χ-boundedness of graph classes excluding wheel vertex-minors

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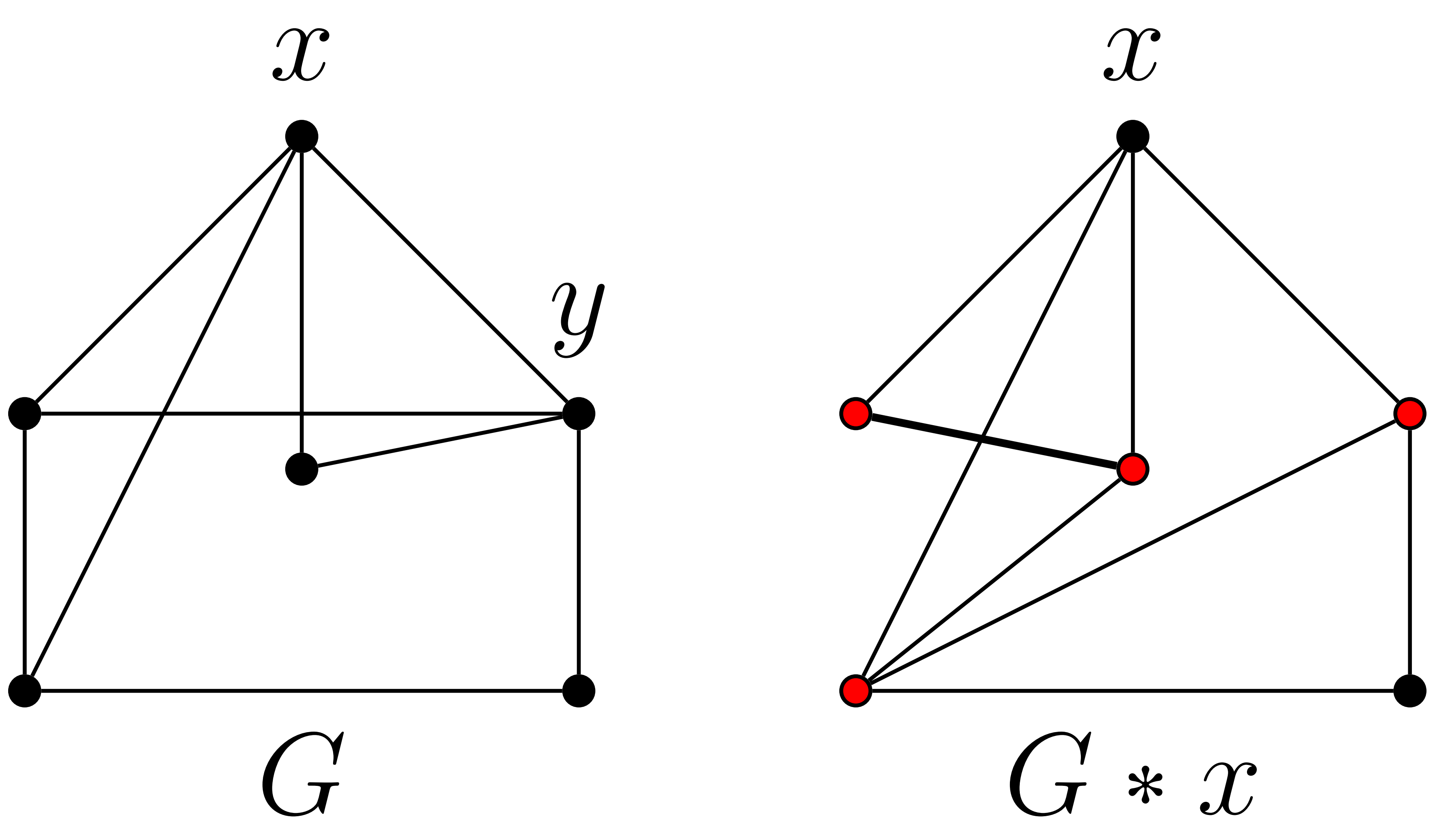
χ-bounded classes of graphs

- A class C of graphs is χ-bounded if $\exists$ function f such that $\forall G \in C$, $\chi(G) \leq f(\omega(G))$.
- Examples:
  - Perfect graphs
    - bipartite graphs or line graphs of bipartite graphs, or their complements
    - interval graphs
  - unit disk graphs $\chi \leq 3\omega-2$ (Peeters, 1991)
  - graphs of bounded rank-width (Dvořák and Kráľ 2012)

Local complementation and vertex-minors



(local complementation at x)

H is locally equivalent to G if $H=G*x_1*x_2*x_3\ldots$

vertex-minor=graph obtained by applying a sequence of local complemention and vertex deletions

Geelen's conjecture

- Geelen (2009): Are H-vertex-minor-free graphs χ-bounded for fixed H?
- True if:
  - H=W_5 (Dvořák and Kráľ 2012)
  - H=Fan graph (Choi, Kwon, and O. 2017)
- If graphs with no H-subdivision (as an induced subgraph) are χ-bounded, then H-vertex-minor-free graphs are χ-bounded.
  - H=long cycle (Gyárfás conj. 1985, solved by Chudnovsky, Scott, Seymour 2016)
  - H=tree (Scott 1997)
  - H="banana tree" (Scott, Seymour 2017)

Our Theorem

- If H=wheel graph, then Geelen's conjecture is true.

THM (Choi, Kwon, O., Wollan)

