Holographic Tensors

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Random Vectors, Matrices, Tensors

Vectors	Matrices	Tensors
XIXth century	XXth century	XXIth century

- Each class is richer than the previous one, having more and more invariants
- Each class has a different universality and a different $1/N$ expansion

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	Data size for a $U(N)^d$ symmetry of size $\simeq N^2$	
N	N^2	N^d, $d \geq 3$
	Associated locality	
scalar product	cyclicity	traciality
	Quartic UV behavior	
asymptotic slavery	asymptotic safety	asymptotic freedom

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- Consider N iid random events.
  They generate a vector of random data X_i, $i = 1, \cdots N$.
- Central Limit Theorem: suppose X_i has mean μ and variance σ. Then under mild conditions $\sqrt{N}(\frac{\sum_i X_i}{N} - \mu)$ converges to a normalized Gaussian distribution when $N \to \infty$.
- Remark that the normalized Gaussian distribution $Z^{-1} e^{-N \sum X_i^2} \prod dX_i$ is invariant under $O(N)$. Moreover $\sum X_i^2$ is the only connected $O(N)$ polynomial invariant in the X_i.
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- M an N by N random Hermitian matrix. $\exists$ unique Gaussian $U(N)$ invariant measure (GUE)

$$e^{-N\mathrm{Tr}M^2} dM = \prod_i e^{-N\sum \lambda_i^2} d\lambda_i \prod_{i\neq j}(\lambda_i - \lambda_j)^2$$

- Universality (eg of Wigner's semi circle law), many many applications: heavy nuclei, perching birds, stability of species, neurosciences, optimal control, Riemann zeta function...
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Random Matrices and Random Surfaces

Consider eg GUE model perturbed by a connected interaction

$$Z(\lambda, N) = Z_0^{-1} \int e^{-N(\mathrm{Tr} M^2 + \lambda \mathrm{Tr} M^p)} dM$$

$$\log Z(\lambda, N) = \sum_{V \geq 1} \lambda^V a(V, N)$$

$$a(V, N) = \sum_{g \geq 0} N^{2-2g} a(g, V)$$

where $a(g, V)$ is the number of connected graphs embedded on a genus g surface with V p-valent vertices, so that $2 - 2g = V - L + F$.

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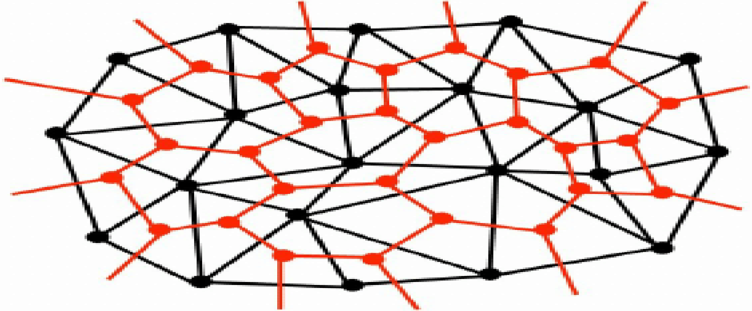
Why Random Surfaces?



Matrix Feynman Graphs are dual to triangulated (or p-angulated) surfaces => dynamical triangulations.

Planar $g = 0$ graphs lead the $1/N$ matrix expansion.

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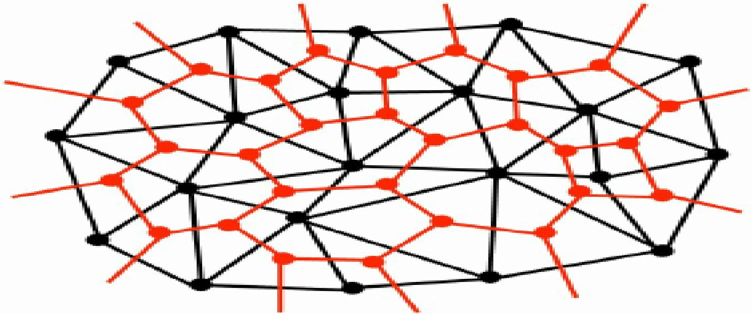

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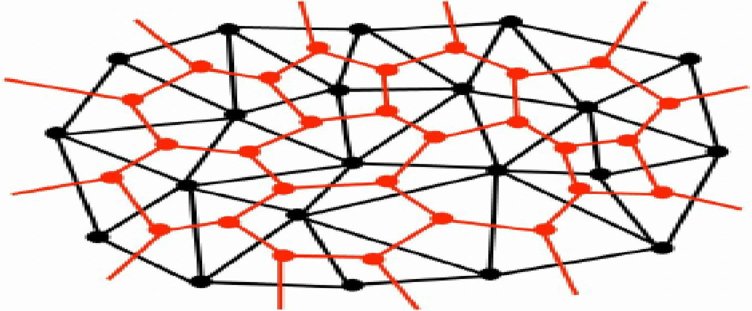

