The Haag-Kastler Axioms on Two-dimensional de Sitter Space

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The two-dimensional de Sitter space

- De Sitter space

$$dS \doteq \left\{x \in \mathbb{R}^{1+2} \mid x \cdot x = x_0^2 - x_1^2 - x_2^2 = -1\right\}.$$

- Wedges: let $W_1 \doteq \left\{x \in dS \mid x_2 > |x_0|\right\}$,

$$W = \Lambda W_1 \subset dS, \qquad \Lambda \in SO_0(1,2).$$

  The set of all wedges is denoted by $\mathcal{W}$.
- Lorentz Boosts (hyperbolic subgroups)

$$\Lambda_W(t) = \Lambda \Lambda_1(t) \Lambda^{-1}, \quad \Lambda_1(t) \doteq \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

Wedge

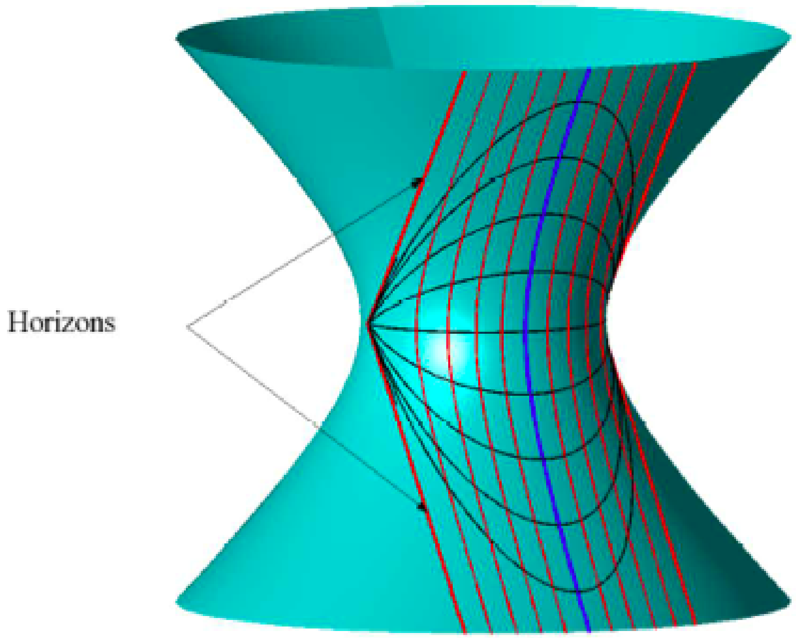

Figure: Wedge

- $\Lambda_W(t)W = W$, $t \in \mathbb{R}$, and, for all $t \in \mathbb{R}$,

$$\Lambda_{\Lambda' W}(t) = \begin{cases} \Lambda' \Lambda_W(t) \Lambda'^{-1} & \text{se } \Lambda' \in SO_0(1,2) \\ \Lambda' \Lambda_W(-t) \Lambda'^{-1} & \text{se } \Lambda' \in O_+^\downarrow(1,2) \,. \end{cases} ,$$

- Rotations (elliptic subgroups)

$$\alpha \mapsto R_0(\alpha) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} , \quad \alpha \in [0, 2\pi) \,.$$

- Horospheric Translations (parabolic subgroups)

$$q \mapsto D(q) \doteq \begin{pmatrix} 1 + \frac{q^2}{2} & q & \frac{q^2}{2} \\ q & 1 & q \\ -\frac{q^2}{2} & -q & 1 - \frac{q^2}{2} \end{pmatrix} , \quad q \in \mathbb{R} \,.$$

