1 Introduction

The goal of this talk is

- Large N duality (Conifold transition): the relation between quantum knot invariants and enumerative invariants. One way to understand modularity
- Refinement of LMOV and Positivity Conjecture of refined Chern-Simons invariants

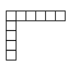
1.1 Notation

- $\mathbb{K}_0 = \mathbb{C}[q^{\pm\frac{1}{2}}, t^{\pm\frac{1}{2}}]$ denotes the ring of Laurent polynomials
- $\mathbb{K} = \mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ denotes the field of rational functions
- $\mathbb{K}[X_1, \ldots, X_N]^{S_n}$ denotes the ring of symmetric functions
- For $f \in \mathbb{K}_0[Y_1, \ldots, Y_N]^{\mathrm{sym}}$, we define a Macdonald polynomial $P_\lambda(X)$ of GL_N-type with a dominant weight $\lambda \in P_+$ by

  $$p(f) \cdot P_\lambda(X) = f(\mathfrak{t}^\rho \mathfrak{q}^\lambda) P_\lambda(X) \ , \quad P_\lambda(X) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu,$$

  where m_ν is the sum of the elements in S_N orbit of X^ν and $<$ is the dominance partial order on the partitions.
- We denote the *unreduced* invariants by $\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t)$ and the *reduced* invariants by $\mathrm{rCS}_\lambda(T_{m,n}; a, q, t)$ where they are related by

  $$\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t) = \overline{\mathrm{rCS}}_\lambda(\bigcirc; a, q, t) \ \mathrm{rCS}_\lambda(T_{m,n}; a, q, t) \ .$$
- Assigning the Young diagram with h boxes of one row to the trivial representation $|0\rangle$, the irreducible representations $\wedge^d V$ of $\mathfrak{S}_h$ are called *hook representations* since their Young tableau are of the form with $(h-d)$-boxes in the first row

   .

2 Refined CS invariants

We follow the definition in [Che13].

2.1 Double affine Hecke algebra of GL_N

Topological interpretation. Let E be a 2-torus. Consider the N-fold product E^N, and let $(E^N)^{\mathrm{reg}} := \{(x_1, \ldots, x_N) \in E^N : x_i \neq x_j \quad \text{if} \quad i \neq j\}$, $C := (E^N)^{\mathrm{reg}}/S_N$. The fundamental group $\pi_1(C)$ is known as the *elliptic braid group* or *double affine braid group*.

Lemma 2.1 *We have* $\pi_1(C) = \langle T_1^{\pm}, \dots, T_{N-1}^{\pm}, X_1^{\pm}, \dots, X_N^{\pm}, Y_1^{\pm}, \dots, Y_N^{\pm}\rangle$ *with relations*

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_iX_iT_i = X_{i+1}, \qquad {T_i}^{-1}Y_i{T_i}^{-1} = Y_{i+1}$$
$$[T_i, T_k] = 0, \qquad [T_i, X_k] = 0, \qquad [T_i, Y_k] = 0, \qquad \text{for } |i-k| > 1$$
$$[X_j, X_k] = 0, \qquad [Y_j, Y_k] = 0,$$
$$Y_jX_1 \dots X_N = X_1 \dots X_NY_j, \quad {X_1}^{-1}Y_2 = Y_2{X_1}^{-1}{T_1}^{-2}$$

The generator X_i corresponds to the i-th point going around a loop in the horizontal direction on E; Y_i corresponds to the i-th point going around in the vertical direction on E; while T_i corresponds to the transposition of the i-th and $(i+1)$-th points.

One can form a twisted group algebra, which is a deformation of the group algebra $\pi_1(C)$

$$Y_jX_1 \dots X_N = q^{\frac{1}{2}}X_1 \dots X_NY_j$$

arising from a central extension of $\pi_1(C)$ (so that the central element z becomes q in the twisted group algebra). The *double affine Hecke algebra* (=DAHA) of GL_n is obtained by

$$\ddot{\boldsymbol{H}}_N := \mathbb{K}^{\text{tw}}\pi_c(C)/((T_i + t^{-\frac{1}{2}})(T_i - t^{\frac{1}{2}}))_{i=1,\dots,N-1} \ .$$

In fact, it contains two copies of the affine Hecke algebras generated by (T_i, X_j) and (T_i, Y_j).