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- Random tensor T of rank D have N^D components $T_{i_1,\cdots i_D}$. Simplest case: complex, not symmetric model of $D+1$ rank D tensors with complete graph interaction => the colored random tensor model, which has

$$U(N)^{\otimes D(D+1)/2}$$

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- For a single rank D tensor, many connected $U(N)^{\otimes D}$ polynomial invariants => a vast family of uncolored random tensor models,
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- uses $D+1$ random tensors;
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- its Feynman graphs are dual to simplicial (orientable) triangulations

Probability measure

$$d\nu = \prod_{i,n_i} \frac{dT^i_{n_i} d\bar{T}^i_{\bar{n}_i}}{2\pi} e^{-S(T,\bar{T})}$$

$$S = \sum_{i=0}^{D} \bar{T}^i \cdot T^i + \frac{\lambda}{N^{D(D-1)/4}} \sum_{\{n\}} \prod_{i=0}^{D} T^i_{n_i} \prod_{i<j} \delta_{n^{ij},n^{ji}} + cc$$

where $\sum_{\vec{n}}$ denotes the sum over all indices n_{ij} from 1 to N. The $\frac{(D+1)D}{2}$ identifying δ functions follow the pattern of edges of the K_{D+1} complete graph on $D+1$ vertices.

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where $\sum_{\vec{n}}$ denotes the sum over all indices n_{ij} from 1 to N. The $\frac{(D+1)D}{2}$ identifying δ functions follow the pattern of edges of the K_{D+1} complete graph on $D+1$ vertices.

The Colored $U(N)$ Tensor Model

- uses $D+1$ random tensors;
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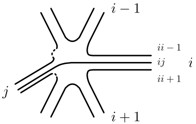
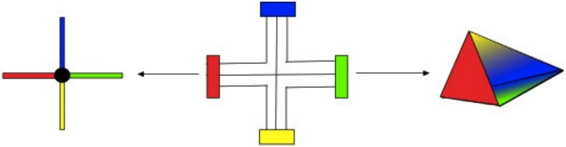


- Colors can conveniently encode strands
- and gluing rules for dual triangulations



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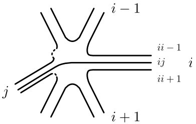

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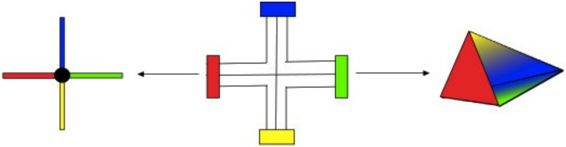

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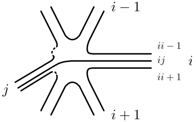

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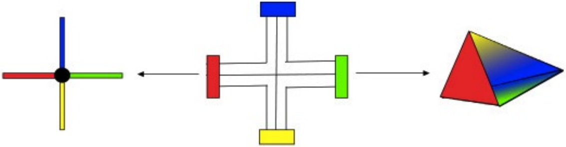

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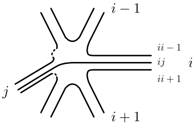

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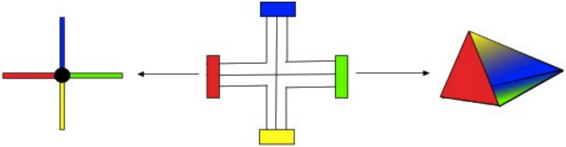

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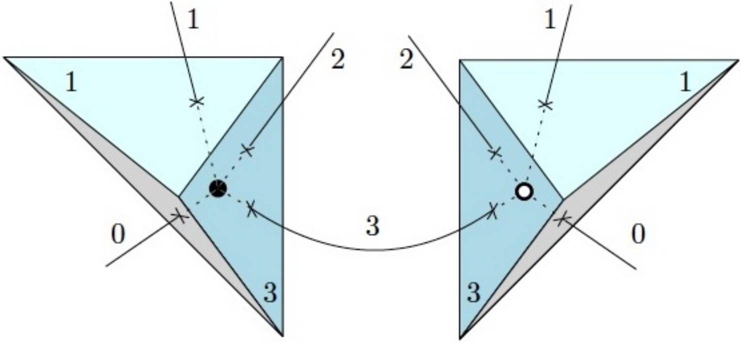

