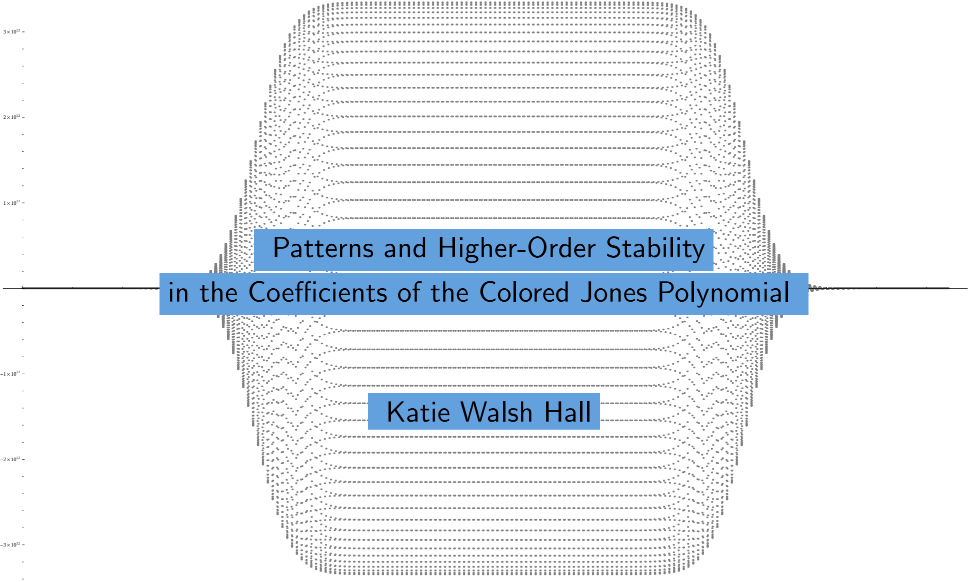
Patterns and Higher-Order Stability in the Coefficients of the Colored Jones Polynomial



Katie Walsh Hall

The Colored Jones Polynomial

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- It assigns to each knot a sequence of polynomials.
- We want to look at the coefficients of these polynomials.

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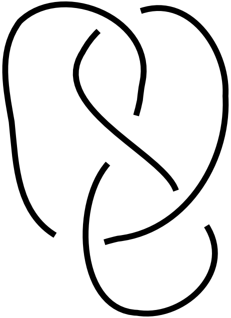

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N	Highest Terms of the Colored Jones Polynomial of 4_1
2	$q^2 - q + 1 - q^{-1} + q^{-2}$
3	$q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 3 - q^{-1} - q^{-2} + \cdots$
4	$q^{12} - q^{11} - q^{10} + 0q^9 + 2q^8 - 2q^6 + 3q^4 - 3q^2 + \cdots$
5	$q^{20} - q^{19} - q^{18} + 0q^{17} + 0q^{16} + 3q^{15} - q^{14} - q^{13} + \cdots$
6	$q^{30} - q^{29} - q^{28} + 0q^{27} + 0q^{26} + q^{25} + 2q^{24} + 0q^{23} + \cdots$
7	$q^{42} - q^{41} - q^{40} + 0q^{39} + 0q^{38} + q^{37} + 0q^{36} + 3q^{35} + \cdots$

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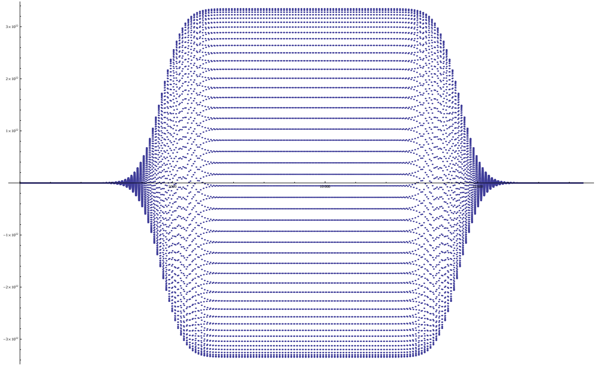

Notes on Normalization

- The $(N+1)^{st}$ colored Jones polynomial of a knot K is the Jones polynomial of K decorated with the $f^{(N)}$, the Jones-Wenzl idempotent in TL_n.

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Normalized colored Jones polynomial:

$$J'_{N,\mathrm{unknot}}(q) = 1$$

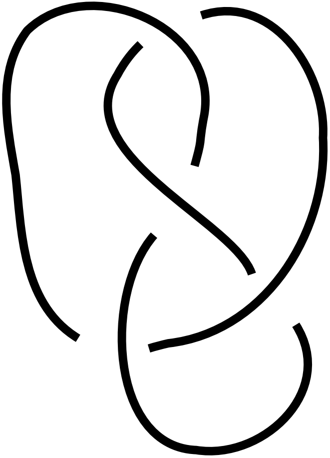
Un-normalized colored Jones polynomial:

$$J_{N,\mathrm{unknot}}(q) = \Delta_{N-1} = (-1)^{N-1}[N]$$

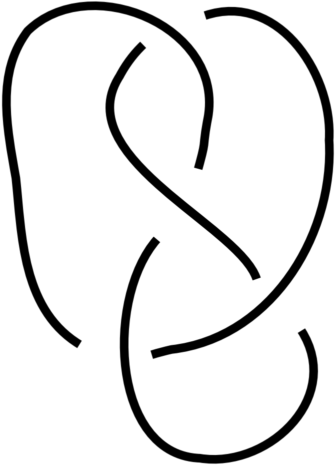
$$J'_{N,K}(q) = \frac{J_{N,K}(q)}{\Delta_{N-1}}$$

We use the convention that $N = 2$ gives the standard Jones polynomial.