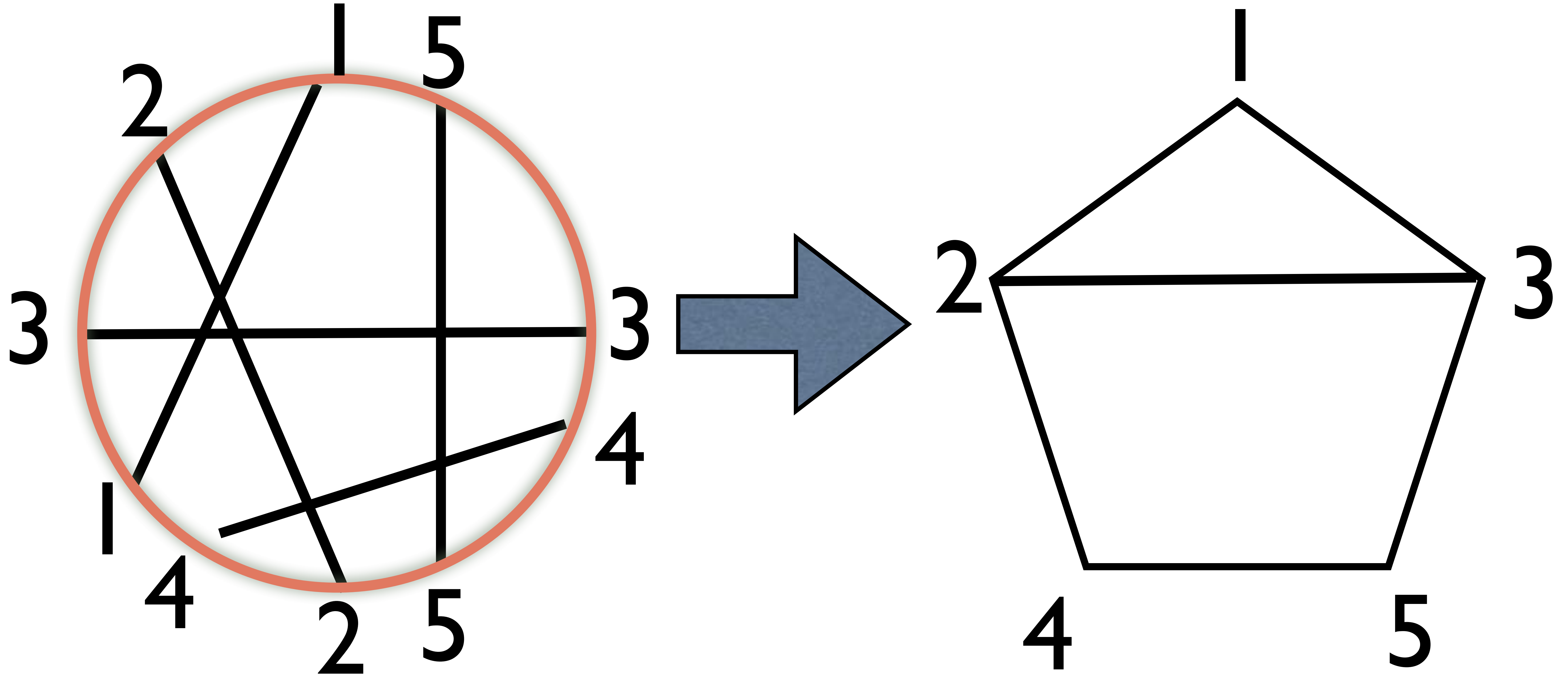
For $n \geq 3$,

the class of graphs with no W_n vertex-minor is χ-bounded

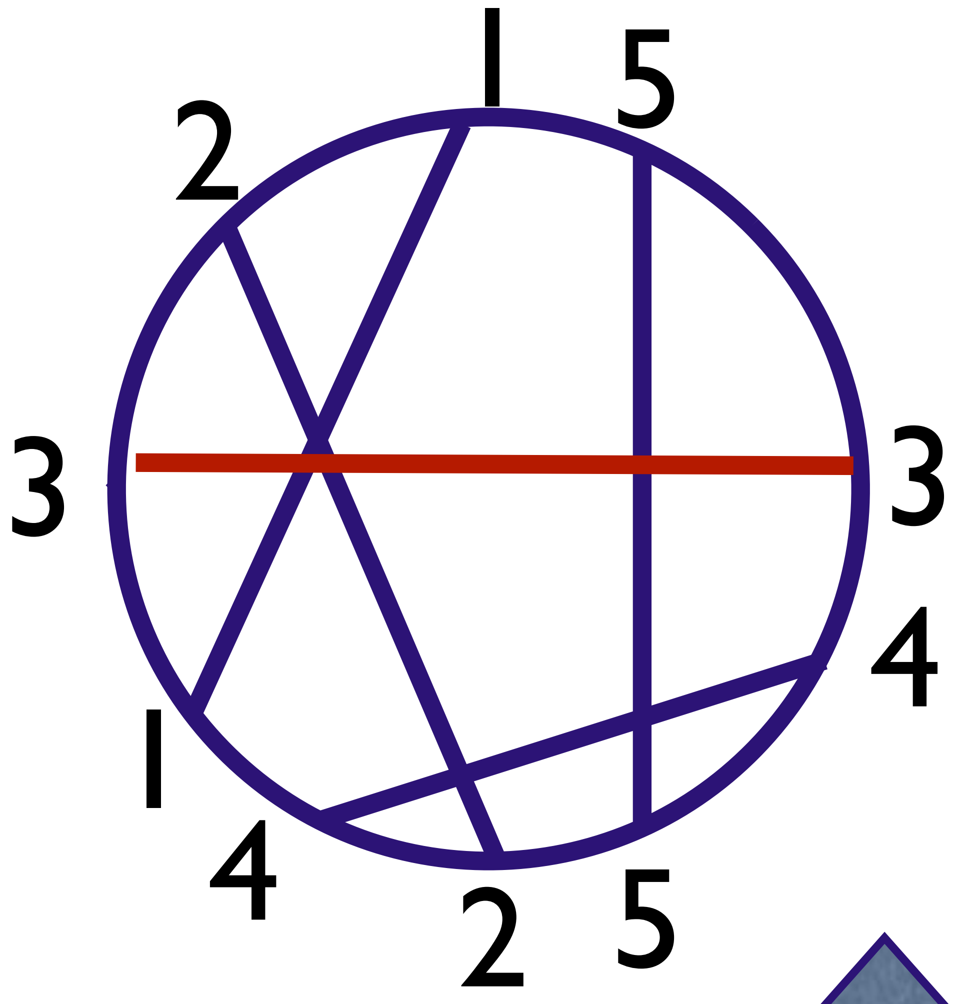
- Corollary
  - Circle graphs are χ-bounded
  - Common generalization of known cases
    - $H=W_5$ (Dvořák and Kráľ 2012) --- depending on χ-boundedness of circle graphs
    - H=Fan graph (Choi, Kwon, and O. 2017)