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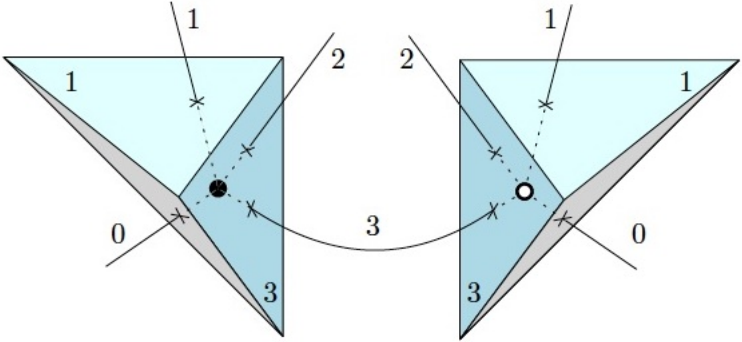

D-Homology

- For D-regular edge-colored graphs there is a simple canonical definition of faces
- k-dimensional objects = connected components with k colors
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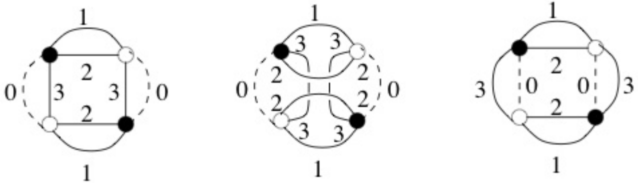
Jackets, Degree, $1/N$ Expansion

Jacket $J =$ color cycle up to orientation ($D!/2$ at rank D)

Defines a ribbon graph G_J with same number of lines and vertices than G. This ribbon graph has a genus g_J.



$A(G) \propto N^{D-\frac{2}{D!}\omega(G)}$, where $\omega = \sum_J g(J) \geq 0$, the Gurau degree, governs the expansion.

For $D \geq 3$ this degree is not a topological invariant of the space dual to G.

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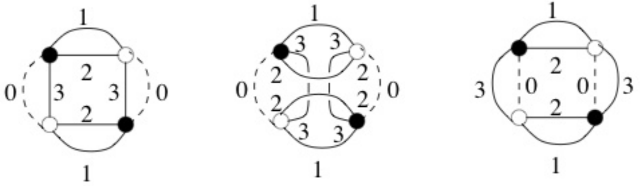

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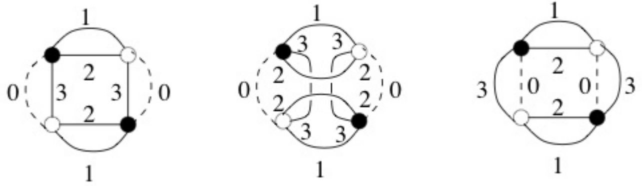

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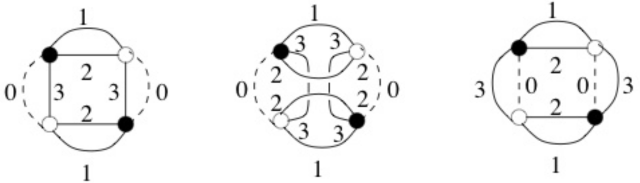

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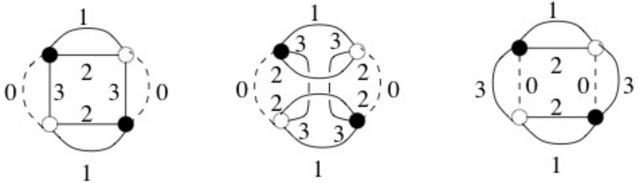

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Counting Faces with Jackets

Each face f_{ij} belongs to $(D-1)!$ jackets (the ones in which i and j are adjacent).

$$2 - 2g_J = V - L + F_J.$$

Since $L = \frac{D+1}{2}V$, summing over all jackets we get

$$\sum_J F_J = (D-1)!F = -2\sum_J g_J + \frac{D!}{2}(2 + \frac{D-1}{2}V)$$

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Elementary Melon

Vacuum elementary melon: two vertices, $D+1$ edges, $F = D(D+1)/2$ faces

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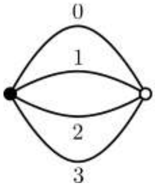

2-point elementary melon of color $i \in \{0, 1, \cdots, D\}$: cut the line of color i.

Definition Vacuum melonic graphs are the graphs obtained from the elementary vacuum melon by finitely many recursive insertions of a 2-point elementary melon of color $i \in \{0, 1, \cdots, D\}$ on any edge of the *same* color i.