Set-Up



Set-Up



5

The 5^{th} colored Jones Polynomial for figure 8 knot is:

$$\frac{1}{q^{20}} - \frac{1}{q^{19}} - \frac{1}{q^{18}} + \frac{3}{q^{15}} - \frac{1}{q^{14}} - \frac{1}{q^{13}} - \frac{1}{q^{12}} - \frac{1}{q^{11}} + \frac{5}{q^{10}} - \frac{1}{q^{9}} - \frac{2}{q^{8}} - \frac{2}{q^{7}} - \frac{1}{q^{6}}$$

$$+\frac{6}{q^5} - \frac{1}{q^4} - \frac{2}{q^3} - \frac{2}{q^2} - \frac{1}{q} + 7 - q - 2q^2 - 2q^3 - q^4 + 6q^5 - q^6 - 2q^7 - 2q^8 - q^9 + 5q^{10}$$

$$-q^{11} - q^{12} - q^{13} - q^{14} + 3q^{15} - q^{18} - q^{19} + q^{20}$$

This has coefficients:

$$\{1, -1, -1, 0, 0, 3, -1, -1, -1, -1, 5, -1, -2, -2, -1, 6, -1, -2, -2, -1, 7,$$

$$-1, -2, -2, -1, 6, -1, -2, -2, -1, 5, -1, -1, -1, -1, 3, 0, 0, -1, -1, 1\}$$

$$\{1, -1, -1, 0, 0, 3, -1, -1, -1, -1, 5, -1, -2, -2, -1, 6, -1, -2, -2, -1, 7,$$
$$-1, -2, -2, -1, 6, -1, -2, -2, -1, 5, -1, -1, -1, -1, 3, 0, 0, -1, -1, 1\}$$

We can plot these:

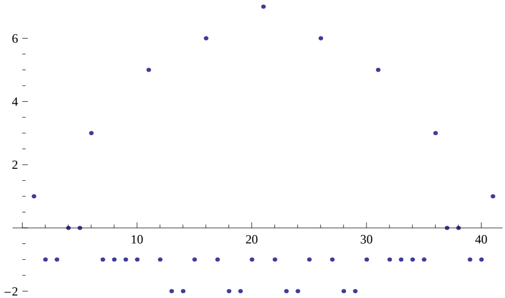


Figure: Coefficients of the 5^{th} Colored Jones Polynomial for the Figure Eight Knot

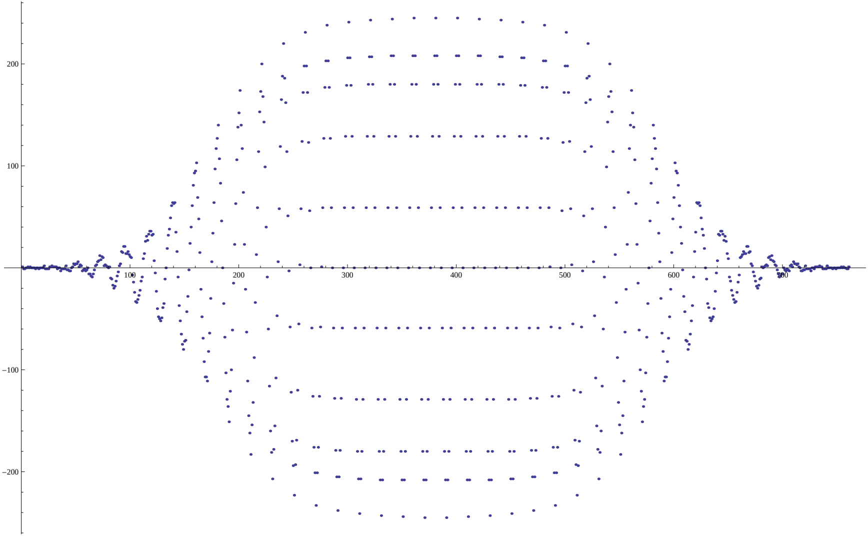


Figure: Coefficients of the 20^{th} Colored Jones Polynomial for the Figure Eight Knot

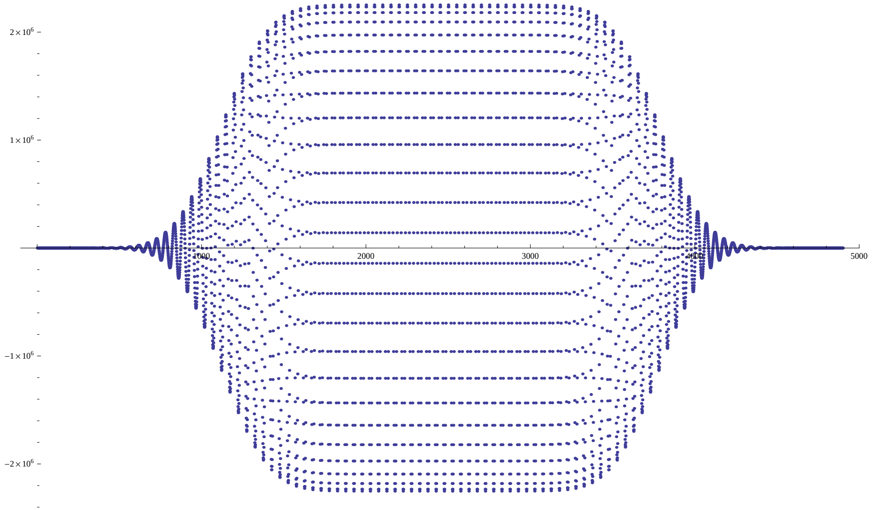


Figure: Coefficients of the 50^{th} Colored Jones Polynomial for the Figure Eight Knot