2.1.1 Modular transformation

There is an action of $\text{PSL}(2,\mathbb{Z}) = \langle \tau_\pm : \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1}\rangle$ generated by

$$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \qquad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} ,$$

on $\ddot{\boldsymbol{H}}_N$, which can be explicitly written as

$$\tau_- : \begin{cases} X_i \mapsto X_iY_i(T_{i-1} \cdots T_i)(T_i \cdots T_{i-1}) \\ T_i \mapsto T_i \\ Y_i \mapsto Y_i \end{cases} \qquad \tau_+ : \begin{cases} X_i \mapsto X_i \\ T_i \mapsto T_i \\ Y_i \mapsto Y_iX_i(T_{i-1}^{-1} \cdots T_i^{-1})(T_i^{-1} \cdots T_{i-1}^{-1}) \end{cases}$$

2.1.2 PBW theorem and evaluation coinvaraint

For an element $w \in W = S_N$ of the Weyl group and its representation $w = s_{i_1} \cdots s_{i_j}$ by transpositions ($s_i = (i, i+1)$), we define $T_w := T_{i_1} \cdots T_{i_j}$. From the definition of $\ddot{\boldsymbol{H}}_N$, this is independent of a representation w. In addition, for a set of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_N)$, we define $X_\lambda := \prod_{i=1}^N X_i^{\lambda_i}$ and $Y_\lambda := \prod_{i=1}^N Y_i^{\lambda_i}$.

Theorem 2.2 (PBW Theorem) *Any* $h \in \ddot{\boldsymbol{H}}_N$ *can be written uniquely in the form*

$$h = \sum_{\lambda,w,\mu} c_{\lambda,w,\mu}X_\lambda T_w Y_\mu,$$

for $c_{\lambda,w,\mu} \in \mathbb{K}_0$.

Writing an element $h \in \ddot{\boldsymbol{H}}_N$ in the form of $h = \sum_{\lambda,w,\mu} c_{\lambda,w,\mu}X_\lambda T_w Y_\mu$ via the PBW Theorem 2.2, we define a map $\{\cdot\}_{ev} : \ddot{\boldsymbol{H}}_N \to \mathbb{K}_0$ called the *evaluation coinvariant* by substituting

$$X_i \mapsto t^{-\frac{N+1-2i}{2}}, \quad T_i \mapsto t^{\frac{1}{2}}, \quad Y_i \mapsto t^{\frac{N+1-2i}{2}}. \tag{2.1}$$

2.1.3 Spherical subalgebra

Using a central idempotent in the group algebra of the Weyl group W

$$\mathbf{e} := \sum_{w\in W} \frac{t_w T_w}{t_w^2} .$$

we can define the *spherical DAHA* $\boldsymbol{S\ddot{H}}_N := \mathbf{e}\boldsymbol{\ddot{H}}_N\mathbf{e} \subset \boldsymbol{\ddot{H}}_N$. Note that given a reduced decomposition $w = s_{i_1}\cdots s_{i_k}$ of an element $w \in W$, we define $t_w := t^{\frac{k}{2}}$. For instance, the idempotents for $N = 2$ and $N = 3$ are expressed by, respectively,

$$\mathbf{e} = \frac{t^{\frac{1}{2}}T_1 + 1}{t+1} , \qquad \mathbf{e} = \frac{1 + t^{\frac{1}{2}}T_1 + t^{\frac{1}{2}}T_2 + tT_1T_2 + tT_2T_1 + t^{\frac{3}{2}}T_1T_2T_1}{1 + 2t + 2t^2 + t^3} .$$

2.2 Refined Chern-Simons invariants

Now let us define DAHA-Jones polynomials. Indeed, Macdonald polynomials $P_\lambda(X) \in \mathbb{K}[X_1,\ldots,X_N]^{S_n}$ is an element of the spherical DAHA $\boldsymbol{S\ddot{H}}_N$ (precisely speaking, up to the idempotent $\mathbf{e}$). Therefore, for the (m,n) torus knot, we choose an element $\gamma_{m,n} \in \mathrm{PSL}(2,\mathbb{Z})$

$$\gamma_{m,n} = \begin{pmatrix} m & * \\ n & * \end{pmatrix} ,$$

such that *reduced* DAHA-Jones polynomial is defined by

$$\overline{\mathrm{rCS}}_{sl(N),\lambda}(T_{m,n}; q,t) := \{\gamma_{m,n}(P_\lambda)\}_{ev} .$$