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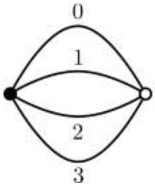

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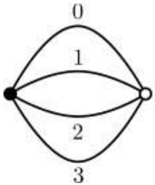

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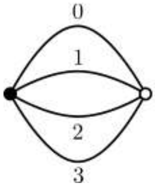

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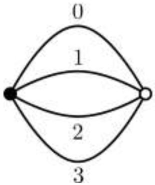

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The Result of the Recursion

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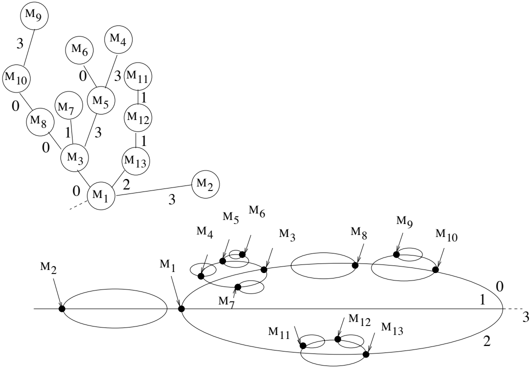

The Basic Theorem (Colored Model Case)

Zero degree graphs (ZDG) are graphs with $\omega = 0$. Equivalently they are planar in each jacket. They form the leading sector of the tensorial $1/N$ expansion.

Zero Degree Graphs = Melonic Graphs (Bonzom, Gurau, Riello, R., 2011)

Let's prove it!

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Melons are ZDG

Recall that $F = D + \frac{D(D-1)}{4} V - \frac{2}{D!}\omega$ hence

$$\omega = 0 <=> F = D + \frac{D(D-1)}{4} V \qquad (A)$$

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By induction, since melonic insertion increases V by 2 and F by $\frac{D(D-1)}{2}$, any melonic graph has $F = D + \frac{D(D-1)}{4} V$ hence has $\omega = 0$, hence is a ZDG.

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ZDG are melons

Consider a ZDG. Call F_k the number of its faces of length $2k$. Recall that

$$F = \sum_{k\geq 1} F_k = D + \frac{D(D-1)}{4} V \qquad (A)$$

- Check by edge counting that

$$2F_1 + 4F_2 + \sum_{k\geq 3} 2kF_k = \frac{D(D+1)}{2} V \qquad (B)$$

- Compute $2A - B/2$ to prove that

$$F_1 = 2D + \sum_{k\geq 3} (k-2)F_k + \frac{D(D-3)}{4} V \geq 2D$$

  Conclude that vacuum (and also 2-point) ZDG's have faces of length 2.

- Conclude that any ZDG is a melon, hence $ZDG = \text{Melons}$.

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$$F_1 = 2D + \sum_{k \geq 3} (k-2) F_k + \frac{D(D-3)}{4} V \geq 2D$$

Conclude that vacuum (and also 2-point) ZDG's have faces of length 2.

- Conclude that any ZDG is a melon, hence $ZDG =$ Melons.

ZDG are melons

Consider a ZDG. Call F_k the number of its faces of length $2k$. Recall that

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Klebanov-Tarnopolosky: should melons also dominate at large N for symmetric traceless tensors?

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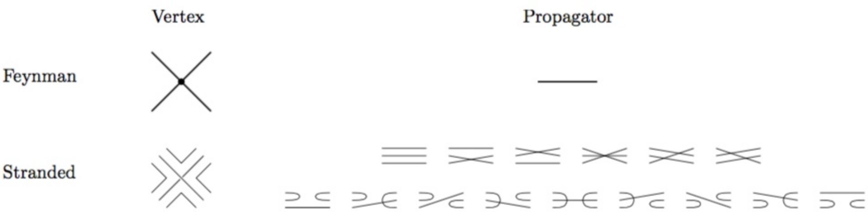
Symmetric Traceless Rank 3 Tensors



Vertex and propagator of the model.

♯1
9
9
B

♯2
8
8
B

♯3
8
7

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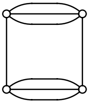

♯2
8
8
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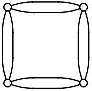
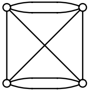
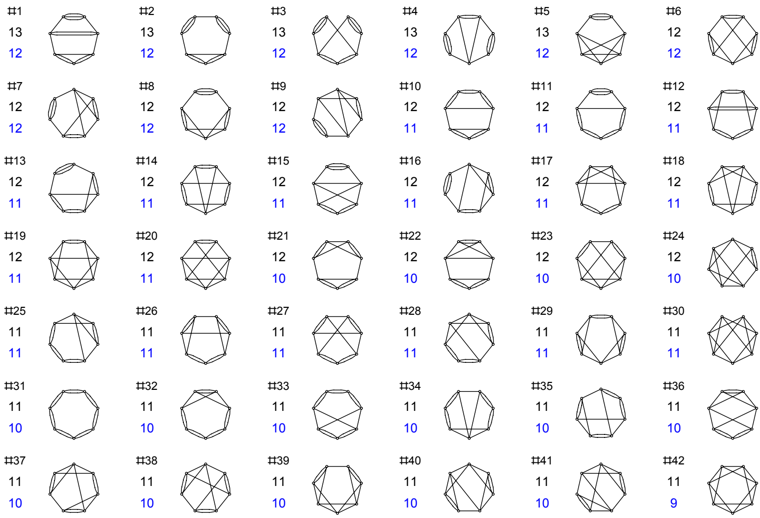


