Feynman Propagators and the Self-Adjointness of the Klein–Gordon Operator

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BERGISCHE UNIVERSITÄT WUPPERTAL

Introduction

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bisolution	$KG = 0, \quad GK = 0$

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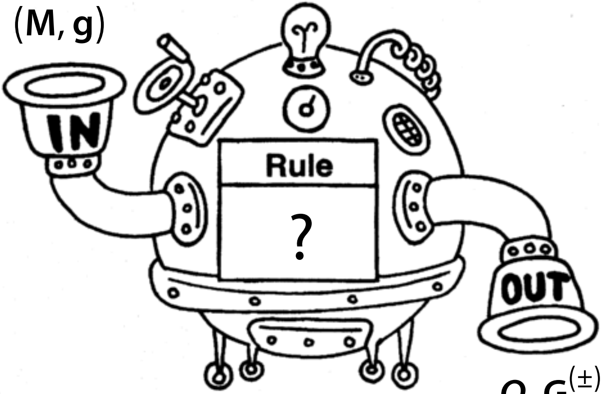
No natural states

Image adapted from: https://transferready.co.uk/worksheet-function-machines/

(M, g)



$\Omega, G^{(\pm)}$

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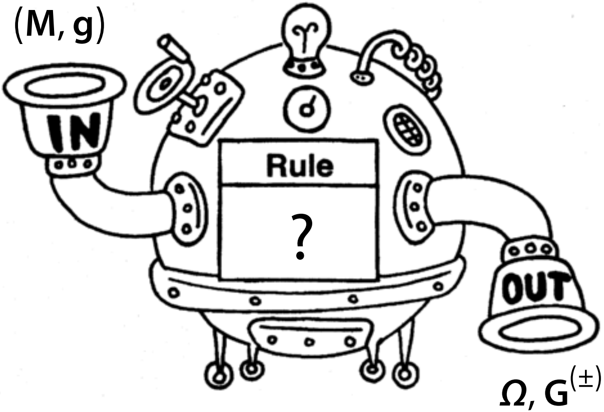

There exists **no natural** (meaning: locally covariant) **construction of states** for all globally hyperbolic spacetimes. [Fewster, Verch]

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Missing: property to replace positive/negative frequency condition

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Properties of **Hadamard** positive/negative frequency bisolutions:

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- $(f|G^{(\pm)}f) \geq 0, \quad f \in C_c^\infty(M)$
- $\mathrm{WF}'(G^{\pm}) \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}$ [Radzikowski, ...]

  ($\mathcal{N}^{\pm}$ = future/past-directed lightlike vectors)

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The Hadamard condition is necessary [Fewster, Verch]
(on spatially compact ultrastatic slab spacetimes)

Examples of Hadamard states

Some **examples** of states satisfying the Hadamard condition:

- Vacuum states and thermal states in **stationary spacetimes**
- **Bunch–Davies state** for de Sitter spacetime
- **Hartle–Hawking–Israel state** for stationary black holes [Kay, Wald, Sanders, ...]
- **States of low-energy** in cosmological spacetimes [Olbermann, ...]
- 'Holographic' states for **asymptotically flat spacetimes** [Moretti, Dappiaggi, Pinamonti, ...]

Constructions are difficult unless the spacetime has symmetries.

For more general spacetimes use e.g. PDO methods of Gérard–Wrochna.

Time-ordered product

What about the **Feynman propagator**?

To each state corresponds a Feynman propagator via

$$G^{\mathrm{F}} := G^{\wedge} + \mathrm{i}G^{(+)} = G^{\vee} + \mathrm{i}G^{(-)}$$
$$G^{\overline{\mathrm{F}}} := G^{\vee} - \mathrm{i}G^{(+)} = G^{\wedge} - \mathrm{i}G^{(-)}$$

Feynman propagators play a subordinate role in the theory...

In-out Feynman propagator

Hamiltonian formalism

Make a $1 + 3$ split of the spacetime $M = \mathbf{R} \times \Sigma$ with time-variable t.

Rewrite the Klein–Gordon equation $K\varphi = 0$ as a (non-autonomous) **evolution equation** on the Cauchy data

$$\big(\partial_t + \mathrm{i}B(t)\big)u(t) = 0, \qquad u(t) = \begin{pmatrix} \varphi(t) \\ \mathrm{i}\partial_t\varphi(t) \end{pmatrix}.$$

The propagators of $\partial_t + \mathrm{i}B(t)$ are directly related to those of K.