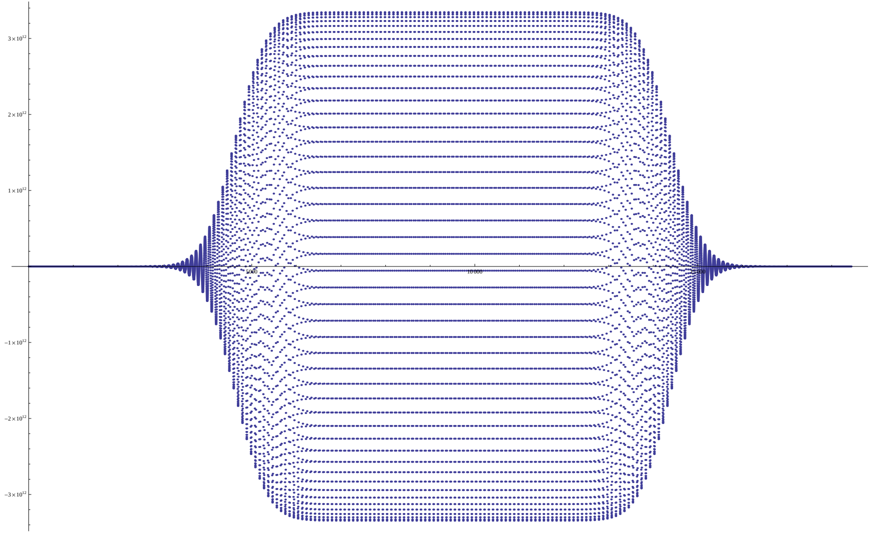


Figure: Coefficients of the 95^{th} Colored Jones Polynomial for the Figure Eight Knot

What is this polynomial?

$$
\begin{aligned}
J'_{N,4_1}(q) &= \sum_{n=0}^{N-1} \prod_{k=1}^{n} \{N+k\}\{N-k\} \\
&= \sum_{n=0}^{N-1} \prod_{k=1}^{n} (q^{-(N+k)/2} - q^{(N+k)/2})(q^{-(N-k)/2} - q^{(N-k)/2}) \\
&= \sum_{n=0}^{N-1} \prod_{k=1}^{n} (q^{N} - q^{k} - q^{-k} + q^{-N})
\end{aligned}
$$

What about other knots? See Mathematica Demo....

But what about the middle?

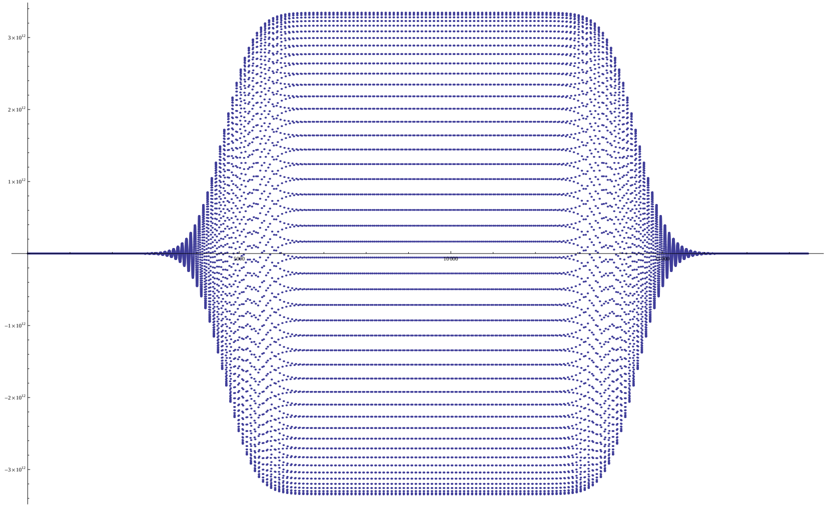


Figure 8 Knot Middle Coefficients Observed Properties

- The middle coefficient is the constant term.
- The maximum coefficient is the coefficient of constant term.
- Some sort of N-periodicity in the middle coefficients.

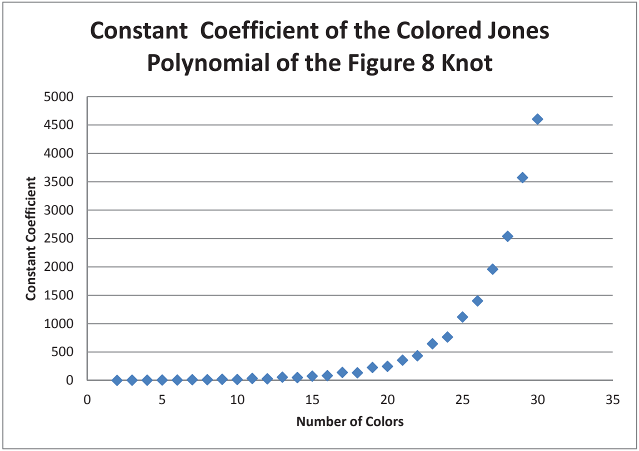


Figure: The Maximum Coefficient of the N Colored Jones Polynomial of the Figure 8 Knot as a function of N.

Assumption

The maximum coefficient takes the form Ae^{bN} where N is the number of colors and A and b depend on the knot.

Proposition

The colored Jones polynomial of a knot K satisfies

$$\lim_{N\to\infty} \frac{\log |J_k(N)(e^{2\pi i/N})|}{N} \leq \lim_{N\to\infty} \frac{\log m_K(N)}{N}.$$

If the above assumption holds, then the colored Jones polynomial satisfies

$$\lim_{N\to\infty} \frac{\log |J_k(N)(e^{2\pi i/N})|}{N} \leq b.$$

Assumption

The maximum coefficient takes the form Ae^{bN} where N is the number of colors and A and b depend on the knot.

So for knots where the Hyperbolic Volume Conjecture holds we get to following

Proposition

For knots for which the Hyperbolic Volume conjecture holds

$$\frac{vol(S^3\backslash K)}{2\pi} \leq \lim_{N\to\infty} \frac{\log m_K(N)}{N}.$$

Now, if we include the above assumption, so that $m(k)(N) = Ae^{bN}$, we get

$$\frac{vol(S^3\backslash K)}{2\pi} \leq b.$$

Assumption

The maximum coefficient takes the form Ae^{bN} where N is the number of colors and A and b depend on the knot.

Assumption

The coefficients take the form of a normal distribution times a sine wave of period $2N$.