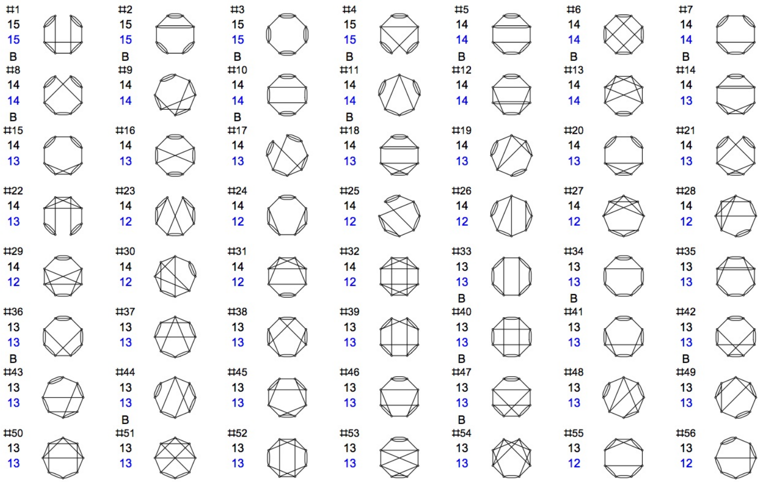
♯3
8
7



Klebanov-Tarnopolosky: Computer Study, order 7



Klebanov-Tarnopolosky: Computer Study, order 8, Part 1



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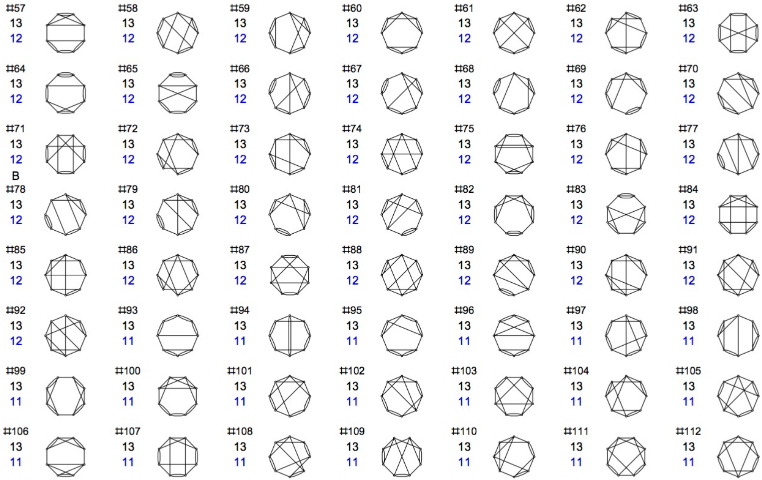

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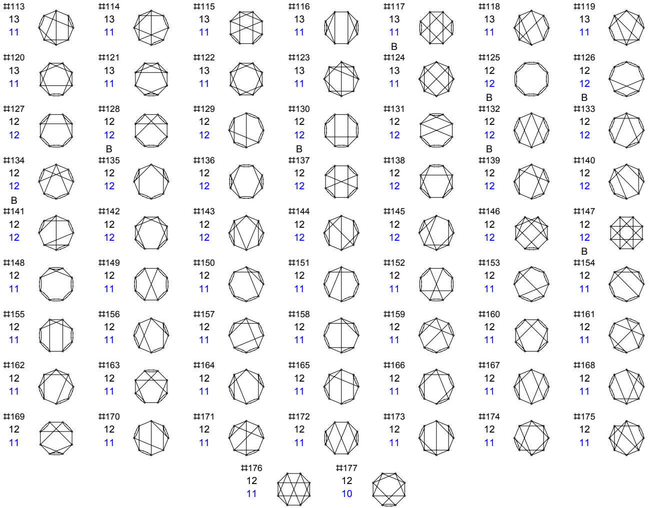

Complete Proof

Carrozza et al (2017- 2018) : melons dominate all rank-three irreducible representations of $O(N)$