Evolution

Solve the Cauchy problem: There is an **evolution operator** $R(t,s)$ with

- $R(t,t) = \mathbf{1}$
- $R(t,r)R(r,s) = R(t,s)$
- $\mathrm{i}\partial_t R(t,s) = +B(t)R(t,s)$
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This immediately yields the **kernels of the classical propagators**:

$$E^{\mathrm{PJ}}(t,s) := R(t,s) \qquad \textbf{(Pauli–Jordan propagator)}$$

$$E^{\vee}(t,s) := \theta(t-s)R(t,s) \qquad \text{(retarded propagator)}$$

$$E^{\wedge}(t,s) := -\theta(s-t)R(t,s) \qquad \text{(advanced propagator)}$$

Charge form and admissible involutions

For $u, v \in H^{\frac{1}{2}}(\Sigma) \oplus H^{-\frac{1}{2}}(\Sigma)$ define the **charge form**

$$(u|Qv), \quad Q = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

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We call S an **admissible involution** if $S^2 = \mathbf{1}$ and

$$(u|v)_S = (u|QSv) = (Su|Qv)$$

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Note that, to each S are associated **two projections**

$$\Pi^{(\pm)} := \frac{1}{2}(\mathbf{1} \pm S).$$

Instantaneous non-classical propagators

Given $R(t,s)$, an admissible involution S and any $\tau \in \mathbf{R}$, we can write **kernels for non-classical propagators**:

$$E_\tau^{(+)}(t,s) := R(t,\tau)\Pi^{(+)}R(\tau,s) \qquad \text{(positive freq. bisolution)}$$

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$$E_\tau^{\mathrm{F}}(t,s) := \theta(t-s)E_\tau^{(+)}(t,s) + \theta(s-t)E_\tau^{(-)}(t,s) \qquad \text{(Feynman propagator)}$$

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Example: $S(\tau) = \operatorname{sgn} B(\tau)$ corresponds to the projections

$$\Pi_\tau^{(\pm)} = \mathbf{1}_{[0,\infty)}\big(\pm B(\tau)\big).$$

It yields the **instantaneous** positive/negative freq. bisolutions, which typically **do not satisfy the Hadamard condition**.

Pairs of projections and admissible involutions

Theorem

Let $\Pi_{+}^{(+)}, \Pi_{-}^{(+)}$ be **projections** on $\mathcal{H}$. Set $\Pi_{\pm}^{(-)} = \mathbf{1} - \Pi_{\pm}^{(+)}$. Suppose that

$$\Upsilon = \mathbf{1} - \left(\Pi_{-}^{(+)} - \Pi_{+}^{(+)}\right)^2$$

is **invertible**.

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$$\Lambda^{(+)} = \Pi_-^{(+)} \Upsilon^{-1} \Pi_+^{(+)}, \qquad \Lambda^{(-)} = \Pi_+^{(-)} \Upsilon^{-1} \Pi_-^{(-)}$$

are **complementary projections**, i.e. $\Lambda^{(+)} + \Lambda^{(-)} = \mathbf{1}$.

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Theorem ('asymptotic complementarity')

Let S_{-}, S_{+} be **admissible involutions** with associated projections $\Pi^{(\pm)}_{\bullet}$. Then $\Upsilon = \mathbf{1} - \left(\Pi^{(+)}_{-} - \Pi^{(+)}_{+}\right)^2$ is **invertible**.

In-out Feynman propagator

Set $S_\pm := \operatorname{sgn} B(t_\pm)$ for $t_+, t_- \in \mathbf{R}$. Define the **time-evolved projections**

$$\Pi_\pm^{(+)}(t) := R(t, t_\pm)\Pi_\pm^{(+)}R(t_\pm, t)$$
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Define the **(in-out) Feynman propagator**

$$E^{\mathrm{F}}(t, s) := \theta(t - s)R(t, s)\Lambda^{(+)}(s) + \theta(s - t)R(t, s)\Lambda^{(-)}(s)$$

NB: Generally not associated to a state! Positivity fails.

In-out?