Rotations and Horospheric Translations

Cauchy surfaces Horospheres

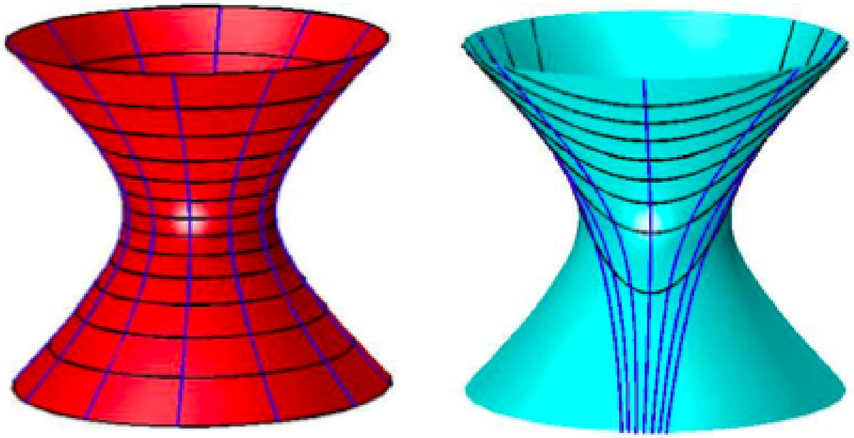

Figure: $dS \doteq \left\{ x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -1 \right\}$.

- Space-time reflections

$$T \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(1,2).$$

- The reflection at the edge of the wedge

$$\Theta_{\Lambda W_1} = \Lambda(P_1 T)\Lambda^{-1}, \qquad \Lambda \in SO_0(1,2).$$

We have

$$\Theta_W W = W', \qquad \Theta_W \mathcal{W} = \mathcal{W}.$$

Free Massive & Massless Bosons

Bargmann's classification of the UIRs of $SO_0(1,2)$

For $m^2 > 0$, the bosonic *one-particle Hilbert space* $\mathcal{H}$ is the completion of the linear span of the eigenfunctions of the *angular momentum operator*,

$$h_k(\psi) \doteq \frac{e^{ik\psi}}{\sqrt{r\,\pi}} \, , \qquad k \in \mathbb{Z} \, , \quad \psi \in [0, 2\pi) \, ,$$

with respect to the scalar product

$$\langle h, h' \rangle_{\mathcal{H}} = \Big\langle h, \frac{1}{2\omega} h' \Big\rangle_{L^2(S^1, r d\psi)} \, .$$

The Fourier coefficients of the strictly positive self-adjoint operator ω are expressed in terms of Γ functions:

$$\widetilde{\omega}(k) = \frac{k+s^+}{r} \frac{\Gamma\left(\frac{k+s^+}{2}\right)}{\Gamma\left(\frac{k-s^+}{2}\right)} \frac{\Gamma\left(\frac{k+1-s^+}{2}\right)}{\Gamma\left(\frac{k+1+s^+}{2}\right)} .$$

Note that $\widetilde{\omega}(k) \sim \sqrt{\frac{k^2}{r^2} + m^2}$ for k large, and that the constant $m^2 > 0$ enters through the parameter

$$s^{\pm} = -\frac{1}{2} \mp i\nu \, , \quad \nu = \begin{cases} i\sqrt{\frac{1}{4} - m^2 r^2} \, , & 0 < m^2 < \frac{1}{4r^2} \, , \\ \sqrt{m^2 r^2 - \frac{1}{4}} \, , & m^2 \geq \frac{1}{4r^2} \, . \end{cases}$$

In the limit $m^2 \to 0$, the Fourier coefficients $\widetilde{\omega}(k)$ become

$$\widetilde{\omega}(k) = \tfrac{|k|}{r} \quad \forall k \neq 0 \ .$$

In fact, one is confronted with two one-particle spaces $\mathcal{H}^{\pm}$, which are given by the completion of the linear span of the eigenfunctions

$$\big\{ h_k \mid k \in \mathbb{N} \big\} \quad \text{and} \quad \big\{ h_k \mid -k \in \mathbb{N} \big\} \ ,$$

respectively. The zero mode (the constant function on the geodesic Cauchy surface corresponding to $k = 0$) no longer appears in the massless case.