- traceless symmetric: arXiv:1712.00249
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- mixed: arXiv:1803.02496

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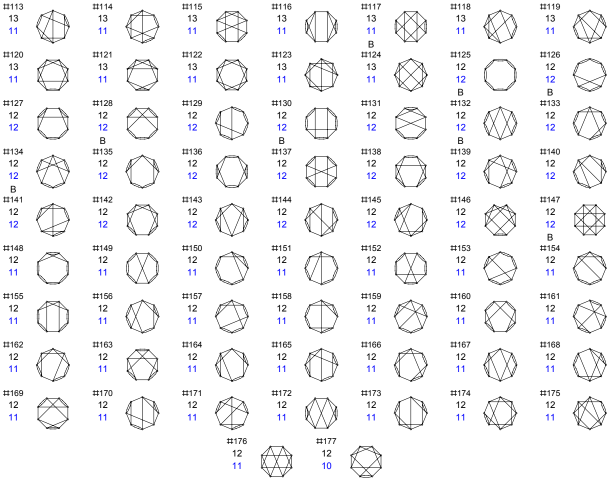
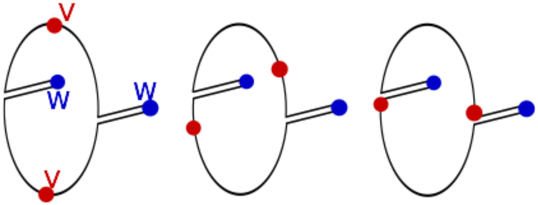


The MSS bound

In March 2015 Maldacena, Shenker and Stanford wrote that in any quantum system at temperature T the Lyapunov exponent for transient chaos in a four point correlator maximally spaced on the thermal circle



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is bounded by $\lambda_L \leq 2\pi T/\hbar$ under very general assumptions (analyticity in a strip of width $\beta/2$ in complex time and reasonable decay at infinity).
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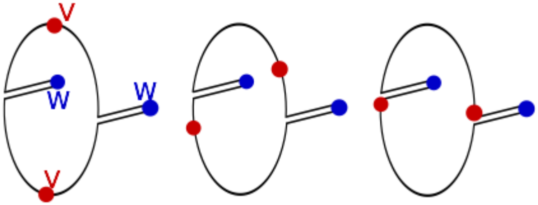

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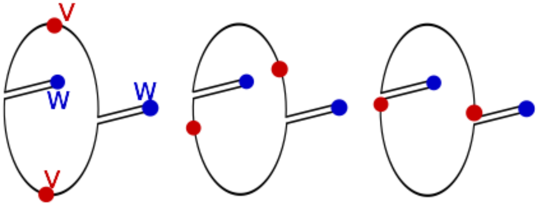

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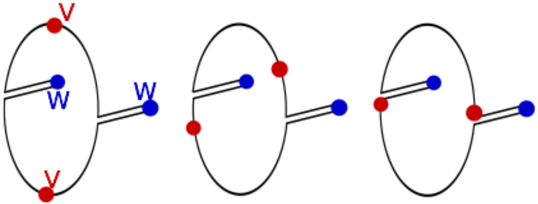

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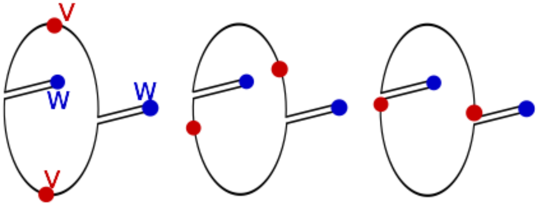

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More precisely they found that

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The Sachdev-Ye-Kitaev Model

In 2015 Kitaev found a very simple quasi-conformal one dimensional quantum mechanics model which saturates the MSS bound, indicating the surprising presence of a gravitational dual.

The action is

$$I = \int dt \left(\frac{i}{2} \sum_i \psi_i \frac{d}{dt} \psi_i - i^{q/2} \sum_{1 \le i_1 < \cdots < i_q \le N} J_{i_1, \cdots, i_q} \psi_{i_1} \cdots \psi_{i_q} \right) \tag{3.1}$$

with J a quenched iid random tensor ($< J_I J_{I'} > = \delta_{II'} J^2 (q-1)! N^{-(q-1)}$), and ψ an N-vector Majorana Fermion.