Proposition

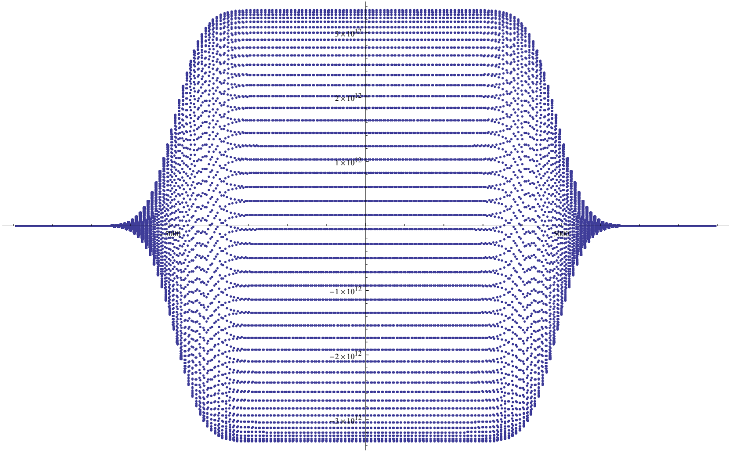
If the colored Jones polynomial of a knot satisfies these two assumptions then

$$b = \lim_{N \to \infty} \frac{\log |(f_N(e^{2\pi i/N}))|}{N}.$$

If this is a knot for which the Hyperbolic Volume Conjecture holds,

$$b = \frac{vol(S^3 \backslash K)}{2\pi}.$$

Further Analysis on the Coefficients of the Figure 8 Knot



semi-(un)normalized colored Jones polynomial: $\mathrm{s}J_{N,K}(q) = \{N\}J'_{N,K}(q)$.

$$\pm J_{N,K}(q)\{1\} = \mathrm{s}J_{N,K}(q) = J'_{N,K}(q)\{N\}$$

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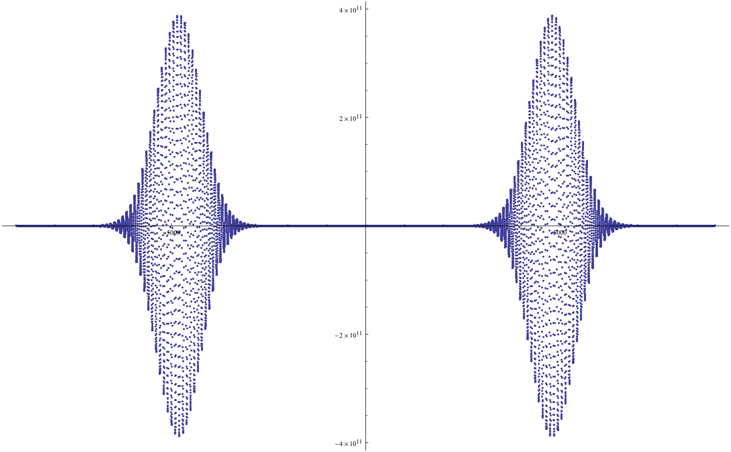

Figure: The coefficients of the 95 colored semi-(un)normalized Jones polynomial of the figure 8 knot.

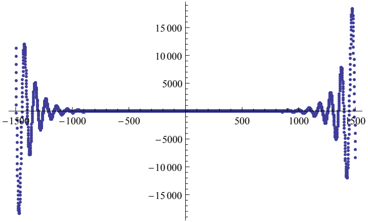

Figure: The middle 3000 coefficients of the $\mathrm{s}J_{95,4_1}(q)$.

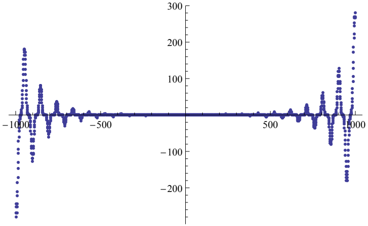

Figure: The middle 2000 coefficients of the $\mathrm{s}J_{95,4_1}(q)$.

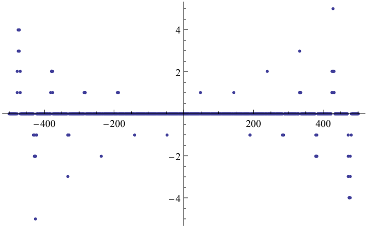

Figure: The middle 1000 coefficients of the $\mathrm{s}J_{95,4_1}(q)$.

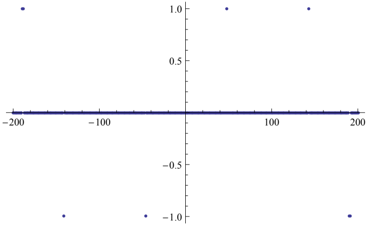

Figure: The middle 400 coefficients of the $\mathrm{s}J_{95,4_1}(q)$.

Conjecture

Let $c(q^i)$ be the coefficient of the q^i term of $\mathcal{s}J_{N,4_1}(q)$. When N is odd,

$$c(q^i) = \begin{cases} \pm 1 & i = \pm N/2 \text{ or } \pm 3N/2 \\ 0 & |i| < 2N - 1/2 \text{ and } i \neq \pm N/2 \text{ or } \pm 3N/2 \end{cases}$$

When N is even,

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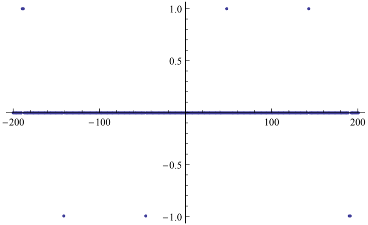

Head and Tail Stabilty

- In 2006, Dasbach and Lin conjectured that the first and last coefficients of the colored Jones polynomial stabilize for alternating knots.
- In 2011, Armond proved this for alternating links and for adequate links, using skein theoretical techniques.
- The head and tail do not exist for all knots, however. Armond and Dasbach showed that the head and tail does not exist for the $(4,3)$ torus knot.
- It was also independently proven by Garoufalidis and Lê. In fact, they proved a stronger version of this stability.