The one-particle space $\mathcal{H}$ carries a UIR of $SO_0(1,2)$ for $m^2 > 0$ and spin zero, generated by the rotations and boosts,

$$\big(u(R_0(\alpha))h\big)(\psi) = h(\psi - \alpha)\,, \quad \alpha \in [0, 2\pi)\,,$$

and

$$u(\Lambda_1(t)) = e^{it\omega\, r\cos}, \qquad t \in \mathbb{R}\,.$$

These representations extend to (anti-) unitary representations of $O(1,2)$. To be able to implement the reflections for $m^2 = 0$, we will use the direct sum $\mathcal{H}^+ \oplus \mathcal{H}^-$ in the massless case.

Localized Cauchy data

We may assume that the Cauchy data have their support contained in a connected open interval $I \subset S^1$. This leads us to consider an $\mathbb{R}$-linear subspace

$$\mathcal{H}(\mathcal{O}_I) \doteq \overline{\left\{h \in \mathcal{H} \mid supp\, \Re h \subset I,\; supp\, \omega^{-1} \Im h \subset I\right\}}\,.$$

Modular localisation

Let ℓ_W be the self-adjoint generator of the one-parameter subgroup $t \mapsto u\big(\Lambda_W(t)\big)$, and let

$$\delta_W \doteq e^{-2\pi \ell_W}, \qquad j_W \doteq u(\Theta_W).$$

δ_W is a densely defined, closed, positive non-singular linear operator on $\mathcal{H}$; j_W is an anti-unitary operator on $\mathcal{H}$.

These properties allow one to introduce the operator

$$s_W \doteq j_W \delta_W^{1/2} \, ,$$

s_W is a densely defined, antilinear, closed operator on $\mathcal{H}$ with $\mathscr{R}(s_W) = \mathscr{D}(s_W)$ and $s_W^2 \subset \mathbb{1}$. Moreover,

$$u(\Lambda) s_W u(\Lambda)^{-1} = s_{\Lambda W} \, , \qquad \Lambda \in SO_0(1,2).$$

i.) For the wedge W_1, we set

$$\mathcal{H}(W_1) \doteq \left\{ h \in \mathscr{D}\left(s_{W_1}\right) \mid s_{W_1} h = h \right\} .$$

ii.) For an arbitrary wedge $W = \Lambda W_1$, $\Lambda \in SO_0(1,2)$, we set

$$\mathcal{H}(W) \doteq u(\Lambda) \mathcal{H}(W_1) \, .$$

iii.) For a causally complete, open and bounded region $\mathcal{O}$, we set

$$\mathcal{H}(\mathcal{O}) \doteq \bigcap_{\mathcal{O} \subset W} \mathcal{H}(W) \, .$$

Localization of Cauchy data = Modular localization

For I a bounded open interval of length $|I| \leq \pi\, r$ in S^1 there holds

$$\mathcal{H}(\mathcal{O}_I) = \bigcap_{\mathcal{O}_I \subset W} \mathcal{H}(W)\ ,$$

where $\mathcal{O}_I = I''$ denotes the *causal completion* of the interval I in dS. This follows from the fact that $\Gamma(W') \cap S^1$ is in the interior $I^c \doteq S^1 \setminus \overline{I}$ of the complement of I within S^1.

Remark: in the massless case $m = 0$, modular localization and the localization of Cauchy data still coincide, there exist perfectly well-behaved Haag-Kastler nets, but no point-like fields!

Fock Space

- Fock space $\Gamma(\mathcal{H}) \doteq \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s^n}$,
- Coherent vectors

$$\Gamma(h) = \oplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \underbrace{h \otimes_s \cdots \otimes_s h}_{n-vezes}$$

- Second quantisation of operators (‘exponentiation’): let A be a closed linear operator, densely defined on $\mathcal{H}$. Then,

$$\Gamma(A) \colon \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$$

is the closure of the linear operator acting on the linear combinations of coherent vectors with exponent in $\mathscr{D}(A)$ such that:

$$\Gamma(A)\Gamma(h) = \Gamma(Ah).$$

‘Exponentiation’ preserves self-adjointness, positivity and unitarity.

