\, v = \int_{\Omega_f} \mathrm{grad}\, u \cdot \mathrm{grad}\, v$$

for all $v \in \mathcal{P}_1\Lambda^0(\mathcal{T}_{f,h})$ with mean value zero.

Commuting projections

To obtain commuting projections, need to define π_h^1 into space $\mathcal{P}_1^-\Lambda^1(\mathcal{T}_h)$ such that

$$\operatorname{grad} \pi^0 u = \pi^1 \operatorname{grad} u.$$

In particular, have to express

$$\operatorname{grad} \pi^0 u = \operatorname{grad} \Big[\sum_{f \in \Delta_0(\mathcal{T}_h)} \Big(\int_\Omega u \cdot \mathrm{vol}_{\Omega_f} \, dx + (Q_f u)(f) \Big) \lambda_f \Big]$$

in terms of grad u.

Since $Q_f u$ only depends on grad u, need to express:

$$\operatorname{grad} M_h^0 u \equiv \sum_{f \in \Delta_0(\mathcal{T}_h)} \Big[\int_\Omega u \cdot \mathrm{vol}_{\Omega_f} \, dx \Big] \operatorname{grad} \lambda_f$$

in terms of grad u.

The δ operator

If $f = [x_0, \dots, x_{m+1}] \in \Delta_{m+1}(\mathcal{T}_h)$, define

$$(\delta u)_f = \sum_{j=0}^{m+1} (-1)^j u_{f_j},$$

where $f_j = [x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{m+1}]$. So if $f = [x_0, x_1]$, $f_0 = x_1$, $f_1 = x_0$ and

$$(\delta u)_f = u_{f_0} - u_{f_1} = u_{x_1} - u_{x_0}.$$

Key properties:

$$d \circ \delta = \delta \circ d, \qquad \delta \circ \delta = 0.$$

If we let $z_f^0 \in \mathcal{P}_0\Lambda^n(\mathcal{T}_{f,h})$ be defined by $z_f^0 = \mathrm{vol}_{\Omega_f}$, then

$$(\delta z_f^0)_f = \mathrm{vol}_{\Omega_{x_1}} - \mathrm{vol}_{\Omega_{x_0}}.$$

.

The M_h^0 operator

As above, if $M_h^0 : L^2(\Omega) \to$ continuous, piecewise linear functions given by: (replaced f by g in sum)

$$(M_h^0 u)(x) = \sum_{g \in \Delta_0(\mathcal{T}_h)} \Big(\int_\Omega u \cdot \mathrm{vol}_{\Omega_g} \, dx \Big) \lambda_g(x),$$

need to express $\operatorname{grad} M_h^0 u$ in terms of $\operatorname{grad} u$.

If $f = [x_0, x_1]$, $\operatorname{grad} \lambda_g \cdot (x_1 - x_0) = [\lambda_g(x_1) - \lambda_g(x_0)]$. Then

$$\begin{aligned} \mathrm{tr}_f \operatorname{grad} M_h^0(u) \cdot (x_1 - x_0) &= \int_\Omega u(\mathrm{vol}_{\Omega_{x_1}} - \mathrm{vol}_{\Omega_{x_0}}) \, dx \\ &= \int_\Omega u(\delta z_f^0)_f \, dx = -\int_\Omega u(\operatorname{div} z_f^1) \, dx = \int_\Omega \operatorname{grad} u \cdot z_f^1 \, dx. \end{aligned}$$

where $z_f^1 \in \mathring{\mathcal{P}}_1^- \Lambda^{n-1}(\mathcal{T}_{f,h}^e)$ satisfies $\operatorname{div} z_f^1 = -(\delta z^0)_f$ and has zero normal components on the boundary of Ω_f^e. Note: $\int_\Omega (\mathrm{vol}_{\Omega_{x_1}} - \mathrm{vol}_{\Omega_{x_0}}) \, dx = 0$.