We have defined the Feynman propagator

$$E^{\mathrm{F}}(t,s) := \theta(t-s)R(t,s)\Lambda^{(+)}(s) + \theta(s-t)R(t,s)\Lambda^{(-)}(s).$$

Although one might be tempted to write

$$G^{\mathrm{F}}(x,x') = \frac{\langle \Omega_{\mathrm{out}} \,|\, \varphi(x)\varphi^*(x')\,\Omega_{\mathrm{in}}\rangle}{\langle \Omega_{\mathrm{out}} \,|\, \Omega_{\mathrm{in}}\rangle},$$

this wrong because Ω_{in} and Ω_{out} are generally not in the same Hilbert space (infinite particle production).

Self-adjointness

Why is self-adjointness relevant?

In **Minkowski spacetime**, the (anti-)Feynman propagator is defined by

$$\hat{G}^{\mathrm{F}/\overline{\mathrm{F}}}(k) = \left(k^2 + m^2 \mp \mathrm{i}0\right)^{-1}.$$

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Translated to position space, this means studying two questions:

1. the **resolvent** of K at $\pm\mathrm{i}\varepsilon$ ($\varepsilon > 0$)

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2. the **boundary value** of the resolvent

$$(K \mp \mathrm{i}0)^{-1} = \lim_{\varepsilon \to 0^+} (K \mp \mathrm{i}\varepsilon)^{-1}.$$

Simple example: ultrastatic spacetimes

Consider an **ultrastatic spacetime** $M = \mathbf{R} \times \Sigma$, $g = -\,\mathrm{d}t^2 + g_\Sigma$.

The Klein–Gordon operator attains the form

$$K = \partial_t^2 - \Delta_\Sigma + Y = \partial_t^2 \otimes \mathbf{1} + \mathbf{1} \otimes \left(-\Delta_\Sigma + Y\right)$$

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Interpreted as an operator on $L^2(M) = L^2(\mathbf{R}) \otimes L^2(\Sigma)$, it is **essentially self-adjoint** on $C_c^\infty(M)$ if the Schrödinger operator $-\Delta_\Sigma + Y$ is essentially self-adjoint $C_c^\infty(\Sigma)$.

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$-\Delta_\Sigma + Y$ is self-adjoint if (Σ, g_Σ) is a **complete Riemannian manifold** or, equivalently, if (M, g) is **globally hyperbolic**.

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The **resolvent limits exist**: if $Y > 0$ and $s > \frac{1}{2}$

$$\lim_{\varepsilon \to 0^+} \langle t \rangle^{-s} (K \mp \mathrm{i}\varepsilon)^{-1} \langle t \rangle^{-s} \in B\big(L^2(M)\big).$$

Generalizations

Question

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Theorem [Vasy, 1712.09650]

Suppose that (M, g) is a 'generalized' asymptotically Minkowski spacetime. Then $\Box_g$ is essentially self-adjoint on $C_c^\infty(M^\circ)$ and

$$\operatorname*{w-lim}_{\varepsilon \to 0^+} (\Box_g - \lambda \mp i\varepsilon)^{-1}, \quad \lambda \in \mathbf{R} \setminus \{0\},$$

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Can this be extended to (non-trapping) **asymptotically static spacetimes**?

Relation to in-out propagator

(Partially) open question

How is the in-out Feynman propagator related to the self-adjointness of the Klein–Gordon operator (and the resolvent limit)?

There are some results by Rumpf ('78, '80) in this direction.
Results by Vasy can also be interpreted in this way.

Relation to Fredholm properties

There are a number of recent results on **Fredholm properties** of the d'Alembert and Klein–Gordon operator:

1. Gell-Redman, Haber and Vasy '14 – The Feynman Propagator on Perturbations of Minkowski Space
2. Vasy '14 – On the Positivity of Propagator Differences
3. Bär, Strohmaier '15 – An Index Theorem for Lorentzian Manifolds with Compact Spacelike Cauchy Boundary [For the Dirac operator]
4. Gérard and Wrochna '16 – The Massive Feynman Propagator on Asymptotically Minkowski Spacetimes
5. Vasy '17 – Essential self-adjointness of the wave operator and the limiting absorption principle on Lorentzian scattering spaces
6. Gérard and Wrochna '18 – The Massive Feynman Propagator on Asymptotically Minkowski Spacetimes II

These works study the (Fredholm) invertiblity directly **without a limiting procedure**.

Summary

- Many spacetimes have distinguished Feynman inverses
- The Klein–Gordon operator is self-adjoint for many spacetimes
- Self-adjointness is closely related to Feynman inverses