Bracket width of simple Lie algebras

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Definition of prime numbers

Let $A = \mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, \dots\}$. Equip A with usual multiplication. Then $a \in A$ is prime if the equation

$$xy = a$$

has no solutions $(x, y) \in A \times A$.

Illustration for young children

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Illustration for young children

Erase the first row and the first column. The numbers that do not appear any more are the prime numbers.

Definition of prime elements in general algebras

Let A be an algebra equipped with a binary operation. Then we say that $a \in A$ is prime if the equation

$$xy = a$$

has no solutions $(x, y) \in A \times A$.

Example: group commutators

Let G be a group, and let A be the underlying set of G with operation $[x, y] := xyx^{-1}y^{-1}$.

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For some perfect groups G the question is also answered in the affirmative but not in an obvious way.
The smallest perfect group containing a prime element is of order 960 (see Malle's Bourbaki 2013 talk).

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The cases of finite and infinite groups should be considered separately.
In the case where G is finite, each element is a single commutator (i.e., the algebra A has no prime elements). This was conjectured by Ore in 1950's. The proof required lots of various techniques. Most groups of Lie type were treated by Ellers and Gordeev in 1990's. The proof was finished by Liebeck, O'Brien, Shalev and Tiep in 2010. See Malle's Bourbaki talk for details.

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There are several cases where each element of G is a single commutator:

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- G is the automorphism group of some nice topological or combinatorial object (e.g., the Cantor set).

So in all these cases A has no prime elements, as in the finite case.

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These groups are indeed very different from "nice" groups discussed above in the following sense.

Commutator width

For any group G one can introduce the following notions.
For any $a \in G$ define its length $\ell(a)$ as the smallest number k of commutators needed to represent it as a product

$$a = [x_1, y_1] \dots [x_k, y_k].$$

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It turns out that for a simple group G the commutator width $\mathrm{wd}(G)$ may be as large as we wish, or even infinite (such examples appear in the papers of Barge–Ghys and Muranov).

Prime elements in Lie algebras

Let now L be a Lie algebra defined over a field k. As above, we say that $a \in L$ is prime if it cannot be represented as a single Lie bracket.

As for groups, if L is not perfect, it contains prime elements (those lying outside the derived subalgebra $[L, L]$).

Main questions

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then define the bracket width of L as

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If Question (i) is answered in the affirmative, one can ask the next question:
(ii) Does there exist a simple Lie algebra L of infinite bracket width?

Where to look for counter-examples?

Throughout below L is a *simple* Lie algebra over a field k.
First suppose that L is *finite-dimensional*.
In the following cases it is known that L does not contain prime elements (i.e., every element is a single bracket, i.e., $\mathrm{wd}(L) = 1$):

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- $k = \mathbb{R}$, L is compact (Djokovic–Tam (2003), Neeb (2007), Akhiezer (2015), D'Andrea–Maffei (2016), Malkoun–Nahlus (2017));

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- $k = \mathbb{R}$, L is compact (Djokovic–Tam (2003), Neeb (2007), Akhiezer (2015), D'Andrea–Maffei (2016), Malkoun–Nahlus (2017));
- some non-compact algebras L over $\mathbb{R}$ (Akhiezer).

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Working hypothesis. All these algebras are of bracket width 1.

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Suppose now that L is *infinite-dimensional*.
There are several natural families of simple infinite-dimensional Lie algebras. Here are some of them:

- four families W_n, H_n, S_n, K_n of algebras of Cartan type;
- (subquotients of) Kac–Moody algebras;
- algebras of vector fields on smooth affine varieties.

Where to look for counter-examples?

Observation (due to Zhihua Chang):
A theorem of Rudakov (1969) shows that $\mathrm{wd}(L) = 1$ for all algebras L of Cartan type.

Back to the origins

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Sophus Lie
(1842–1899)

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Élie Cartan
(1869–1951)

Main result

Among Lie algebras of vector fields on smooth affine varieties there are algebras L with $\mathrm{wd}(L) > 1$
(B.K. and Andriy Regeta, work in progress).