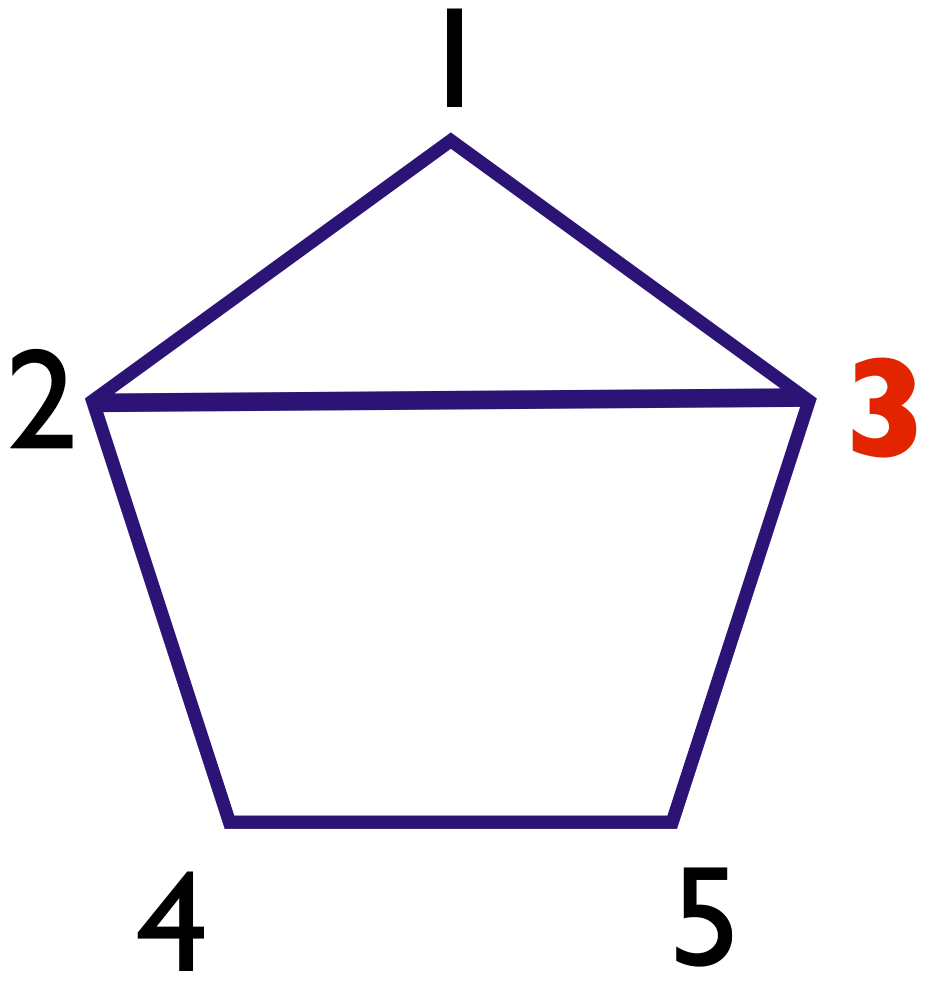
Circle graphs are χ-bounded

Intersection graphs of chords in a circle

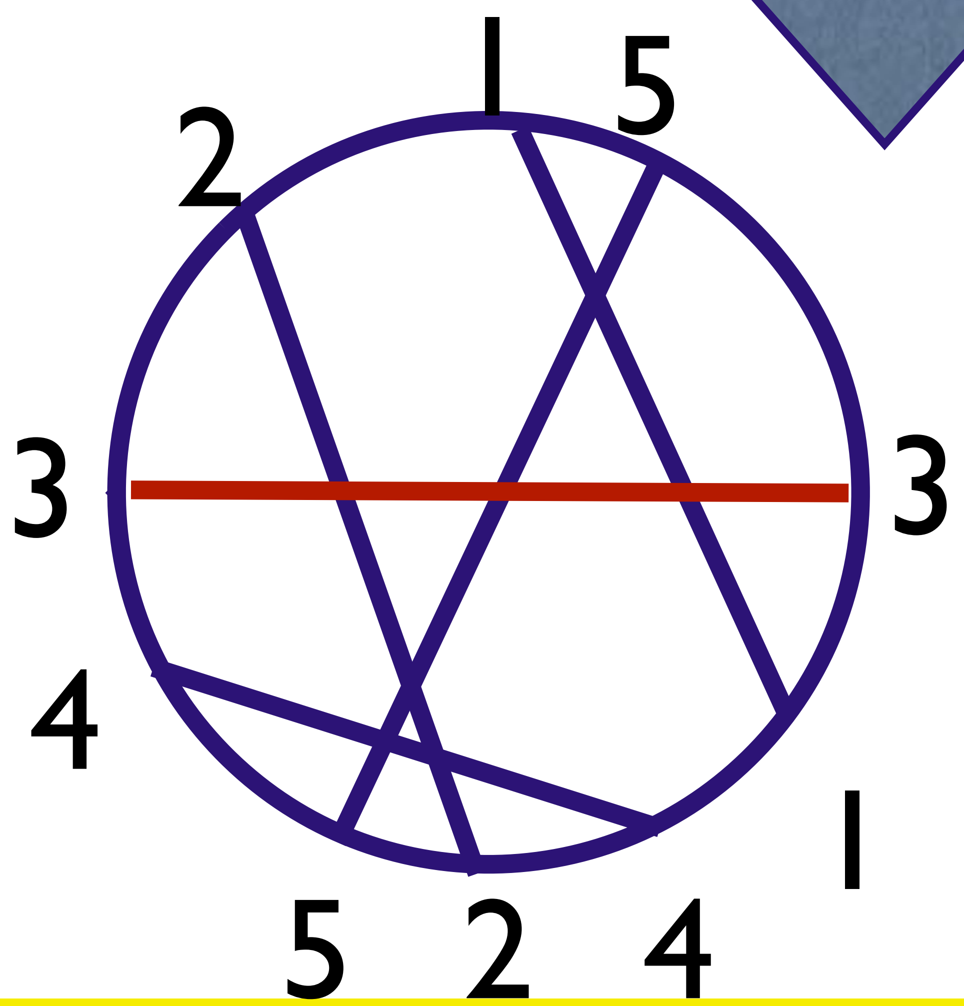


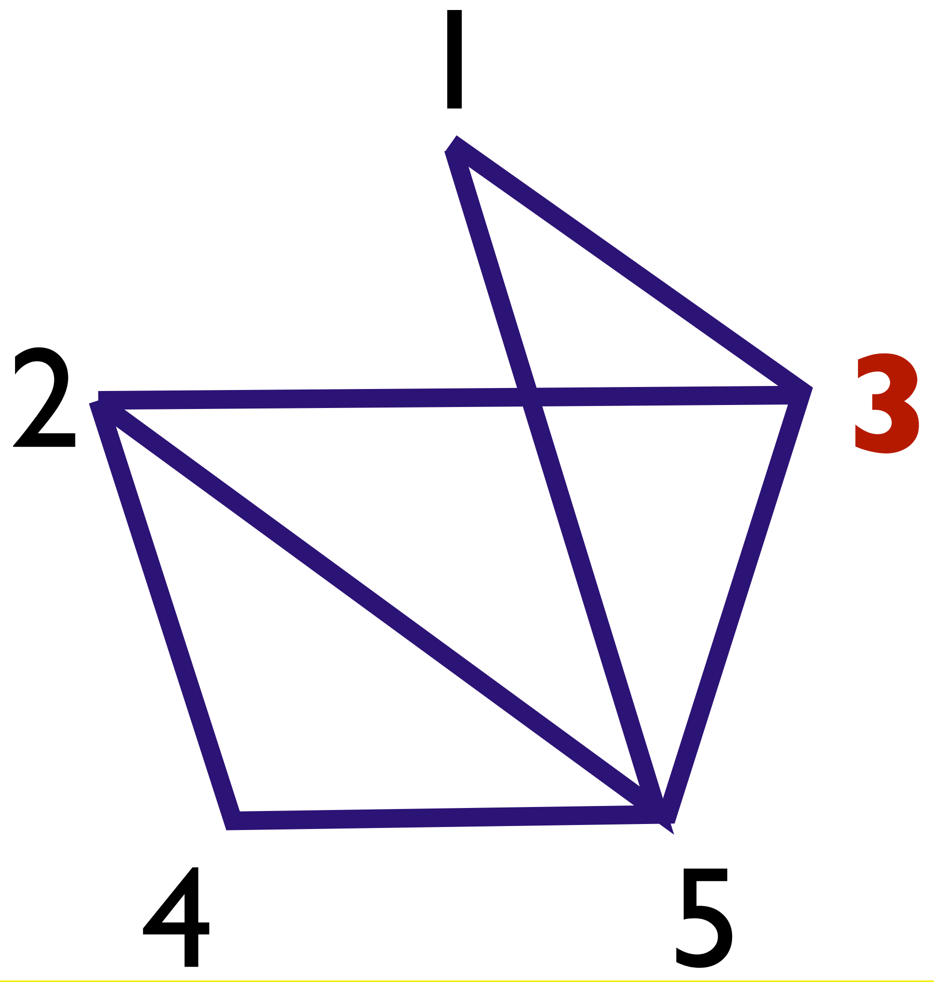
Gyárfás (1985): circle graphs are χ-bounded