The M_h operator (continued)

Let $f = [x_0, x_1]$, $\lambda_i = \lambda_{x_i}$, and

$$\phi_f = \lambda_0(\operatorname{grad} \lambda_1) - \lambda_1(\operatorname{grad} \lambda_0).$$

Then $\operatorname{tr}_f[\phi_f \cdot (x_1 - x_0)] = \operatorname{tr}_f[\lambda_0 + \lambda_1] = 1$. Can conclude:

$$\operatorname{grad} M_h^0 u = \sum_{f \in \Delta_1(\mathcal{T}_h)} \Big(\int_\Omega \operatorname{grad} u \cdot z_f^1 \, dx\Big)\phi_f.$$

Combining results, get

$$\operatorname{grad} \pi^0 u = \operatorname{grad} \sum_{f \in \Delta_0(\mathcal{T}_h)} \Big(\int_\Omega u \cdot \operatorname{vol}_{\Omega_f} dx + (Q_f u)(f)\Big)\lambda_f$$

$$= \operatorname{grad} M_h^0 u + \sum_{f \in \Delta_0(\mathcal{T}_h)} (Q_f u)(f) \operatorname{grad} \lambda_f$$

$$= \sum_{f \in \Delta_1(\mathcal{T}_h)} \Big(\int_\Omega \operatorname{grad} u \cdot z_f^1 \, dx\Big)\phi_f + \sum_{f \in \Delta_0(\mathcal{T}_h)} (Q_f u)(f) \operatorname{grad} \lambda_f.$$

The general case: First define z_f^k

For each $f \in \Delta_0(\mathcal{T}_h)$, $z_f^0 \in \mathring{\mathcal{P}}\Lambda^n(\Omega_f^e)$ defined by $z_f^0 = \mathrm{vol}_{\Omega_f}$.

For $f \in \Delta_k(\mathcal{T}_h)$, define $z_f^k \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k}(\mathcal{T}_f^e)$ inductively by

$$dz_f^k = (-1)^k(\delta z^{k-1})_f$$

$$\int_{\Omega_f^e} z_f^k \wedge d\tau = 0, \qquad \tau \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k-1}(\mathcal{T}_f^e).$$

Construction justified (inductively) by

$$d(\delta z_f^{k-1}) = \delta(dz_f^{k-1}) = (-1)^{k-1}(\delta \circ \delta)z_f^{k-2} = 0.$$

The general case: Next define M_h^k

Let $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ denote family of spaces of form $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, such that corresponding polynomial sequence $(\mathcal{P}\Lambda^k, d)$ is an exact complex.

Using functions $z_f^k \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k}(\mathcal{T}_f^e)$, define $M_h^k : L^2 \to \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ by

$$M_h^k u = \sum_{f \in \Delta_k(\mathcal{T}_h)} \Big(\int_{\Omega_f^e} u \wedge z_f^k \Big) k! \phi_f^k,$$

where ϕ_f^k is Whitney form associated to f, i.e,

$$\phi_f^k = \sum_{i=0}^{k} (-1)^i \lambda_i d\lambda_0 \wedge \cdots \wedge \widehat{d\lambda_i} \wedge \cdots \wedge d\lambda_k.$$

Key result: For any $v \in H\Lambda^{k-1}(\Omega)$, $dM_h^{k-1} v = M^k dv$.

Local bounded cochain projections

Projection $\pi^k = \pi^k_n$ defined by recursion

$$\pi^k_m u = \pi^k_{m-1} u + \sum_{f \in \Delta_m(\mathcal{T}_h)} E^k_f \circ \mathrm{tr}_f \circ P^k_f (u - \pi^k_{m-1} u), \qquad k \le m \le n,$$

where $P^k_f : H\Lambda^k(\Omega_f) \to \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ defined by:

$$\langle P^k_f u, d\tau \rangle_{\Omega_f} = \langle u, d\tau \rangle_{\Omega_f}, \quad \tau \in \breve{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}),$$
$$\langle dP^k_f u, dv \rangle_{\Omega_f} = \langle du, dv \rangle_{\Omega_f}, \quad v \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}).$$