Definition

The *head* of the Colored Jones Polynomial of a knot K - if it exists - is a polynomial whose first N terms (highest powers of q) have the same coefficients as the first N terms of $J'_{N,K}$.

Definition

The *tail* of the Colored Jones Polynomial of a knot K - if it exists - is a polynomial whose last N terms (lowest powers of q) have the same coefficients as the last N terms of $J'_{N,K}$.

N	Highest Terms of the Colored Jones Polynomial of 4_1
2	$q^2 - q + 1 - q^{-1} + q^{-2}$
3	$q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 3 - q^{-1} - q^{-2} + \cdots$
4	$q^{12} - q^{11} - q^{10} + 0q^9 + 2q^8 - 2q^6 + 3q^4 - 3q^2 + \cdots$
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6	$q^{30} - q^{29} - q^{28} + 0q^{27} + 0q^{26} + q^{25} + 2q^{24} + 0q^{23} + \cdots$
7	$q^{42} - q^{41} - q^{40} + 0q^{39} + 0q^{38} + q^{37} + 0q^{36} + 3q^{35} + \cdots$
8	$q^{56} - q^{55} - q^{54} + 0q^{53} + 0q^{52} + q^{51} + 0q^{50} + q^{49} + \cdots$

The Head of the Colored Jones Polynomial of 4_1

$$J'_{N,4_1}(q) = \sum_{n=0}^{N-1} \prod_{k=1}^{n} q^N - q^k - q^{-k} + q^{-N}$$

The max degree of each summand is Nn so decreasing the n by 1 changes the max degree by N thus only $n = N-1$ contributes to the head and tail.

$$\begin{aligned} J'_{N,4_1}(q) &\overset{HT}{=} \prod_{k=1}^{N-1} q^N - q^k - q^{-k} + q^{-N} \\ &\overset{HT}{=} \prod_{k=1}^{N-1} q^N - q^k \\ &= q^N \prod_{k=1}^{N-1} 1 - q^{k-N} \end{aligned}$$

reindex: $k' = N - k$

$$J'_{N,4_1}(q) \overset{HT}{=} \prod_{k'=1}^{N-1} (1 - q^{-k'})$$

Theorem (Euler's Pentagonal Number Theorem)

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}$$
$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$$

A similar arguments shows that the head of twists knots is the same polynomial.

Theorem (Armond)

The head and tail of the colored Jones polynomial exist for alternating and adequate links.

Theorem (Armond and Dasbach)

The tail and head of the colored Jones polynomial of adequate links only depend on a certain reduced checkerboard graph of a diagram of the link.

Figure: The Knot 6_2

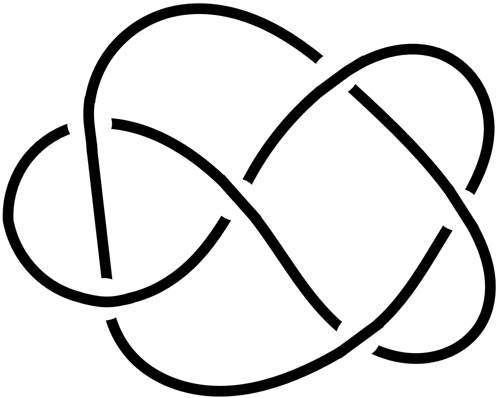


Figure: The Knot 6_2 with a checkerboard coloring

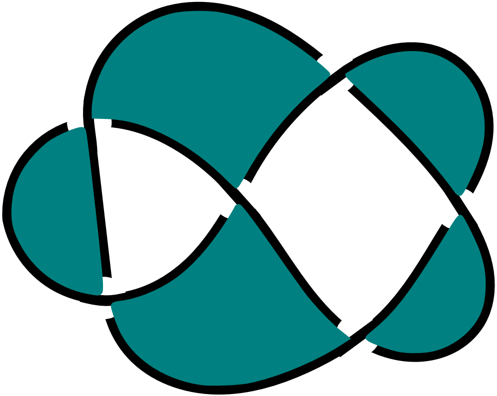


Figure: The Knot 6_2 with one of its associated graph

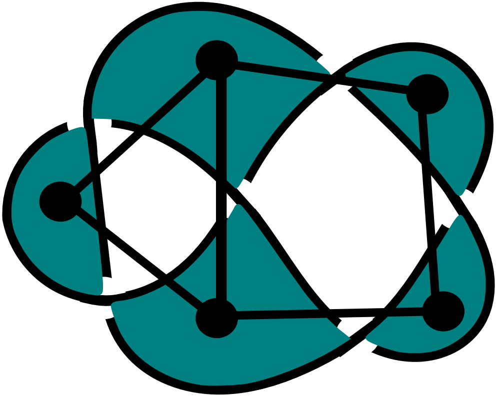
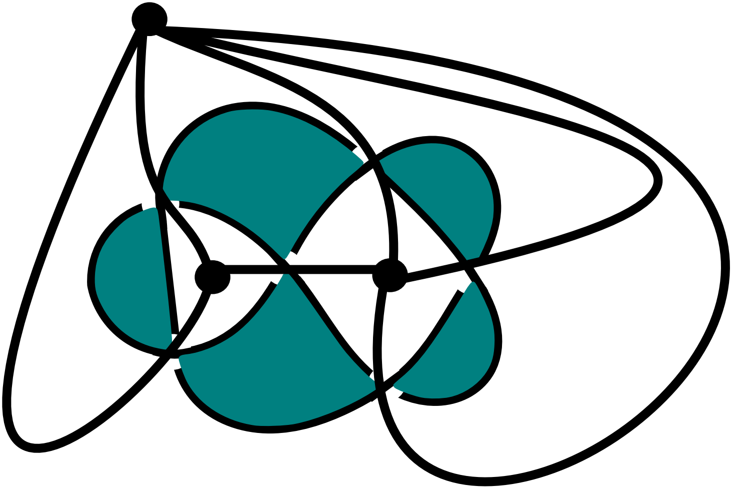
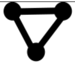
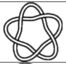
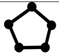
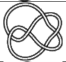
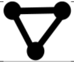
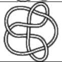
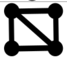
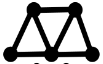
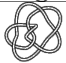
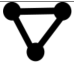
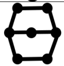