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This model is solvable as $N \to \infty$, being approximately conformal and reparametrization invariant in the infra-red limit. For instance the two point function in that limit reads

$$G(\tau) = b_q[\frac{\pi}{\beta \sin(\pi\tau/\beta)}]^{2/q} \mathrm{sgn}\tau.$$

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The Schwinger Dyson Equation in Melonic limit is

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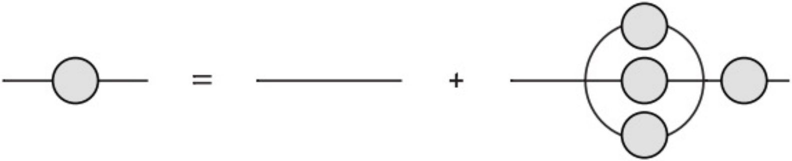

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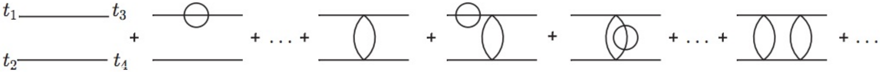

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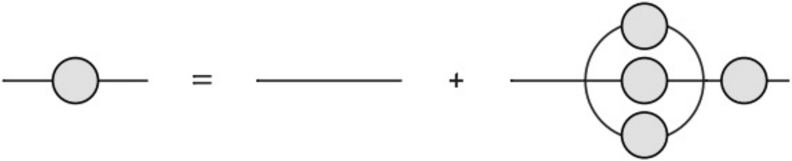

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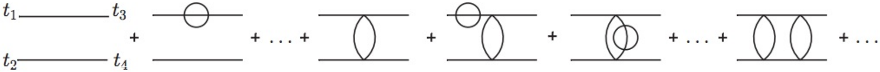

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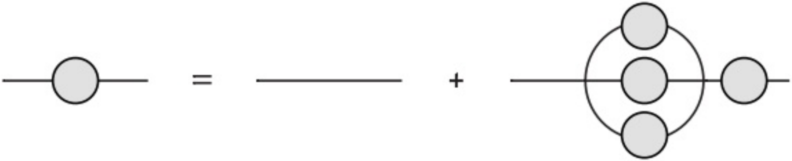

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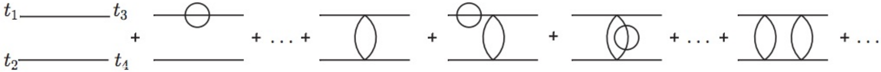

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suggests to search for a particular solution of type

$$G_c(\tau) = b|\tau|^{-2\Delta} \mathrm{sign}\ \tau, \quad J^2 b^q \pi = (\frac{1}{2} - \Delta) \tan(\pi\Delta)$$

The equation for b comes from the formula

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Some Remarks

The conformal decay $G_c(\tau) \simeq \tau^{2/q}$ at small ω corresponds to the theory being a just renormalizable tensor field theory (Ben Geloun, Dine, Carrozza, Oriti, R. Toriumi...)

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He proposed a modification to eliminate the quenched disorder with action

$$I = \int dt \left(\frac{i}{2} \sum_i \psi_i \frac{d}{dt} \psi_i - i^{q/2} j \psi_0 \psi_1 \cdots \psi_D \right) \tag{3.2}$$

where ψ's are $D+1$ fermionic tensors and the pattern of index contraction is exactly the one of Gurau's initial colored tensor model, hence this new model is called the Gurau-Witten model.

An uncolored, i.e. single tensor model with similar properties was soon developped by Klebanov and Tarnopolsky (arXiv:1611.08915), based on the three dimensional $O(N)$ tensor model of Carrozza and Tanasa. (arXiv:1512.06718).

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The SYK 4-point-Function and MSS Bound

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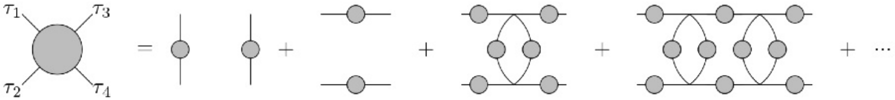
The 4-point Function in Melonic Limit

$$\frac{1}{N^2}\sum_{1\leq i,j\leq N} \exp T\left(\psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\right) = G(\tau_{12})G(\tau_{34}) + \frac{1}{N}\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) + \cdots$$

where we write $\tau_{12} = \tau_1 - \tau_2$.

The function $\mathcal{F}$ then develops graphically as



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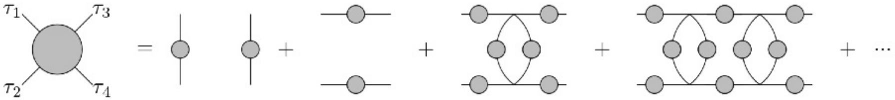

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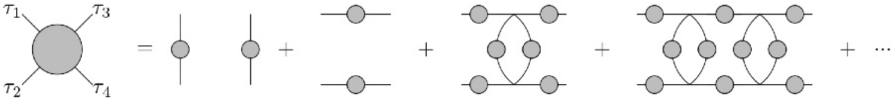


Ladders and Rungs

Calling $\mathcal{F}_n$ the ladder with n "rungs" we have $\mathcal{F} = \sum_{n\geq 0} \mathcal{F}_n$.