The specialization $t = q$ leads to λ-colored $sl(N)$ quantum invariants of the (m,n) torus knot. In addition, the existence of stabilization (DAHA-superpolynomials) $\mathrm{rCS}_\lambda(T_{m,n}; a,q,t)$ has been proven:

Theorem 2.3 (Stabilization) *[GN15] There exists a unique polynomial* $\overline{rCS}_\lambda(T_{m,n}; a,q,t)$ *such that:*

$$\overline{rCS}_{sl(N),\lambda}(T_{m,n}; q,t) = \overline{rCS}_\lambda(T_{m,n}; a = t^N, q,t) .$$

The invariant is proven to be equivalent to refined Chern-Simons invariants formulated in [AS15], using

$$S_{\lambda\mu} = S_{00}\ P_\lambda(t^\rho q^\mu) P_\mu(t^\rho) , \qquad T_{\lambda\mu} = \delta_{\lambda\mu} q^{\frac{1}{2}\sum_i \lambda_i(\lambda_i - 1)} t^{\sum_i \lambda_i(i-1)} .$$

2.3 Properties

When colors are specified by rectangular Young diagrams, therefined Chern-Simons invariants with the change of variables

$$a = -\mathbf{a}^2 , \quad q = \mathbf{q}^2\mathbf{t}^2 , \quad t = \mathbf{q}^2 , \tag{2.2}$$

conjecturally coincide with Poincaré polynomial polynomials of colored HOMFLYPT homology of the corresponding torus knot. For non-rectangular Young diagrams, it is known that the DAHA-superpolynomials include both positive and negative signs after the change of the variables (2.2).

It turns out that refined Chern-Simons invariants have surprisingly rich properties. Especially, it is proven in [Che16] that the reduced invariants the following properties:

- mirror/transposition symmetry

$$\mathrm{rCS}_{\lambda^T}(T_{m,n};a,q,t)=\mathrm{rCS}_{\lambda}(T_{m,n};a,t^{-1},q^{-1})\ . \tag{2.3}$$

- refined exponential growth property

$$\mathrm{rCS}_{\sum_{i=1}^{\ell}\lambda_i\omega_i}(T_{m,n};a,q=1,t)\ =\prod_{i=1}^{\ell}\left[\mathrm{rCS}_{\omega_i}(T_{m,n};a,q=1,t)\right]^{\lambda_i}\ , \tag{2.4}$$

  where ω_i are the fundamental weights of $\mathfrak{sl}(N)$.

For instance, (r)-colored refined CS invariants of the trefoil admit cyclotomic expansions

$$\mathrm{rCS}_{(r)}(T_{2,3};a,q,t):=a^r q^{-\frac{r}{2}}t^{-\frac{r}{2}}\sum_{k\geq 0}q^{kr}t^{-k}\binom{r}{k}_q\left(\frac{a}{t};q\right)_k$$

Question: How are they related to modular forms? (Tails, refinement of modular forms, etc)

3 Large N duality

Conjecture 3.1

$$\sum_{\lambda}\overline{rCS}_{\lambda}(T_{m,n};a,q,t)\ g_{\lambda}(q,t)P_{\lambda}(x;q,t)=\exp\left(\sum_{d=1}^{\infty}\sum_{\mu}\frac{1}{d}\frac{f_{\mu}^{q}(T_{m,n};a^d,q^d,t^d)}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}s_{\mu}(x^d)\right)\ , \tag{3.1}$$

$$\sum_{\lambda}\overline{rCS}_{\lambda}(T_{m,n};a,q,t)\ P_{\lambda^T}(-x;t,q)=\exp\left(\sum_{d=1}^{\infty}\sum_{\mu}\frac{1}{d}\frac{f_{\mu}^{\bar t}(T_{m,n};a^d,q^d,t^d)}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}}s_{\mu}(x^d)\right). \tag{3.2}$$