Figure: The Knot 6_2 with the other associated graph



Knot	Knot Diagram	"White" Checkerboard Graph	"Black" Checkerboard Graph	Tail	Head
3_1				1	h_3
4_1				h_3	h_3
5_1				h_5	1
5_2				h_3	h_4^*
6_2				h_3	$h_3h_4^*$
7_4				h_3	$(h_4)^2$
7_7				$(h_3)^2$	$(h_3)^3$
8_5				h_3	???

$$h_b(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{bn(n+1)/2-n} \qquad h_b^*(q) = \sum_{n \in \mathbb{Z}} \epsilon(n) q^{bn(n+1)/2-n}$$

Definition (Garoufalidis and Lê)

A sequence $(f_n(q)) \in \mathbb{Z}[[q]]$ is *k-stable* if there exist $\Phi_j(q) \in \mathbb{Z}((q))$ for $j = 0, \ldots, k$ such that

$$\lim_{n\to\infty} q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) = 0.$$

We call $\Phi_k(q)$ the k-limit of $(f_n(q))$. We say that $(f_n(q))$ is *stable* if it is k-stable for all k.

For example, a sequence $(f_n(q))$ is *2-stable* if

$$\lim_{n\to\infty} q^{-2(n+1)} \left(f_n(q) - \left(\Phi_0(q) + q^{(n+1)}\Phi_1(q) + q^{2(n+1)}\Phi_2(q) \right) \right) = 0.$$

For the knot 8_5:

$$\Phi_0 = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k}{2}(3k-1)}.$$

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0
$N=5$	1	-1	-1	0	0	5	-1	-3	-3	-5	11	4	1	$\cdots$	
$N=6$	1	-1	-1	0	0	1	4	0	-4	-3	-3	-1	9	8	1
$N=7$	1	-1	-1	0	0	1	0	5	-1	-4	-3	-3	0	-2	14

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0
$N = 5$	1	-1	-1	0	0	5	-1	-3	-3	-5	11	4	1	$\cdots$	
$N = 6$	1	-1	-1	0	0	1	4	0	-4	-3	-3	-1	9	8	1
$N = 7$	1	-1	-1	0	0	1	0	5	-1	-4	-3	-3	0	-2	14

Now, since we know all of Φ_0, we can subtract it from the shifted colored Jones polynomials. Now are coefficients are:

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0
$N = 5$	0	0	0	0	0	4	-1	-4	-3	-5	11	4	2	$\cdots$	
$N = 6$	0	0	0	0	0	0	4	-1	-4	-3	-3	-1	10	8	1
$N = 7$	0	0	0	0	0	0	0	4	-1	4	-3	-3	1	-2	14

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0
$N = 5$	0	0	0	0	0	4	-1	-4	-3	-5	11	4	2	$\cdots$	
$N = 6$	0	0	0	0	0	0	4	-1	-4	-3	-3	-1	10	8	1
$N = 7$	0	0	0	0	0	0	0	4	-1	4	-3	-3	1	-2	14

Shifting these sequences back so that they start with a non-zero term, we can see that they again stabilize. The sequence they stabilize to is Φ_1.

Φ_1	4	-1	-4	-3	-3	1	0	4	3	3	3	$3\cdots$
$N = 5$	4	-1	-4	-3	-5	11	4	2	$\cdots$			
$N = 6$	4	-1	-4	-3	-3	-1	10	8	1	-4	$\cdots$	
$N = 7$	4	-1	4	-3	-3	1	-2	14	7	1	-4	$-9\cdots$

Φ_1	4	-1	-4	-3	-3	1	0	4	3	3	3	$3\cdots$
$N = 5$	4	-1	-4	-3	-5	11	4	2	$\cdots$			
$N = 6$	4	-1	-4	-3	-3	-1	10	8	1	-4	$\cdots$	
$N = 7$	4	-1	4	-3	-3	1	-2	14	7	1	-4	$-9\cdots$

Subtracting and shifting these sequences back so that they start with a non-zero term, we can see that they again stabilize. The sequence they stabilize to is Φ_{2*}.

Φ_{2*}	-2	10	4	-2	-7	$-12\cdots$
$N = 5$	-2	10	4	-2	$\cdots$	
$N = 6$	-2	10	4	-2	-7	$\cdots$
$N = 7$	-2	10	4	-2	-7	$-12\cdots$

Theorem (Garoufalidis and Lê)

For every alternating link K, the sequence $f_N(q) = (\hat{J}_{N+1,K}(q))$ is stable and its associated k-limit $\Phi_{K,k}(q)$ can be effectively computed from any reduced alternating diagram D of K.

A $k+1$-stable sequence satisfies:

$$\lim_{N\to\infty} q^{-(k+1)(N+1)}\left(J_{N+1,K}(q) - \sum_{j=0}^{k+1} q^{j(N+1)}\Phi_j(q)\right) = 0.$$

- The first $k(N+1)$ coefficients of $J_{N+1,K}(q)$ match $\sum_{j=0}^{k} q^{j(N+1)}\Phi_j$ for large enough N.
- This does not guarantee this property from the beginning.

Order	Stabilization of Coefficients	Which knots stabilize to the same sequence?	What do they stablize to?
1st Order	✓ adequate links	✓	✓ in some cases
Higher Order	✓ for alternating links	?	?

Corollary

Let m be the minimum number of parallel edges in a diagram. In addition to the first $N + 1$ terms only depending on the overall graph structure, the next $(m - 1)N$ terms also depend only on the graph structure.

Corollary

To find Φ_k, we can consider the graph diagram reduced so that multiples edges with multiplicity greater than k are reduced to k multiple edges.

Order	Stabilization of Coefficients	Which knots stabilize to the same sequence?	What do they stablize to?
1st Order	✓ adequate links	✓	✓ in some cases
Higher Order	✓ for alternating links	✓	?