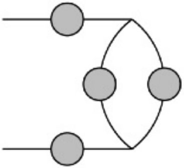
$$\mathcal{F}_0 = -G(\tau_{13})G(\tau_{24}) + G(\tau_{14})G(\tau_{23}).$$

$$\begin{aligned}
\mathcal{F}_{n+1}(\tau_1,\tau_2,\tau_3,\tau_4) &= J^2(q-1)\int d\tau d\tau' G(\tau_1-\tau)G(\tau_2-\tau')G^{q-2}(\tau-\tau') \\
&\quad G(\tau-\tau_3)G(\tau'-\tau_4) - [\tau_3 <-> \tau_4] \\
&= \int \partial\tau\partial\tau' K(\tau_1,\tau_2,\tau,\tau')\mathcal{F}_n(\tau,\tau',\tau_3,\tau_4)
\end{aligned}$$

The rung operator

K is the "rung" operator adding one rung to the ladder. It acts on the space of "bilocal" functions with kernel

$$K(\tau_1, \tau_2, \tau_3, \tau_4) = -J^2(q-1)G(\tau_{13})G(\tau_{24})G(\tau_{34})^{(q-2)}.$$



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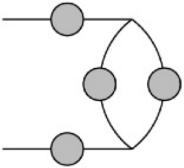

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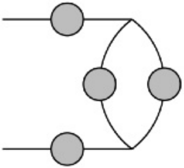

The rung operator, II

The main problem is therefore to diagonalize this rung operator K. In particular if 1 is an eigenvalue of K, it signals a divergent mode. Indeed

$$\mathcal{F} = \sum_{n \geq 0} \mathcal{F}_n = \sum_{n \geq 0} K^n \mathcal{F}_0 = \frac{1}{1-K} \mathcal{F}_0$$

or, more explicitly

$$\mathcal{F}(\tau_1, \tau_2, \tau_3, \tau_4) = \int \partial\tau \partial\tau' \frac{1}{1-K}(\tau_1, \tau_2, \tau, \tau') \mathcal{F}_0(\tau, \tau', \tau_3, \tau_4).$$

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The rung kernel

Recalling the formula for the two point function in the approximate conformal (infrared) limit at zero temperature

$$G_c(\tau) = \frac{b}{|\tau|^{2\Delta}} \mathrm{sign}\tau$$

with

$$b^q J^2 \pi = (\frac{1}{2} - \Delta) \tan(\pi\Delta)$$

we find that in this limit the kernel K becomes

$$K_c(\tau_1, \tau_2, \tau_3, \tau_4) = -\frac{1}{\alpha_0} \frac{\mathrm{sign}(\tau_{13})\mathrm{sign}(\tau_{24})}{|\tau_{13}|^{2\Delta}|\tau_{24}|^{2\Delta}|\tau_{34}|^{2-4\Delta}}$$

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Reformulation with cross ratios

Conformal invariance allows us to simplify the problem by reexpressing K as a function of the cross ratio $\chi = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}$ acting on single variable rung functions

$$\mathcal{F}_{n+1}(\chi) = \int \frac{d\tilde{\chi}}{\tilde{\chi}^2} K_c(\chi, \tilde{\chi}) \mathcal{F}_n(\tilde{\chi})$$

To further simplify the diagonalization it is important to find out operators commuting with K. The Casimir operator $C = \chi^2(1-\chi)\partial_\chi^2 - \chi^2\partial_\chi$ is such an operator, with a known complete set of eigenvectors $\Psi_h(\chi)$ with eigenvalues $h(h-1)$, which are therefore also the eigenvectors of $K_c(\chi, \tilde{\chi})$.

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The strategy

The strategy to compute $\mathcal{F}$ can then be summarized as

- Find properties of $\mathcal{F}_n(\tilde{\chi})$ and the eigenvectors $\Psi_h(\chi)$ of the Casimir operator C with these properties.
- Deduce conditions on h. One finds two families, $h = 2n$ with $n \in \mathbb{N}^\star$ and $h = \frac{1}{2} + is$, $s \in \mathbb{R}$
- Compute the eigenvalues $k_c(h)$ of the kernel K_c and the inner products $< \Psi_h, \mathcal{F}_0 >$ and $< \Psi_h, \Psi_h >$.
- Conclude that the 4 point function is

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- But... one finds a single $h = 2$ mode with $k_c(h) = 1$. It requires special desingularization!

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Conclusion

Thank you for your attention!