Local complementation
at 3



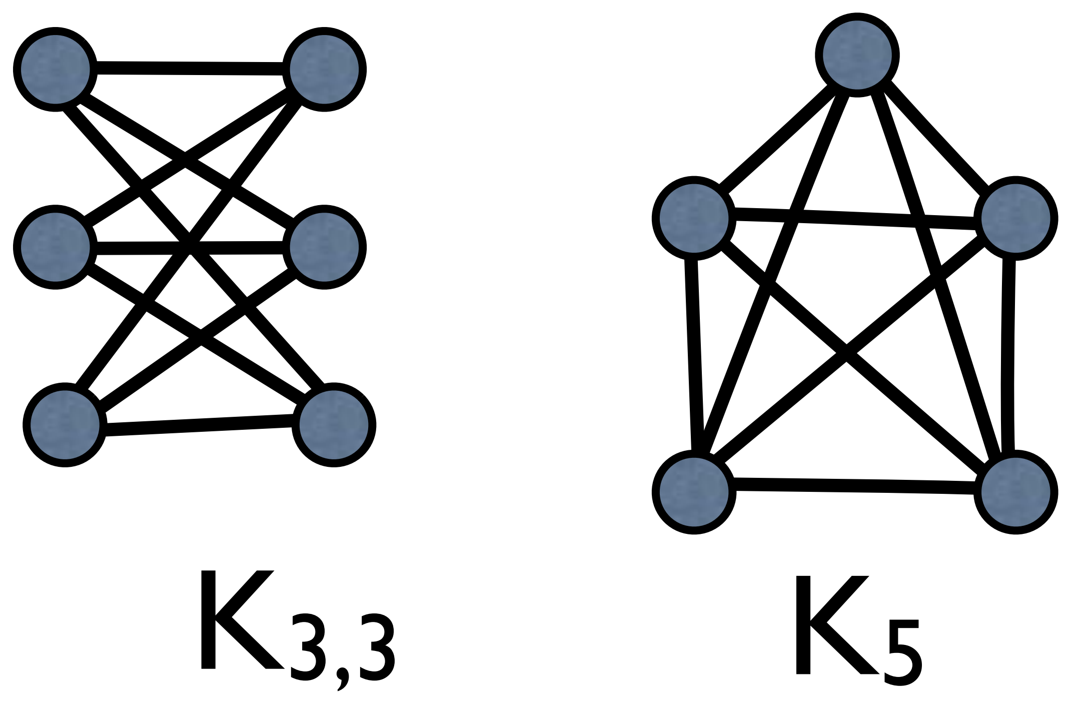


"Local complementation" preserves
the property of being circle graphs

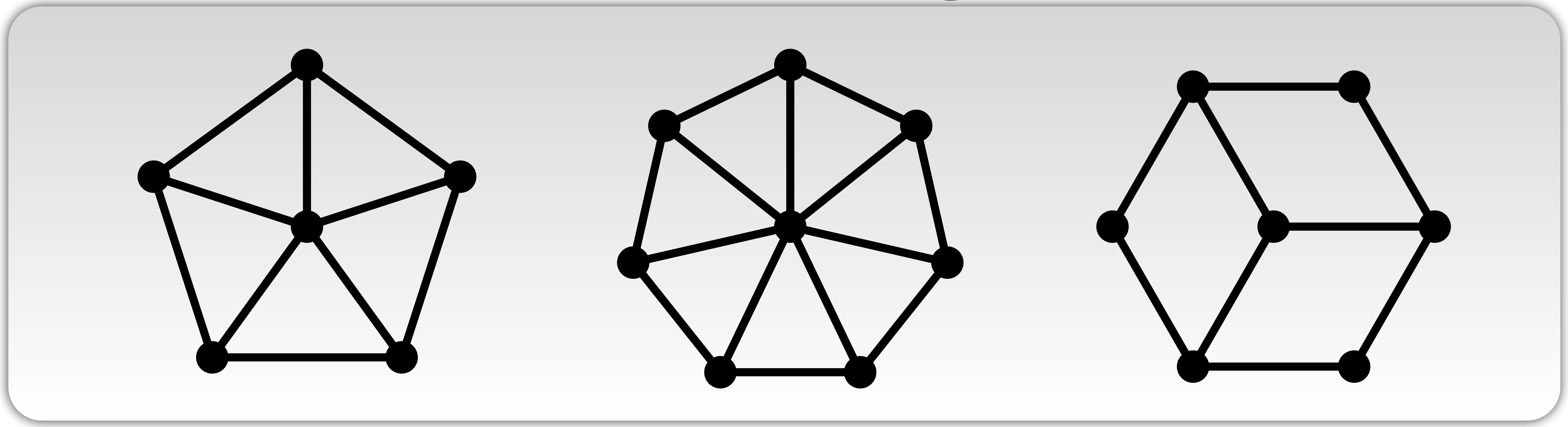
Kuratowski-Wagner

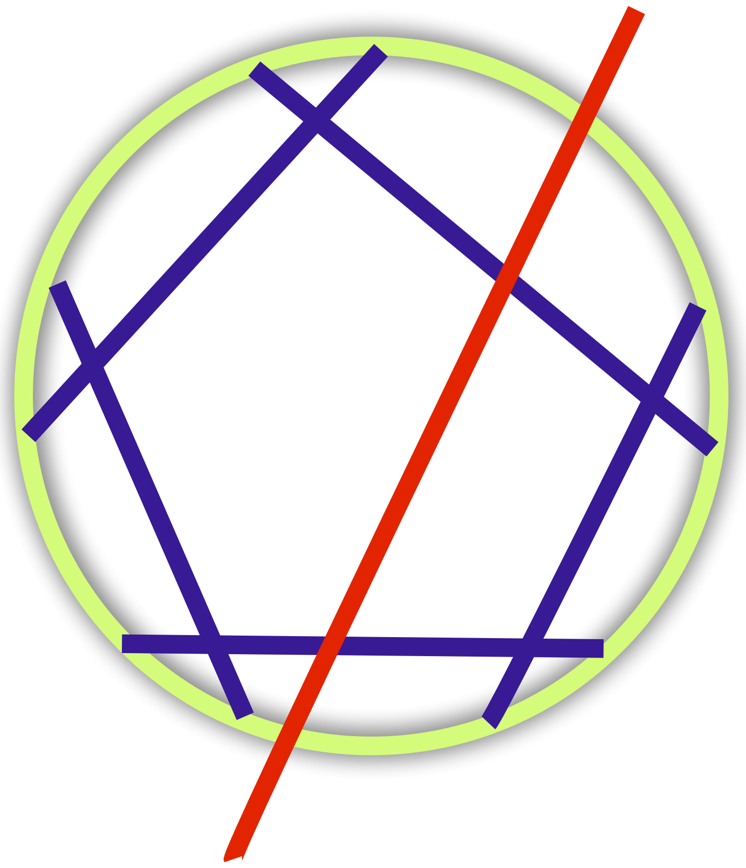
- "G is planar" ⇔
  "G has no minor isomorphic to K_5 or $K_{3,3}$"

Can we characterize circle graphs in terms of forbidden structures?



Non-circle graphs



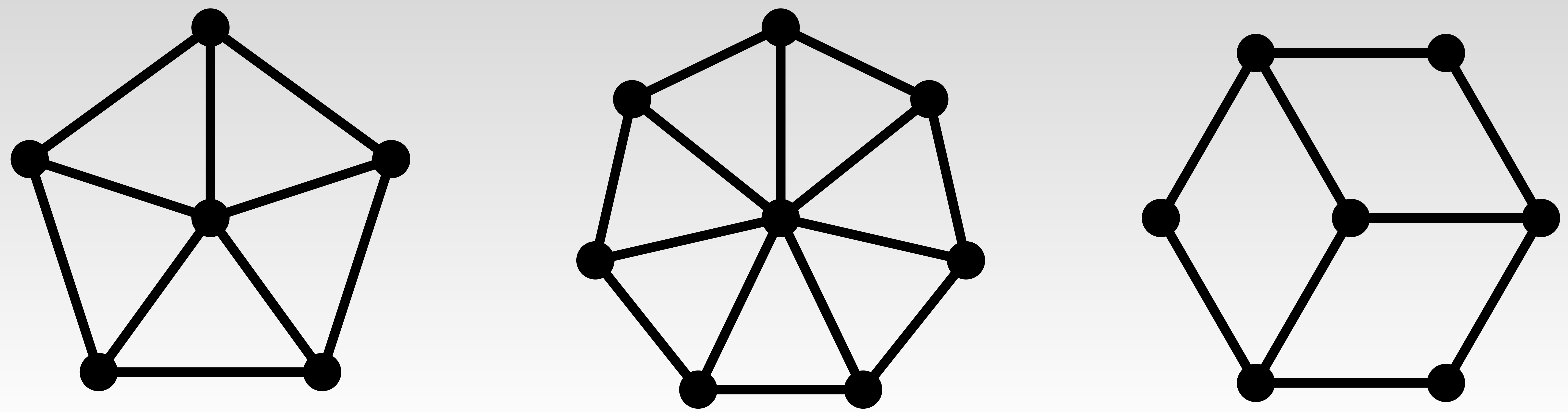


They are not circle graphs

Circle graphs via vertex-minors

Bouchet (1994)