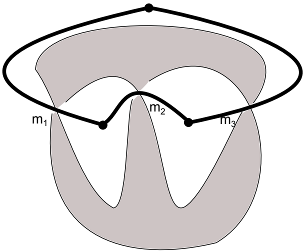
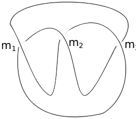
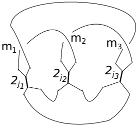


Figure: A knot with its checkerboard graph.

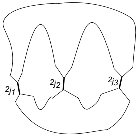
$$J_{N+1,K}(q) = \left\langle \quad \right\rangle$$



$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \left\langle \quad \right\rangle$$

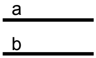


$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \left\langle \quad \right\rangle$$

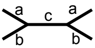


$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,j_2,j_3)}$$

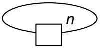
Fusion



$$= \sum_c \frac{\Delta_c}{\theta(a,b,c)} \qquad (1)$$



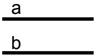
$$\Delta_n = \left\langle \quad \right\rangle = (-1)^n [n+1] \qquad (2)$$



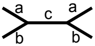
where

$$[n] = \frac{\{n\}}{\{1\}},\ \{n\} = A^{2n} - A^{-2n} \text{ and } A^{-4} = a^{-2} = q. \qquad (3)$$

Fusion



$$= \sum_{c} \frac{\Delta_c}{\theta(a,b,c)}$$

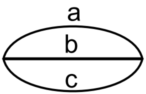


(4)

Assume (a, b, c) is an admissible triple, then let i, j, k be the internal colors, in particular

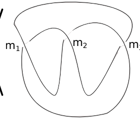
$$i = (b + c - a)/2 \qquad j = (a + c - b)/2 \qquad k = (a + b - c)/2. \quad (5)$$

$$\theta(a, b, c) = \Big\langle$$



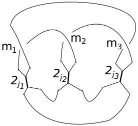
$$\Big\rangle = (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!}. \quad (6)$$

$$J_{N+1,K}(q) = \Big\langle$$



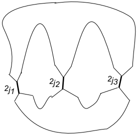
$$\Big\rangle$$

$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Big\langle$$



$$\Big\rangle$$

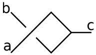
$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Big\langle$$



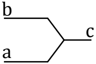
$$\Big\rangle$$

$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,j_2,j_3)}$$

Twist Coefficients

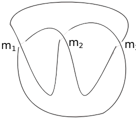


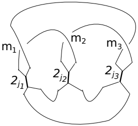
$$= \gamma(a, b, c) \qquad (7)$$

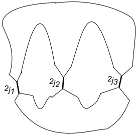


with

$$\gamma(a, b, c) = (-1)^{\frac{a+b-c}{2}} A^{a+b-c+\frac{a^2+b^2-c^2}{2}}. \qquad (8)$$

$J_{N+1,K}(q) = \Big\langle$  $\Big\rangle$

$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Big\langle$  $\Big\rangle$

$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Big\langle$  $\Big\rangle$

$$= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,j_2,j_3)}$$

m-reduced graph Corollary Proof

Label edge sets with m or more parallel edges 1 through b.

$$
\begin{aligned}
J_{N+1,K} &= \sum_{j_1,\dots j_k=0}^{N} \prod_{i=1}^{k} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,\dots,j_k)} \\
&\overset{\cdot(m+1)N}{=} \gamma(N,N,2N)^{\sum_{i=1}^{b} m_i} \\
&\quad \times \sum_{j_{b+1},\dots,j_k=0}^{N} \prod_{i=b+1}^{k} \gamma(N,N,2j_i)^{m_i} \prod_{i=1}^{k} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(N,\dots,N,j_{b+1},\dots,j_k}
\end{aligned} \tag{9}
$$

Again, since $\gamma(N,N,2N)$ only contributes an overall shift, we get the same highest $(m+1)N$ coefficients regardless of the values of $m_1,\dots,m_b$ and thus knots with the same $m+1$-reduced graph structure have the same highest $(m+1)N$ coefficients.

$$J_{N+1,K}(q) \quad = \quad \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,j_2,j_3)}$$

What terms contribute to the first $2N+1$ coefficients?

$$J_{N+1,K}(q) \quad = \quad \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,j_2,j_3)}$$

What terms contribute to the first $2N+1$ coefficients?

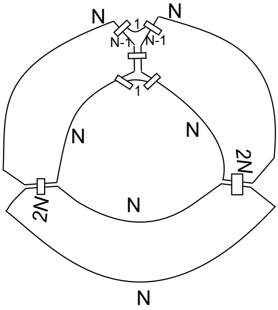
either all $j_i = N$ or exactly one $j_i = N-1$ and the rest are N

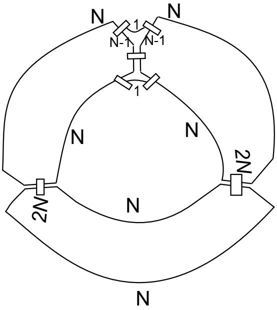
In the case where each m_i is greater than 2, the maximum degree decreases by more than $2N$ when we decrease j_i from N to $N-1$, thus we only need to deal with the case where each $j_i = N$. Thus we get