The Weyl Algebra

For $h, g \in \mathcal{H}$, the relations

$$V(h)V(g) = e^{-i\Im\langle h,g\rangle} V(h+g),$$
$$V(h)\Omega_\circ = e^{-\frac{1}{2}||h||^2}\Gamma(ih), \qquad \Omega_\circ \doteq \Gamma(0),$$

define unitary operators, called the Weyl operators .

They satisfy $V^*(h) = V(-h)$ and $V(0) = \mathbb{1}$. The group $\Lambda \mapsto u(\Lambda)$ induces a group of automorphisms

$$\alpha^\circ_\Lambda(V(h)) \doteq V\big(u(\Lambda)h\big), \qquad h \in \mathcal{H}, \quad \Lambda \in SO_0(1,2),$$

representing the free dynamics.

Definition (A Net of Local Algebras)

We associate v. Neumann algebras to space-time regions in dS:

i.) *for the wedge* W_1,

$$\mathscr{A}_\circ(W_1) \doteq \{V(h) \mid h \in \mathcal{H}(W_1)\}'';$$

ii.) *for an arbitrary wedge* W, *set*

$$\mathscr{A}_\circ(W) \doteq \alpha^\circ_\Lambda\left(\mathscr{A}_\circ(W_1)\right), \qquad W = \Lambda W_1;$$

iii.) *for an arbitrary bounded, causally complete, convex region* $\mathcal{O} \subset dS$, *set*

$$\mathscr{A}_\circ(\mathcal{O}) = \bigcap_{\mathcal{O}\subset W} \mathscr{A}_\circ(W)\,.$$

Theorem

- *i.)* *The map* $\mathcal{O} \mapsto \mathscr{A}_\circ(\mathcal{O})$ *preserves inclusions and respects the causal structure.*
- *ii.)* *The algebras* $\mathscr{A}_\circ(\mathcal{O})$ *are hyperfinite type* III_1.
- *iii.)* *The automorphisms act covariantely, i.e.,*

$$\alpha^\circ_\Lambda\left(\mathscr{A}_\circ(\mathcal{O})\right) = \mathscr{A}_\circ(\Lambda\mathcal{O}).$$

- *iv.)* $J_W = \Gamma(j_W)$ *is a modular conjugation for* $(\mathscr{A}_\circ(W), \Omega_\circ)$.
- *v.)* $\Delta_W = \Gamma(\delta_W)$ *is the modular operator for* $(\mathscr{A}_\circ(W), \Omega_\circ)$. *The unitary groups* $t \mapsto \Delta_W^{it}$ *implement the Lorentz boosts.*

The free Fock vacuum vector $\Omega_\circ = \Gamma(0)$.

Remark

We have constructed the net of von Neumann algebras from the representation theory of $SO_0(1,2)$ using modular theory.

For the massive case, we could have arrived at exactly the same algebraic structure by quantizing the Klein-Gordon equation

$$\left(\Box_{dS} + m^2\right) \Phi(x) = 0 \ ,$$

smearing the field operator with test-functions that arise by restricting the Fourier-Helgason transformation to the mass shell. The von Neumann algebras than arise by going over to bounded functions of the unbounded field operators.

Interacting Models

A new proposal (Mund & J., 2015)

We can exploit modular theory for the construction of group representations !