A graph is a circle graph
iff
it has no vertex-minor isomorphic to



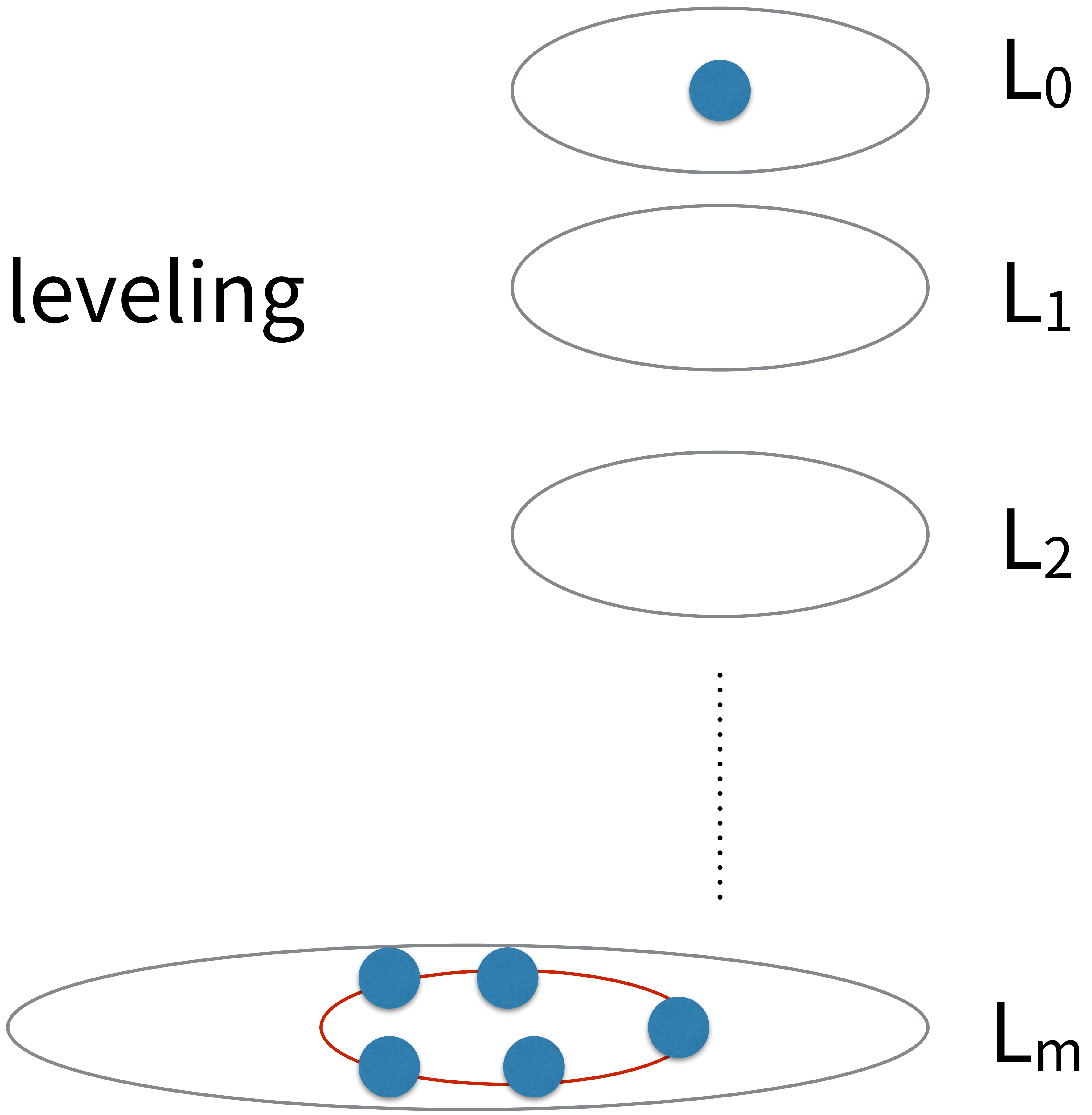
THM (Choi, Kwon, O., Wollan)

The class of graphs with no W_n vertex-minor is χ-bounded

Corollary: Circle graphs are χ-bounded

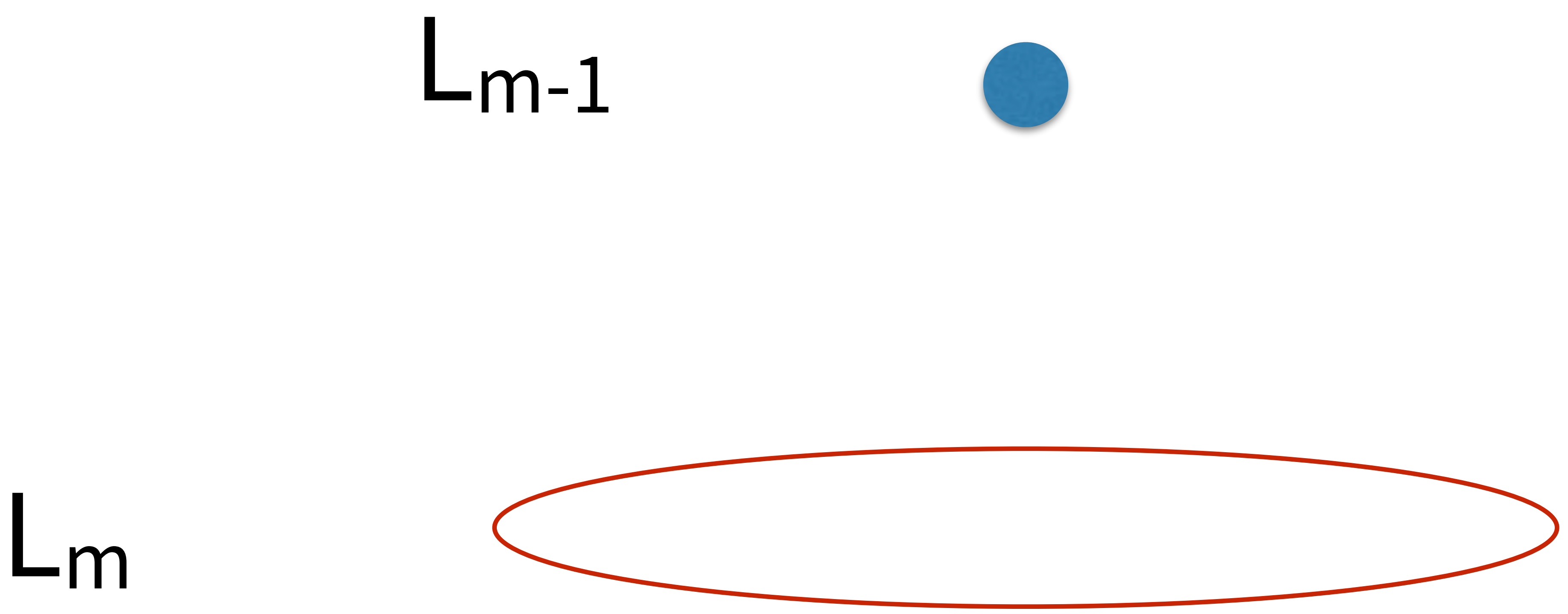
Proof sketch

Initial proof setup (leveling)