The refined reformulated invariants $f^q_{\mu}(T_{m,n})$ and $f^{\bar t}_{\mu}(T_{m,n})$, expressed in terms of refined Chern-Simons invariants of a torus knot $T_{m,n}$ via the geometric transition (3.1) and (3.2) can be written

$$\begin{aligned}f^q_{\mu}(T_{m,n};a,q,t)&=\sum_{\rho}M_{\mu\rho}(t)\widehat{f}_{\rho}(T_{m,n};a,q,t)\ ,\\ f^{\bar t}_{\mu}(T_{m,n};a,q,t)&=\sum_{\rho}M_{\mu\rho}(q^{-1})\widehat{f}_{\rho}(T_{m,n};a,q,t)\ ,\end{aligned} \tag{3.3}$$

where, upon the a-grading shift by $\pm\frac{1}{2}$, $\widehat{f}_{\rho}(T_{m,n})$ takes the form

$$\widehat{f}_{\rho}(T_{m,n};a,q,t)=\sum_{charges}(-1)^{2J_r}\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})(q^{\frac12}-q^{-\frac12})^g(t^{-\frac12}-t^{\frac12})^g\left(\frac{q}{t}\right)^{J_r-\frac{\beta}{2}}a^{\beta}\ , \tag{3.4}$$

with non-negative *integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})\in\mathbb{Z}_{\geq 0}$. Note that we define an invertible symmetric matrix*

$$M_{\mu\rho}(t):=\sum_{\sigma}C_{\mu\sigma\rho}B_{\sigma}(t)\ ,$$

where the Clebsch-Gordon coefficients $C_{\mu\sigma\rho}$ of the permutation group $\mathfrak{S}_h$ are

$$C_{\mu\sigma\rho} = \sum_{\vec{k}} \frac{|C(\vec{k})|}{k!} \chi_\mu(C(\vec{k}))\chi_\sigma(C(\vec{k}))\chi_\rho(C(\vec{k})) \ , \tag{3.5}$$

and physics tells us

$$B_\sigma(t) = \begin{cases} (-t)^d t^{-\frac{|\sigma|-1}{2}} & \sigma : \text{hook rep for } \wedge^d V \\ 0 & \sigma : \text{otherwise} \end{cases} .$$

Furthermore, for ρ, g, β fixed, the $2J_r$ charges of non-zero (hence positive) integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ are either all even or all odd so that no cancellation occurs in the unrefined limit and therefore the LMOV invariant is

$$\widehat{N}_{\rho,g,\beta}(T_{m,n}) = \pm \sum_{J_r \in \frac{1}{2}\mathbb{Z}} \widehat{N}_{\rho,g,\beta,J_r}(T_{m,n}) \ . \tag{3.6}$$

The relation between (3.1) and (3.2) can be explained from the mirror/transposition symmetry (2.3).

Conjecture 3.2 *Moreover, $\widehat{f}_\rho(T_{m,n}; a, q, t)$ exhibit the other positivity in the following expansion*

$$\widehat{f}_\rho(T_{m,n}; a, q, t) = \sum_{charges} \widehat{N}^{PT}_{\rho,J_1,J_2,\beta}(T_{m,n})\, q^{J_1} t^{J_2} (-a)^\beta$$

where $\widehat{N}^{PT}_{\rho,J_1,J_2,\beta}(T_{m,n})$ are non-negative integers. These can be regarded as open analogues of refined Pandharipande-Thomas invariants.

3.0.1 Remark

The BPS states that contribute to the refined index are fermion zero modes on an M2-brane wrapped on a holomorphic curve $\Sigma_{g,h} \subset X$ whose boundary is on L. The fermion zero modes on an M2-brane can be associated to cohomology groups of the moduli space

$$\begin{array}{ccc} \mathrm{Jac}(\Sigma_{g,h}) & \longrightarrow & \widehat{\mathcal{M}}_{\rm op} \\ & & \big\downarrow \pi \\ & & \mathcal{M}_{g,h,\beta} \end{array} ,$$

where the moduli space $\mathcal{M}_{g,h,\beta}$ parametrizes deformations of $\Sigma_{g,h} \subset X$ that preserve a half of supersymmetry. Since the moduli spaces $\mathcal{M}_{g,h,\beta}$ are in general singular, there has yet to be a definition. Although the PT/GV invariants (closed version) are related to modular forms, the relation of its open analogues discussed here to modular forms has not understood at al.