Some preliminaries

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$$[\xi, \eta] := \xi \circ \eta - \eta \circ \xi.$$

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- two normal affine varieties are isomorphic if and only if $\mathrm{Vec}(X)$ and $\mathrm{Vec}(Y)$ are isomorphic as Lie algebras (Janusz Grabowski (1978) for smooth varieties, Thomas Siebert (1996) in general);
- X is smooth if and only if $\mathrm{Vec}(X)$ is simple (David Alan Jordan (1986), Siebert (1996); see also Kraft's notes (2017) and a new proof due to Billig and Futorny (2017)).

Some preliminaries

There is also a structure of an $\mathcal{O}(X)$-module on $\mathrm{Vec}(X)$. For $x \in X$ we define $(f \cdot \xi)_x := f(x)\xi_x$. The two structures are related by the formula

$$[\xi, f \cdot \eta] = \xi(f) \cdot \eta + f \cdot [\xi, \eta].$$

Let $\varepsilon_x \colon \mathrm{Vec}(X) \to T_x X$ denote the evaluation map, $\xi \mapsto \xi_x$. It is a homomorphism of $\mathcal{O}(X)$-modules.

Examples

Example 1. $X = \mathbb{A}^n$.
$\mathrm{Vec}(\mathbb{A}^n)$ is a free $\mathcal{O}(\mathbb{A}^n) = k[x_1, \ldots, x_n]$-module of rank n generated by $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$.

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Example 2. $X = \{y^2 = x^3\}$, the cuspidal curve. $\mathrm{Vec}(X)$ is generated by $\xi_1 = 2\bar{x}\partial_x + 3\bar{y}\partial_y$, $\xi_2 = 2\bar{y}\partial_x + 3\bar{x}^2\partial_y$, with the relation $\bar{x}^2\xi_1 - \bar{y}\xi_2 = 0$.

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Example 3. $X = \{x^2 + y^2 + z^2 = 0\}$, a normal surface with isolated singularity. $\mathrm{Vec}(X)$ is generated by $\xi_1 = \bar{y}\partial_x - \bar{x}\partial_y$, $\xi_2 = \bar{z}\partial_x - \bar{x}\partial_z$, $\xi_3 = \bar{z}\partial_y - \bar{y}\partial_z$, with the relation $\bar{z}\xi_1 - \bar{y}\xi_2 + \bar{x}\xi_3 = 0$.

Main example (Billig–Futorny, 2017)

Let $H = \{y^2 = 2h(x)\}$ where $h(x)$ is a separable monic polynomial of odd degree $2m + 1 \geq 3$,
$A = \mathcal{O}(H) = k[x, y]/\left\langle y^2 - 2h(x)\right\rangle$. As a vector space,
$A \cong k[x] \oplus yk[x]$.
$\mathrm{Vec}(H) = \mathrm{Der}_k(A)$.
Lemma (Billig–Futorny). Vec(H) *is a free A-module of rank 1 generated by*

$$\tau = y\partial_x + h'(x)\partial_y.$$

Filtration on A

Define the degree of a monomial in $A = k[x] \oplus yk[x]$ by

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Let A_s be the space spanned by the monomials of degree $\leq s$. Then we have a filtration

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We have $A_s A_k \subset A_{s+k}$.

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For $f \in A$ define its degree as the degree of its (unique) leading monomial $\mathrm{LT}(f)$ (which is the term of highest degree in the expansion of f in the basis $\{x^k, x^k y\}$).

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For $f \in A$ define its degree as the degree of its (unique) leading monomial $\mathrm{LT}(f)$ (which is the term of highest degree in the expansion of f in the basis $\{x^k, x^k y\}$). We have $\deg(fg) = \deg(f) + \deg(g)$.

Grading on A

Let $\operatorname{gr} A = A_0 \oplus A_1/A_0 \oplus A_2/A_1 \oplus \ldots$ be the associated graded algebra.