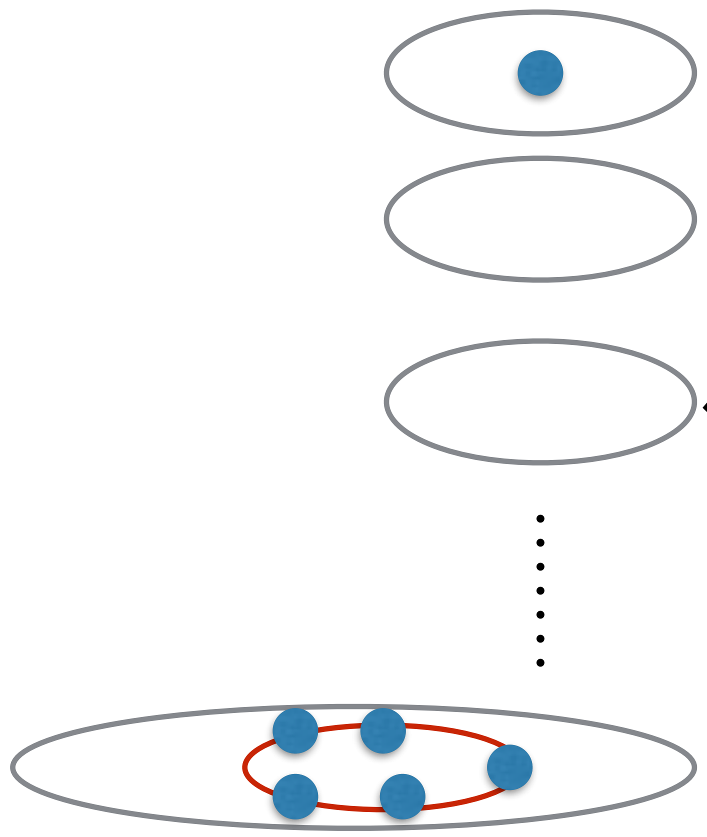


- $|L_0|=1$
- $|L_i|$=vertices having ≥ 1 neighbor in L_{i-1} and no neighbor in $L_0, L_1, \ldots, L_{i-2}$.
- If G has large χ, then some L_m has large χ.
- Choose min m
- Find a long induced cycle in L_m by Chudnovsky, Scott, Seymour
- Choose each $L_0, L_1, \ldots, L_{m-1}$ minimal while keeping this long cycle
- Pray to get a wheel?
  - FAIL

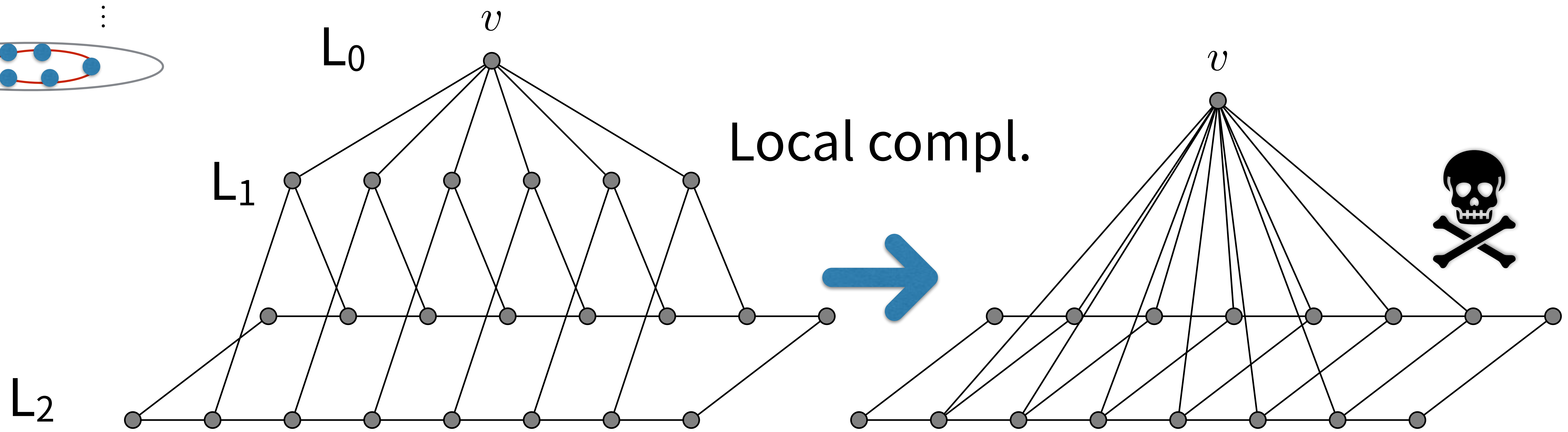
Single leveling --- Lucky case



If a vertex in L_{m-1} has large # neighbors in the cycle of L_m, then we win



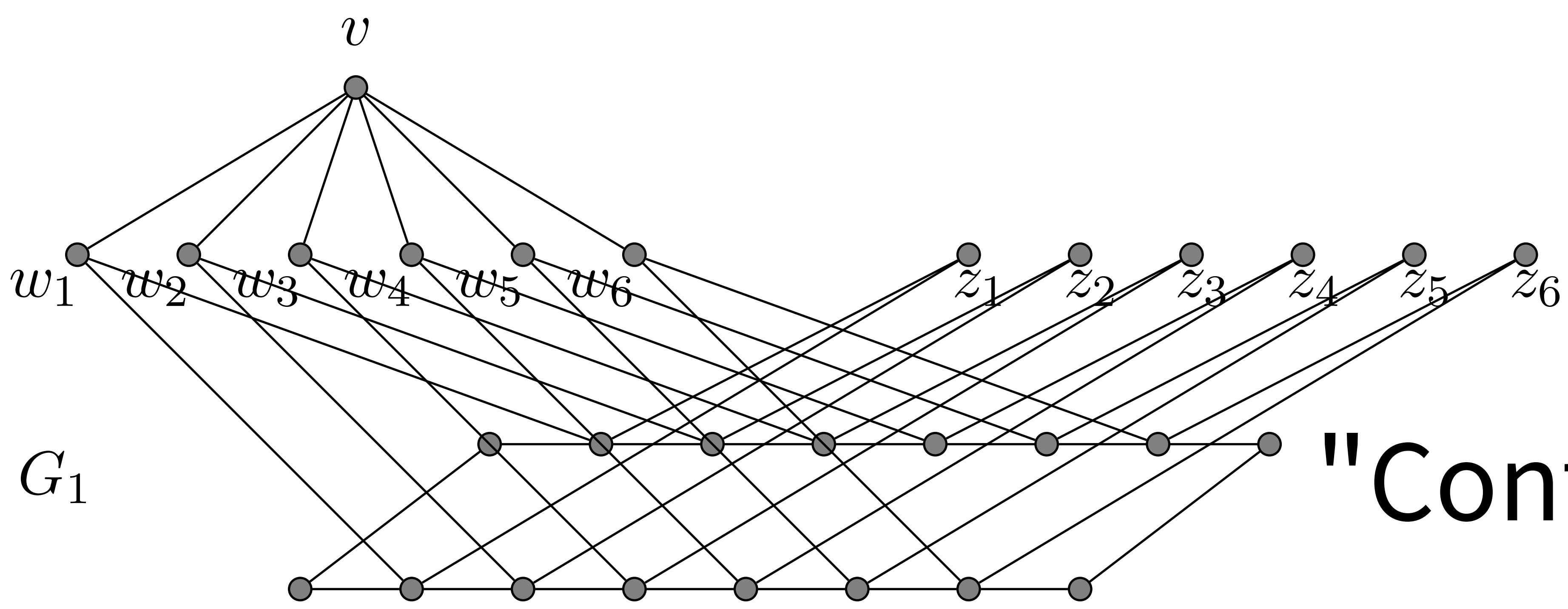
Single leveling --- Trouble



A long cycle with
many attachments from the top

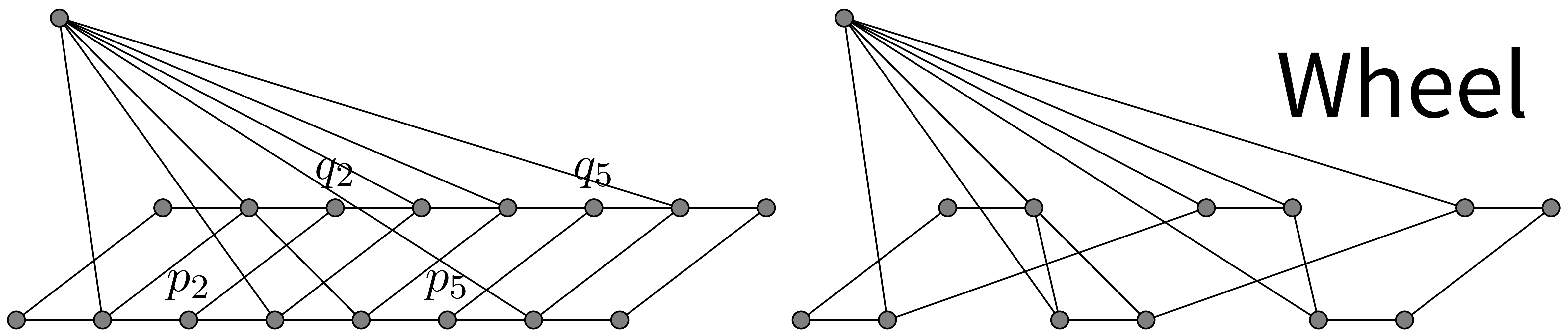
Difficult to remove chords

Fix: Find "controllers"