4 Miscellaneous

Other relations of DAHA

For each $l \geq 2$ we put $\alpha_l = T_{l-1}^{-1} \cdots T_2^{-1} T_1^{-2} T_2 \cdots T_{l-1}$. The following relations hold

$$X_l^{-1} Y_1 X_l = \alpha_l Y_1,$$

$$
\begin{aligned}
Y_l X_1 &= X_1 Y_l + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) T_{l-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{l-1}^{-1} Y_1 X_1, \\
q^{\frac{1}{2}} X_1 Y_1 &= T_1^{-1} \cdots T_{n-2}^{-1} T_{n-1}^{-2} T_{n-2}^{-1} \cdots T_1^{-1} Y_1 X_1. \\
\alpha_2 \cdots \alpha_l &= T_1^{-1} \cdots T_{l-2}^{-1} T_{l-1}^{-2} T_{l-2}^{-1} \cdots T_1^{-1},
\end{aligned}
$$

For $n > j \geq i \geq 1$, we have

$$
\begin{aligned}
Y_{i+1}^{-1} X_i Y_{i+1} X_i^{-1} &= T_i^2 \ , \\
Y_{j+1}^{-1} X_i Y_{j+1} X_i^{-1} &= T_j \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_i^{-1} \\
X_{j+1}^{-1} Y_i X_{j+1} Y_i^{-1} &= T_j \cdots T_{i+1} T_i^{-2} T_{i+1}^{-1} \cdots T_j^{-1}
\end{aligned}
$$

Macdonald functions

The Macdonald functions $P_\lambda(x;q,t)$ are uniquely defined by orthogonality and normalization conditions:

$$
\begin{array}{ll}
\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \ , & \text{if} \quad \lambda \neq \mu, \\
P_\lambda(x;q,t) = m_\lambda(x) + \sum_{\mu<\lambda} u_{\lambda\mu}(q,t)\, m_\mu(x) \ , & u_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t) \ ,
\end{array}
$$

where the inner product is defined by

$$
\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i\geq 1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \ , \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! \ .
$$

At the $q = t$ specialization, the Macdonald functions reduce to the Schur functions. From the definition one can show

$$
\frac{(q/t)^{|\lambda|}}{g_\lambda(q,t)} := \langle P_\lambda, P_\lambda \rangle_{q,t} = \prod_{s\in\lambda} \frac{1 - q^{a(s)+1} t^{l(s)}}{1 - q^{a(s)} t^{l(s)+1}} \ ,
$$

Explicit formulas of refined reformulated invariants

$$
\begin{aligned}
\frac{f^q_{\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} =& \overline{\mathrm{rCS}}_{\square}, \\
\frac{t^{\frac{1}{2}}}{q^{\frac{1}{2}}} \frac{f^q_{\square\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} =& \frac{qt-1}{q^2-1} \overline{\mathrm{rCS}}_{\square\square} - \frac{t-1}{2(q-1)} (\overline{\mathrm{rCS}}_{\square})^2 - \frac{t+1}{2(q+1)} \overline{\mathrm{rCS}}^{(2)}_{\square} \ , \\
\frac{t^{\frac{1}{2}}}{q^{\frac{1}{2}}} \frac{f^q_{\substack{\square\\\square}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} =& \frac{t-q}{q^2-1} \overline{\mathrm{rCS}}_{\square\square} + \frac{t^2-1}{qt-1} \overline{\mathrm{rCS}}_{\substack{\square\\\square}} - \frac{t-1}{2(q-1)} (\overline{\mathrm{rCS}}_{\square})^2 + \frac{t+1}{2(q+1)} \overline{\mathrm{rCS}}^{(2)}_{\square} \ ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{f^{\bar{t}}_{\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} =& \overline{\mathrm{rCS}}_{\square}, \\
\frac{-f^{\bar{t}}_{\square\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} =& \overline{\mathrm{rCS}}_{\substack{\square\\\square}} + \frac{1}{2} \overline{\mathrm{rCS}}^{(2)}_{\square} - \frac{1}{2} (\overline{\mathrm{rCS}}_{\square})^2 \ , \\
\frac{-f^{\bar{t}}_{\substack{\square\\\square}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} =& \overline{\mathrm{rCS}}_{\square\square} + \frac{q-t}{qt-1} \overline{\mathrm{rCS}}_{\substack{\square\\\square}} - \frac{1}{2} \overline{\mathrm{rCS}}^2_{\square} - \frac{1}{2} \overline{\mathrm{rCS}}^{(2)}_{\square}
\end{aligned}
$$

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