Theorem (Tomita-Takesaki)

Given a cyclic and separating vector Ω *(such vectors are dense in* $\mathcal{H}$*) for* $\mathscr{A}_\circ(W_1)$*, the polar decomposition of the map*

$$A\Omega \mapsto A^*\Omega\,, \quad A \in \mathscr{A}_\circ(W_1)\,,$$

gives rise to a one-parameter group

$$t \mapsto \Delta_{W_1}^{it}\,, \qquad t \in \mathbb{R}\,,$$

which leaves $\mathscr{A}_\circ(W_1)$ *and* Ω *invariant. Moreover,* $\Omega \in \mathcal{P}^\sharp\big(\mathscr{A}_\circ(W^{(\alpha)}), \Omega_\circ\big)$ *implies* $J_{W^{(\alpha)}} = J^\circ_{W^{(\alpha)}}$.

$\Rightarrow$ candidate for a new Lorentz boost. To ensure that we get a new representation of $SO_0(1,2)$, we require that

- Ω is cyclic for $\mathscr{A}_\circ(W_1)$.
- Ω is invariant under the rotations $U_\circ(R_0(\alpha))$, $\alpha \in [0, 2\pi)$, which leave the Cauchy surface S^1 invariant.
- Ω lies in the positive cone $\overline{\{AJ^\circ_{W_1} A\Omega_\circ \mid A \in \mathscr{A}_\circ(W_1)\}}$; this ensures that $J_{W_1} = J^\circ_{W_1}$.
- some rather technical properties (which are satisfied in models, but should be eliminated from this list).

Theorem (Mund & J (2015 & work in progress))

The boost $t \mapsto \Delta^{it}_{W_1}$ and the (free) rotations $U_\circ(R_0(\alpha))$, $\alpha \in [0, 2\pi)$, generate a representation $U(\Lambda)$ of $SO_0(1,2)$.

Proof. Extend Ω to a rotation invariant state on the Euclidean sphere and then analytically continue the virtual representation of $SO(3)$ to $SO_0(1,2)$.

Example: The vacuum vector for the $\mathscr{P}(\varphi(\psi))_2$ model

The interacting de Sitter vacuum state for the $\mathscr{P}(\varphi(\psi))_2$ model is induced by a vector in Fock space:

$$\Omega = \frac{e^{-\pi H}\Omega_\circ}{||e^{-\pi H}\Omega_\circ||} ,$$

where

$$H := L_\circ + \int_{-\pi/2}^{\pi/2} r^2 \cos\psi \, d\psi \ :\mathscr{P}(\varphi(\psi)): .$$

Here $L_\circ = d\Gamma(\ell_1)$. Note that Ω induces a geodesic KMS state.

Finite speed of light

The representation $U(\Lambda)$ of $SO_0(1,2)$ is said to satisfy *finite speed of light*, if for any wedge W,

$$\mathscr{A}(W) \subset \mathscr{A}_\circ\big((\Gamma(W) \cap S^1)''\big) \ .$$

If *finite speed of light* holds, the local algebras associated to an interval $I \subset S^1$ on the Cauchy surface coincide with those of the free theory, *i.e.*,

$$\mathscr{A}(\mathcal{O}_I) = \mathscr{A}_\circ(\mathcal{O}_I) \ , \qquad I \subset S^1 \ .$$

Conjecture: the technical assumptions on Ω are not needed.

Definition (A new net of local algebras)

We proceed just as for the free theory: we associate v. Neumann algebras to space-time regions in dS:

- *i.)* *for the wedge W_1, set $\mathscr{A}(W_1) \doteq \mathscr{A}_\circ(W_1)$;*
- *ii.)* *for an arbitrary wedge $W = \Lambda W_1$, $\Lambda \in SO_0(1,2)$, set*
$$\mathscr{A}(W) \doteq U(\Lambda)\, \mathscr{A}_\circ(W_1)\, U^{-1}(\Lambda) \ ;$$
- *iii.)* *for an arbitrary bounded, causally complete, convex region $\mathcal{O} \subset dS$, set with $\mathcal{O}''$ bounded by*
$$\mathscr{A}(\mathcal{O}) = \bigcap_{\mathcal{O} \subset W} \mathscr{A}(W) \, .$$

The map $\mathcal{O} \mapsto \mathscr{A}(\mathcal{O})$ specifies a new quantum theory.