"Controllers"

Wheel



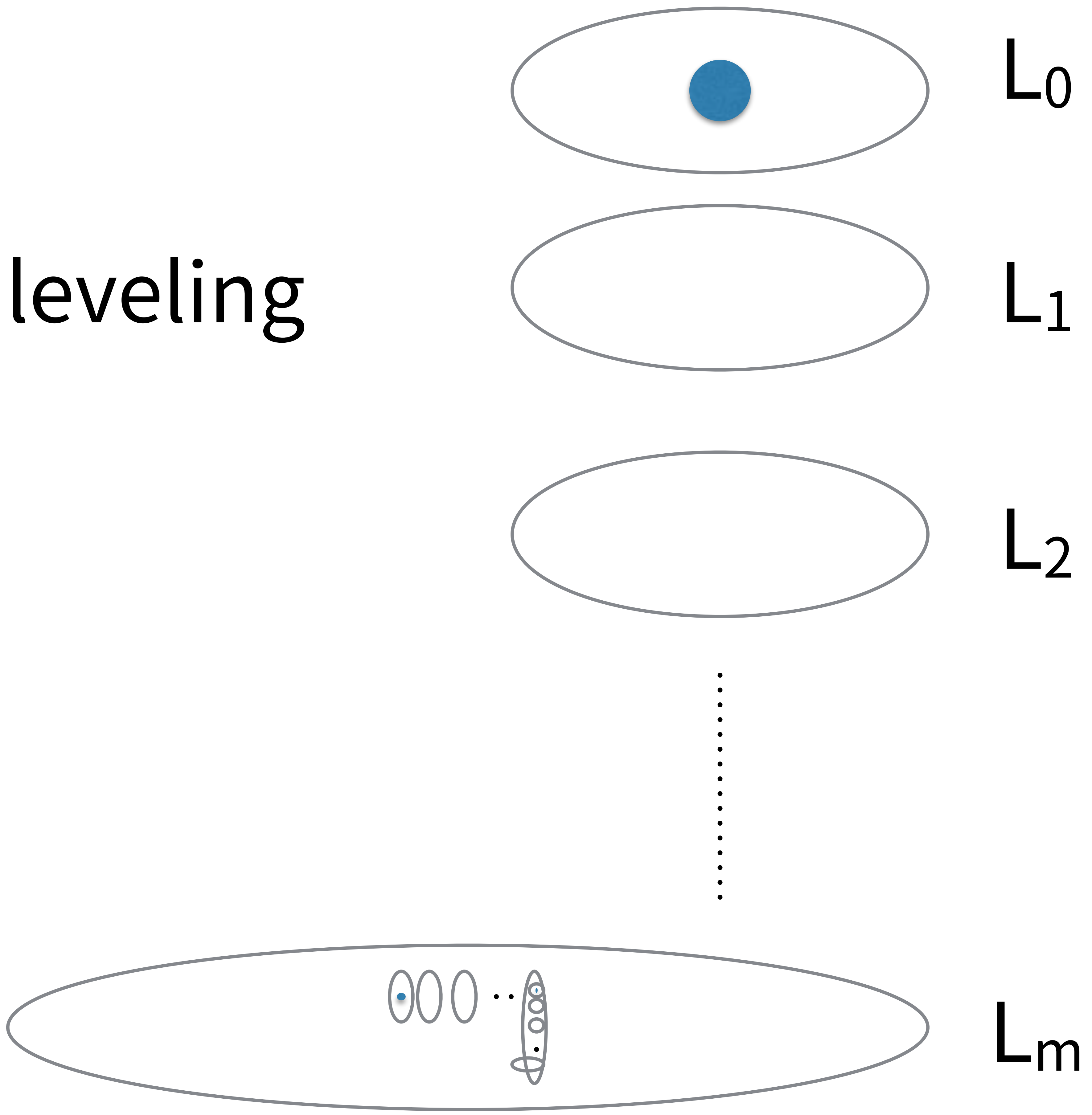
G_2 G_3

$G_1{}^*w_1{}^*w_3{}^*w_4{}^*w_6{}^*z_2{}^*z_5$ $G_3{}^*p_2{}^*q_2{}^*p_2{}^*p_5{}^*q_5{}^*p_5$

"Controllers" allow us to remove chords

- Goal: Find two disjoint large independent sets having "regular" neighbors on the cycle
  - So that one set would become "controllers"
- **How to obtain such a structure?**

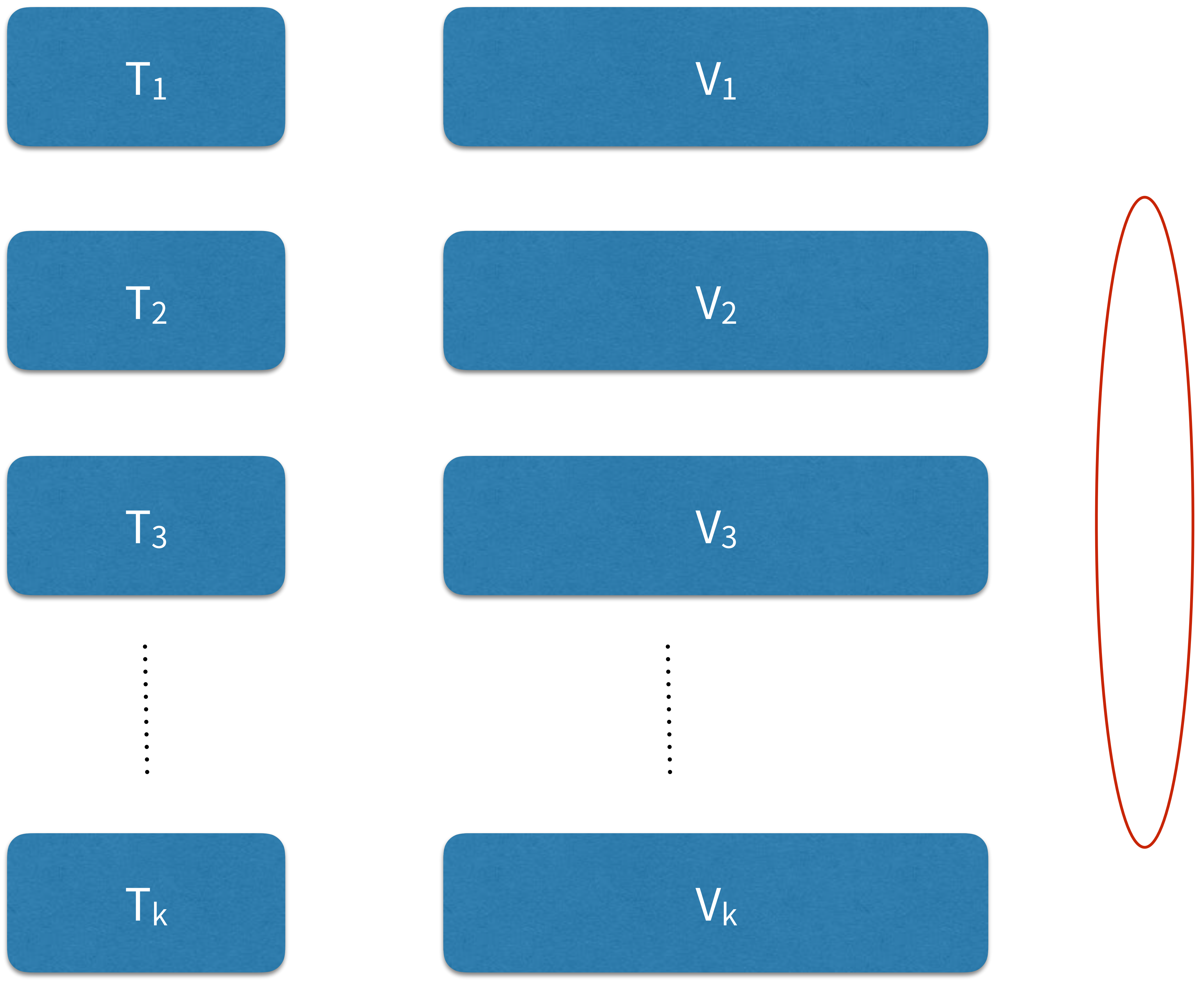
Repeated leveling



- Inside L_m, find another leveling
- Repeat this many times
    - Many layers of leveling
- Apply **Ramsey**!
- Find one vertex from each layer to create "controllers"