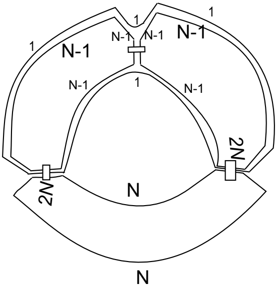
$$
\begin{aligned}
J_{N+1,K}(q) &= \sum_{j_i=0}^{N} \prod_{i=1}^{3} \gamma(N,N,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(N,N,2j_i)} \Gamma_{N,(j_1,j_2,j_3)} \\
&\overset{\cdot 2N+1}{=} \prod_{i=1}^{3} \gamma(N,N,2N)^{m_i} \frac{\Delta_{2N}}{\theta(N,N,2N)} \Gamma_{N,(N,N,N)} \\
&= \gamma(N,N,2N)^{m_1+m_2+m_3} \left(\frac{\Delta_{2N}}{\theta(N,N,2N)} \right)^3 \left(\Gamma_{N,(N,N,N)} \right) \\
&= \cdots \\
&\overset{\cdot 2N+1}{=} \frac{(-1)\{N\}!^3}{\{N\}!^2 \left(1 - \frac{2q^{-N-1}}{1-q^{-1}} \right) \{1\}} \\
&= \frac{(-1)\{N\}!}{\left(1 - \frac{2q^{-N-1}}{1-q^{-1}} \right) \{1\}}.
\end{aligned}
$$

$$
\begin{aligned}
J'_{N+1,K}(q) - \text{stabilized head} &\overset{\cdot 2N+1}{=} (-1)^N\{N\}!\left(\left(1+\frac{2q^{-N-1}}{1-q^{-1}}+q^{-N-1}\right)\right.\\
&\left.-\left(1-\frac{q^{-N-1}}{1-q^{-1}}\right)\right)\\
&= (-1)^N\{N\}!\left(q^{-N-1}+\frac{3q^{-N-1}}{1-q^{-1}}\right)\\
&= (-1)^N q^{-N-1}\left(\{N\}!+\frac{3\{N\}!}{1-q^{-1}}\right)\\
&\overset{\infty}{=} \prod_{i=1}^{N}(1-q^{-i})+\frac{3\prod_{i=1}^{N}(1-q^{-i})}{1-q^{-1}}.
\end{aligned}
$$

When an m_i is 1, we do get a term with $j_i = N - 1$.
Need to consider the graph $\Gamma(N+1, N-1, N-1)$ and its evaluation.







$\Gamma_{N,(N-1,N,N)} = \Gamma(N+1, N-1, N-1)).$

Theorem (KPWH)

Let m be the number edges in the checkerboard graph with m_i of 2 or more. The tailneck of knots whose reduced checkboard graph is the triangle graph is:

$$\prod_{n=1}^{\infty}(1-q^n) + m\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q},$$

i.e. the pentagonal numbers plus the m times the partial sum of the pentagonal numbers.

Oliver Dasbach suggested the following:

- Redefine the neck and tail neck by subtracting consecutive terms in sequence, shifted so that they have the same minimum degree.
- This gives us a simpler expression for the higher order stable pieces.

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- This gives us a simpler expression for the higher order stable pieces.

Corollary

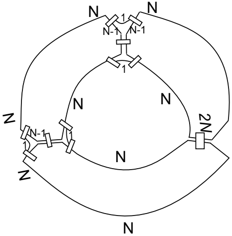
Again, let m be the number edges in the checkerboard graph with m_i of 2 or more. Then we have

$$J'_{N,K} - q^* J'_{N+1,K} \overset{\cdot N}{=} (1 + m - q) \prod_{n=1}^{\infty} (1 - q^n).$$

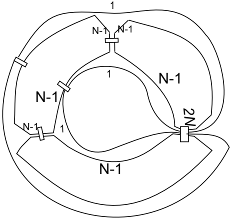
If we want the first $3N+1$ terms, which terms in the sum contribute?

Labeling (up to permutation)	Increase in min degree from (N,N,N)
(N,N,N)	0
$(N,N,N-1)$	at least $N+1$
$(N,N,N-2)$ $(N,N-1,N-1)$	at least $2N+1$ at least $2N+2$
$(N,N,N-3)$ $(N,N-1,N-2)$ $(N-1,N-1,N-1)$	at least $3N-1$ at least $3N+1$ at least $3N+3$

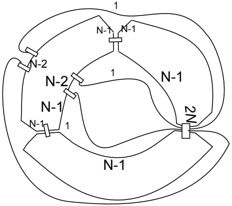
The Graph $\Gamma_{N,(N,N-1,N-1)}$



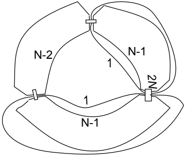
We can absorb the smaller idempotents into the larger ones and combine adjoining ones.



We resolve the remaining N idempotents but only one of the terms is non-zero. We get a factor of $-\left(\frac{\Delta_{N-2}}{\Delta_{N-1}}\right)$ from each resolution.



We can then absorb the remaining $N-1$ idempotents.



The evaluation of $\Gamma_{N,(N,N-1,N-1)} = \left(\frac{\Delta_{N-2}}{\Delta_{N-1}}\right)^2 \Gamma(N,N,N-2)$.

Let's look at the next stable sequence for these knots:

$$q^{-(2N+2)}(q(\hat{J}'_{N,K} - \hat{J}'_{N+1,K}) - (\hat{J}'_{N+1,K} - \hat{J}'_{N+2,K}))/\prod_{i=1}^{\infty}(1-q^i)$$

Values of (m_1, m_2, m_3) (up to permutation)	T_1^*	T_2^*: 1	q	q^2	q^3	q^4
$(1,1,1)$	$1-q$	0	1	-1	-1	1
$(1,1,2)$	$2-q$	0	4	-1	-3	1
$(1,1,3^+)$	$2-q$	-1	4	0	-3	1
$(1,2,2)$	$3-q$	0	7	0	-4	1
$(1,2,3^+)$	$3-q$	-1	7	1	-4	1
$(1,3^+,3^+)$	$3-q$	-2	7	2	-4	1
$(2,2,2)$	$4-q$	0	10	2	-4	1
$(2,2,3^+)$	$4-q$	-1	10	3	-4	1
$(2,3^+,3^+)$	$4-q$	-2	10	4	-4	1
$(3^+,3^+,3^+)$	$4-q$	-3	10	5	-4	1

Question

What information is contained these coefficients?

Question

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More questions:

- Can we prove that these are the stable sequences?
- Can we get a geometric proof of stability for all knots?
- What does the stable sequence look like for other families of knots or for even higher stability?
- Can these sequences help us understand the middle coefficients?

ANY QUESTIONS?