Here

$$\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) = \{u \in \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) : tr_f \in \breve{\mathcal{P}}\Lambda^k(f)\},$$

where $\breve{\mathcal{P}}\Lambda^k(f) = \mathring{\mathcal{P}}\Lambda^k(f)$ if $\dim f > k$, while if $\dim f = k$,

$$\breve{\mathcal{P}}\Lambda^k(f) = \{u \in \mathcal{P}\Lambda^k(f) : \int_f u = 0\}.$$

Local bounded cochain projections (continued)

To start iteration:

$$\pi_m^k u = \pi_{m-1}^k u + \sum_{f \in \Delta_m(\mathcal{T}_h)} E_f^k \circ \operatorname{tr}_f \circ P_f^k (u - \pi_{m-1}^k u), \qquad k \le m \le n,$$

need $\pi_{k-1}^k : H\Lambda^k \to \mathcal{P}_1^- \Lambda^k(\mathcal{T}_h)$.

Two requirements: operators π_{k-1}^k commute with d, and

$$\int_f \operatorname{tr}_f \pi_{k-1}^k u = \int_f \operatorname{tr}_f u, \quad f \in \Delta_k(\mathcal{T}_h),\, u \in \mathcal{P}\Lambda^k(\mathcal{T}_h).$$

Operators M_h^k *essential for construction* of π_{k-1}^k, but need further technical results, since M_h^k not a projection (not an identity on Whitney forms).

A double complex (A. Weil, 1952)

Let $\mathcal{T}_f^e$ denote $\mathcal{T}_h$ restricted to Ω_f^e. For $0 \le m \le n$, consider complexes

$$\bigoplus_{f \in \Delta_m(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^0(\mathcal{T}_f^e) \xrightarrow{d} \bigoplus_{f \in \Delta_m(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^1(\mathcal{T}_f^e) \xrightarrow{d} \cdots$$

$$\cdots \xrightarrow{d} \bigoplus_{f \in \Delta_m(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^n(\mathcal{T}_f^e) \to \bigoplus_{f \in \Delta_m(\mathcal{T})} \mathcal{P}_0(\Omega_f^e)$$

where $d = d_k$ is exterior derivative restricted to each Ω_f^e.

For $f = [x_0, x_1, \ldots x_{m+1}] \in \Delta_{m+1}(\mathcal{T}_h)$, let δ be defined by

$$(\delta u)_f = \sum_{j=0}^{m+1} (-1)^j u_{f_j},$$

where $f_j = [x_0, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots x_{m+1}]$.

Commuting diagram:

Then

$$\delta : \bigoplus_{f \in \Delta_m(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f^e) \to \bigoplus_{f \in \Delta_{m+1}(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f^e)$$

and satisfies $\delta \circ d = d \circ \delta$ and $\delta \circ \delta = 0$.

Get

$$\begin{array}{ccc}
\bigoplus_{f \in \Delta_m(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f^e) & \xrightarrow{d} & \bigoplus_{f \in \Delta_m(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^{k+1}(\mathcal{T}_f^e) \\
\downarrow \delta & & \downarrow \delta \\
\bigoplus_{f \in \Delta_{m+1}(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f^e) & \xrightarrow{d} & \bigoplus_{f \in \Delta_{m+1}(\mathcal{T})} \mathring{\mathcal{P}}_1^- \Lambda^{k+1}(\mathcal{T}_f^e)
\end{array}$$

Summary:

To construct local bounded cochain projections:

- I. Need to define for each k, operators π_h^k that are projections, i.e., are the identity on the subspace (so Clément interpolants don't work).
- II. Need to have π_h^k bounded on $H\Lambda^k$, so can't just use canonical degrees of freedom.
- III. Need to get π_h^k to commute with exterior derivative; i.e., $d\pi_h^k = \pi_h^{k+1} d$; not so easy.
- IV. Need to have π_h^k locally defined.

Key ideas: Use of geometric decompositions of finite element spaces and a double complex involving d and δ.