Producing a wheel vertex-minor

- Prop: For all n and q, there exist k, M such that if G with no clique of size q has an induced cycle C of length $\geq$M and disjoint vertex sets $V_1, V_2, \ldots, V_k$, $T_1, T_2, \ldots, T_k$ disjoint with the cycle such that
  (1) each vertex in C has a neighbor in each V_j,
  (2) each vertex in V_j has at most n-1 neighbors in C
  (3) no edges from T_j to C
  (4) each vertex in V_j has a neighbor in T_j,
  (5) T_j has a "root" vertex that for each vertex in $N(V_j) \cap T_j$, there is a path P in $G[T_j]$ having only one vertex in $N(V_j)$,
  then it contains W_n as a vertex-minor.



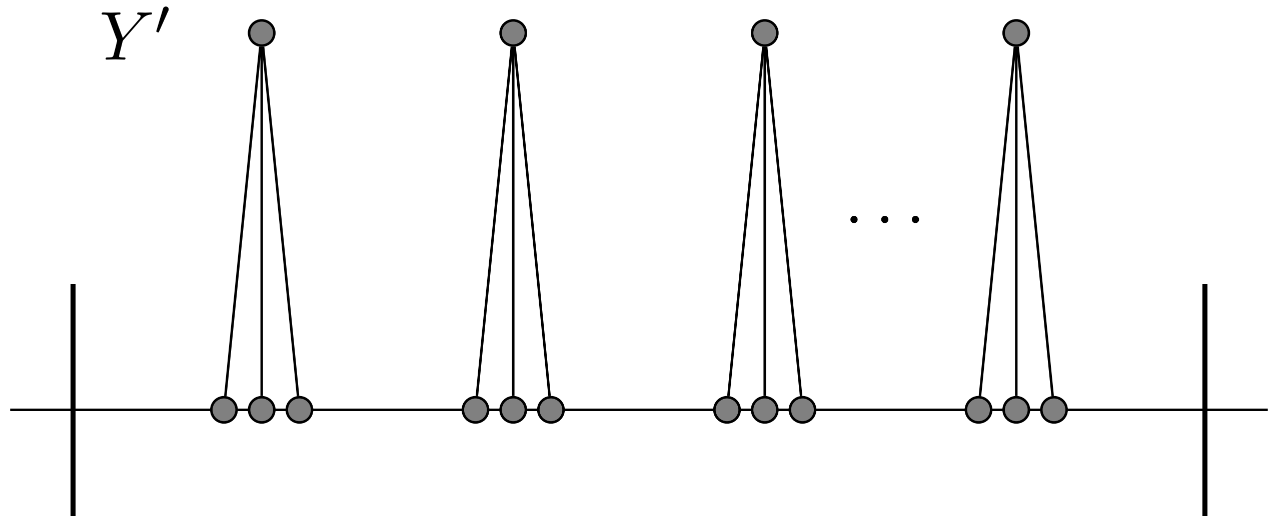
For a sequence $(A_1, \ldots, A_\ell)$ of finite subsets of an interval $I \subseteq \mathbb{R}$, a partition $\{I_1, \ldots, I_k\}$ of I into intervals is called a *regular partition* of I with respect to $(A_1, \ldots, A_\ell)$ if for all $i \in \{1, \ldots, k\}$, either

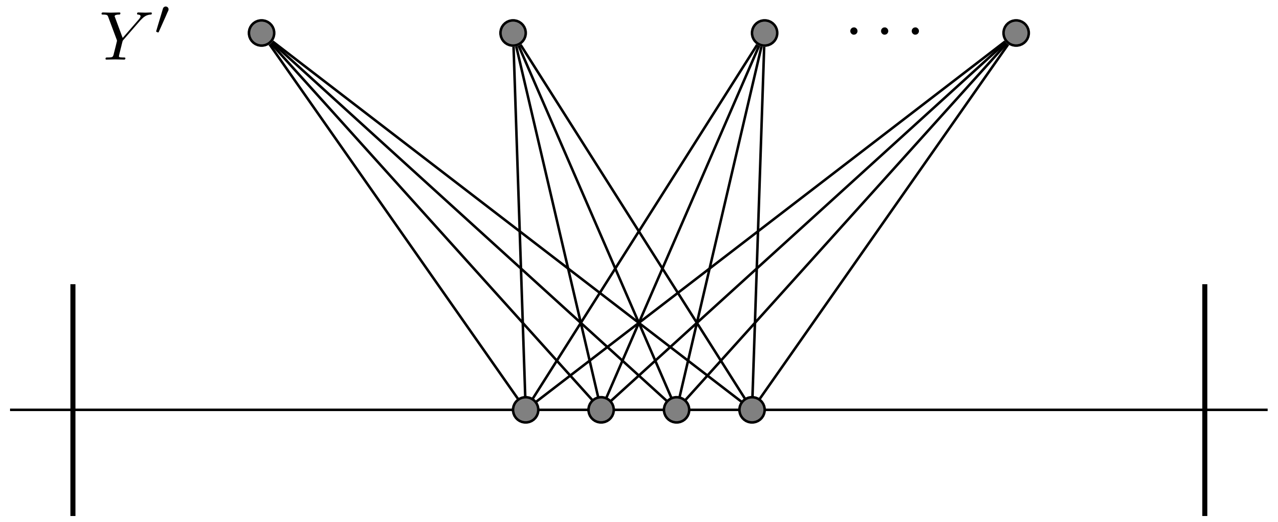
- $A_1 \cap I_i = A_2 \cap I_i = \cdots = A_\ell \cap I_i \neq \emptyset$, or
- $|A_1 \cap I_i| = |A_2 \cap I_i| = \cdots = |A_\ell \cap I_i| > 0$, and for all $j, j' \in \{1, \ldots, \ell\}$ with $j < j'$, $\max(A_j \cap I_i) < \min(A_{j'} \cap I_i)$, or
- $|A_1 \cap I_i| = |A_2 \cap I_i| = \cdots = |A_\ell \cap I_i| > 0$, and for all $j, j' \in \{1, \ldots, \ell\}$ with $j < j'$, $\max(A_{j'} \cap I_i) < \min(A_j \cap I_i)$.

The number of parts k is called the *order* of the regular partition.

Regular partition lemma:

For all m, there exists N such that every sequence $(A_1, \ldots, A_N)$ of k-element sets of reals has a subsequence $(A_1', A_2', \ldots, A_m')$ and a regular partition of $\mathbb{R}$ w.r.t. $(A_1', \ldots, A_m')$ of order $\leq$k.





Further questions

- Q1: Geelen's conjecture.
  Is it true for double wheels?
- Q2: Structure for H-vertex-minor free graphs?
- Q3: Deciding whether H is a vertex-minor in poly time?

Thank you
for your attention!