Lemma (Billig–Futorny). *Each graded component is of dimension at most 1 and*

$$\operatorname{gr} A \cong k[x, y]/\left\langle y^2 - 2x^{2m+1} \right\rangle .$$

There is an embedding

$$\psi \colon k[x, y]/\left\langle y^2 - 2x^{2m+1} \right\rangle \xrightarrow{\sim} k[t],$$

$x \mapsto 2t^2$, $y \mapsto 2^{m+1} t^{2m+1}$*, allowing one to identify* $\operatorname{gr} A$ *with a subalgebra of* $k[t]$ *generated by* t^2 *and* t^{2m+1}*.*

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$x \mapsto 2t^2$, $y \mapsto 2^{m+1}t^{2m+1}$, *allowing one to identify* $\operatorname{gr} A$ *with a subalgebra of* $k[t]$ *generated by* t^2 *and* t^{2m+1}*.*
We have a multiplicative map $\mathrm{LT}\colon A \to \operatorname{gr} A$.

Filtration and grading on $\mathrm{Vec}(H)$

Denote $\mathrm{Vec}(H) = D$ and recall that $D = A\tau$. For any monomial $u \in A$, $u \neq 1$, we have $\tau(A) \neq 0$ and $\deg \tau(u) = \deg(u) + 2m - 1$. Hence for any nonzero $g\tau \in D$ and any nonconstant $f \in A$ we have

$$\deg(g\tau(f)) = \deg(f) + \deg(g) + 2m - 1.$$

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Define $\deg(g\tau) := \deg(g) + 2m - 1$. This allows one to define a filtration

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Lemma (Billig–Futorny). $\mathrm{gr}\, D$ *acts on* $\mathrm{gr}\, A$ *by derivations. One can identify* $\mathrm{gr}\, D$ *with a* $\mathrm{gr}\, A$*-submodule of* $\mathrm{Der}\, k[t]$ *generated by* $t^{2m}\partial_t$.

Some properties of D

Theorem. (Billig–Futorny). *Let* $0 \neq \eta \in D$. *Then*

1. $\mathit{Ker}\,\mathrm{ad}(\eta) = k\eta$.
2. $\eta \notin \mathit{Im}\,\mathrm{ad}(\eta)$.
3. D *has no semisimple elements.*
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(We say that η is semisimple if $\mathrm{ad}(\eta)$ has an eigenvector.)

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Proof. Let $\eta \in D$ be such that $\deg \eta = 2m - 1$. Suppose that there exist $\nu, \xi \in D$ such that $[\nu, \xi] = \eta$. Since $\deg[\nu, \xi] \geq \deg \nu + 2m - 1$ (Billig–Futorny), and $\deg \nu > 0$, we are done.

Another example

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Lemma. (Leuenberger–Regeta, 2017). *L is a simple Lie algebra.*

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Lemma. (Leuenberger–Regeta, 2017). *L is a simple Lie algebra.*
Theorem. *Let $\eta = p'(z)\partial_y + x\partial_z$. We have $\eta \in L$ and there are no $\xi, \nu \in L$ such that $[\xi, \nu] = \eta$.*

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Lemma. (Leuenberger–Regeta, 2017). *L is a simple Lie algebra.*
Theorem. *Let $\eta = p'(z)\partial_y + x\partial_z$. We have $\eta \in L$ and there are no $\xi, \nu \in L$ such that $[\xi, \nu] = \eta$.*
The proof is based on the same paper by Leuenberger and Regeta and uses degree arguments.

Bracket width

Question. *What is the bracket width of the algebras* $\mathrm{Vec}(H)$ *and* $LND(S)$*?*

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Remark. If L is finite-dimensional over any infinite field of characteristic different from 2 and 3, its bracket width is at most two (Bergman–Nahlus, 2011).

Further questions

- What geometric properties of X are responsible for the existence of prime elements in $\mathrm{Vec}(X)$?
- Does there exist a Lie-algebraic counterpart of the Barge–Ghys example? This requires to go over to the category of vector fields on smooth manifolds.

THANKS FOR YOUR ATTENTION!