Theorem (Verification of the Haag-Kastler Axioms)

The representation $\alpha\colon \Lambda \mapsto \alpha_\Lambda$ *of the Lorentz group* $SO_0(1,2)$ *is covariant:*

$$\alpha_\Lambda\big(\mathscr{A}(\mathcal{O})\big) = \mathscr{A}(\Lambda\mathcal{O})\,, \qquad \Lambda \in SO_0(1,2)\,.$$

The local algebras satisfy micro-causality, i.e.,

$$\mathscr{A}(\mathcal{O}_1) \subset \mathscr{A}(\mathcal{O}_2)' \quad se \quad \mathcal{O}_1 \subset \mathcal{O}_2'\,.$$

Here $\mathcal{O}'$ *denotes the space-like complement of* $\mathcal{O}$ *in* dS *and* $\mathscr{A}(\mathcal{O})'$ *is the commutante of* $\mathscr{A}(\mathcal{O})$ *in* $\mathcal{B}(\Gamma(\mathcal{H}))$.

(Additivity).For X a double cone or a wedge, there holds

$$\mathscr{A}(X) = \bigvee_{\mathcal{O} \subset X} \mathscr{A}(\mathcal{O}) \,. \tag{1}$$

The right hand side denotes the von Neumann algebra generated by the local algebras associated to double cones $\mathcal{O}$ contained in X.

(Weak additivity). For each double cone $\mathcal{O} \subset dS$ there holds

$$\bigvee_{\Lambda \in SO_0(1,2)} \mathscr{A}(\Lambda\mathcal{O}) = \mathscr{A}(dS) \quad \big(= \mathscr{B}(\Gamma(\mathcal{H}))\big) \,.$$

The time-slice axiom holds as well.

Theorem (continuation; Mund & J. (2017))

$\Omega \in \mathcal{H}$ is the unique (up to a phase factor) vector which

- *is invariant under the action of $U(SO_0(1,2))$;*
- *for every wedge W, the map*

$$t \mapsto \langle \Omega, A\Delta^{-it} B\Omega \rangle \,, \qquad A, B \in \mathscr{A}(W)\,,$$

allows an analytic continuation to $\{t \in \mathbb{C} \mid 0 < \Im t < 1/2\}$. Moreover, the boundary values satisfies the KMS condition (describing thermalisation due to the curvature of dS).

Remark: In the limit of curvature to zero, these analyticity properties imply that the limiting state is a Poincaré invariant positive energy state in Minkwoski space, i.e., a vacuum state.

Connes' cocycle and non-commutative L^p spaces

The relative modular operator $\Delta_{\Omega,\Omega_\circ} = S^*_{\Omega,\Omega_\circ}\overline{S_{\Omega,\Omega_\circ}}$ arises from the polar decomposition of the anti-linear map

$$S_{\Omega,\Omega_\circ} M\Omega_\circ = M^*\Omega\,, \qquad M \in \mathscr{A}_\circ(W_1)\,.$$

Given Ω and $\Omega_\circ$, the Radon-Nikodym derivative exists as a strongly continuous one-parameter family of unitaries

$$u_t = [D\Omega : D\Omega_\circ]_t = \Delta^{it}_{\Omega,\Omega_\circ}\Delta_\circ^{-it} \in \mathscr{A}_\circ(W_1)\,, \quad t \in \mathbb{R}\,,$$

which intertwines the modular groups for Ω and $\Omega_\circ$, *i.e.*,

$$\sigma_t(M) = u_t\,\sigma^\circ_t(M)\,u_t^* \quad \forall M \in \mathscr{A}_\circ(W_1), \quad \sigma^\circ_t = ad_{\Delta_\circ^{-it}}\,,$$

and satisfies Connes' cocycle relation $u_{t+s} = u_t\,\sigma^\circ_t(u_s)$, $t, s \in \mathbb{R}$.