# Asymptotic expansion of the partition function of one-matrix models

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joint work with T. Claeys (Leuven) and K. McLaughlin (Fort Collins)

based on CMP 2015

# Hermitian Matrix integrals

$$\mathbb{H}_n = \{M \in Mat(n, \mathbb{C}), \quad M = M^*\}, \qquad M_{ij} = x_{ij} + iy_{ij}$$

- Lebesgue measure: $dM = \prod_{i=1}^{n} dx_{ii} \prod_{i<j} dx_{ij} dy_{ij}$.
- Partition function

$$Z_n(\boldsymbol{t}; \epsilon) = \frac{1}{\mathrm{vol}(U_n)} \int_{\mathbb{H}_n} e^{-\frac{1}{\epsilon} Tr V_{\boldsymbol{t}}(M)} dM$$

with

$$V_{\boldsymbol{t}}(M) = \frac{1}{2} M^2 + \sum_{k=1}^{2d} t_k M^k, \qquad t_{2d} > 0,$$

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is a $\tau$-function of the Toda lattice equations.
Goal: rigorous derivation of $Z_n(\boldsymbol{t}; \epsilon)$ for $\epsilon \to 0$ and $n \to \infty$ with $n\epsilon = x > 0$ finite, in the so called multi-cut case.

# Orthogonal polynomials

$$Z_n = n! \prod_{j=0}^{n-1} \sqrt{\kappa_j}$$

with $\kappa_j$ norming constants of orthogonal polynomials $p_j(\lambda) = \kappa_j \lambda^j + \dots$

$$\int_{-\infty}^{+\infty} p_j(\lambda) p_k(\lambda) e^{-\frac{1}{\epsilon} V_t(\lambda)} d\lambda = \delta_{jk},$$

- Three terms recurrence relations: $\lambda p_0(\lambda) = \gamma_1 p_1(\lambda) + \beta_0 p_0(\lambda)$ and

$$\lambda p_j(\lambda) = \gamma_{j+1} p_{j+1}(\lambda) + \beta_j p_j(\lambda) + \gamma_j p_{j-1}(\lambda), \;\; \gamma_j = \frac{\kappa_j}{\kappa_{j+1}}.$$

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- Relevant statistical quantities are described by orthogonal polynomials like the one point function: $\rho_n(\lambda) = \dfrac{1}{n} e^{-\frac{1}{\epsilon} V(\lambda)} \sum_{j=0}^{n-1} p_j(\lambda)^2$ which is related to the distribution of eigenvalues.

# Distribution of eigenvalues

For $n \to \infty$, $n\epsilon = x$ finite, the distribution of eigenvalues $d\mu_{V_t} = \lim \rho_n(\lambda) d\lambda$. The measure $d\mu_{V_t}$ minimizes the variational problem

$$\inf_{\int d\mu = 1} \left[ \iint \log \frac{1}{|s-y|} d\mu(s) d\mu(y) + \frac{1}{x} \int V_{\boldsymbol{t}}(s) d\mu(s) \right].$$

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• For $\boldsymbol{t}$ small, one has that support of $d\mu_{V_{\boldsymbol{t}}}$ equal $[r_-, r_+]$, $r_\pm = r_\pm(\boldsymbol{t}, x)$ and the distribution of eigenvalues is given by a deformation of the Wigner semicircle law $d\mu_{V_{\boldsymbol{t}}} = h(\lambda)\sqrt{(\lambda-r_-)(r_+-\lambda)}d\lambda$.

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• For $\boldsymbol{t} > \boldsymbol{t_c}$, support of $d\mu_{V_{\boldsymbol{t}}}$ consists of more then one interval, (multi-cut case).

# Toda equations

The partition function $Z_n(\boldsymbol{t};\epsilon)$ is also a *tau*-function of the Toda lattice: the dependent variables

$$\gamma_n^2(\boldsymbol{t};\epsilon) = \frac{1}{2}\frac{Z_{n+1}(\boldsymbol{t};\epsilon)Z_{n-1}(\boldsymbol{t};\epsilon)}{Z_n(\boldsymbol{t};\epsilon)^2}$$

$$\beta_n(\boldsymbol{t};\epsilon) = -\epsilon\frac{\partial}{\partial t_1}\log\frac{Z_{n+1}(\boldsymbol{t};\epsilon)}{Z_n(\boldsymbol{t};\epsilon)}$$

solve the Toda equations. The first flow is

$$\epsilon\frac{\partial\gamma_n}{\partial t_1} = \frac{\gamma_n}{2}\left(\beta_{n-1}-\beta_n\right), \qquad \epsilon\frac{\partial\beta_n}{\partial t_1} = \gamma_n^2 - \gamma_{n+1}^2,$$

Initial data: $\beta_n(\boldsymbol{t},\epsilon)|_{\boldsymbol{t}=0} = 0$ and $\gamma_n(\boldsymbol{t},\epsilon)|_{\boldsymbol{t}=0} = \sqrt{n\epsilon}$.

# Perturbative expansion

For $\boldsymbol{t} \ll 1$ (one-cut) the partition function has the following expansion

$$\log \frac{Z_n(\boldsymbol{t};\epsilon)}{Z_n(\boldsymbol{0};\epsilon)} = \sum_{k\geq 0} \frac{1}{k!} \sum_{m\geq 0} \epsilon^m \sum_{i_1+\cdots+i_k=k+2m} t_{i_1}\dots t_{i_k} \langle \mathrm{Tr} M^{i_1} \dots \mathrm{Tr} M^{i_k} \rangle_c$$

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Introducing the 't Hooft coupling parameter $x = N\epsilon$

$$\log \frac{Z_n(\boldsymbol{t};\epsilon)}{Z_n(\boldsymbol{0};\epsilon)} = \sum_{g\geq 0} \epsilon^{2g-2} F_g(x,\boldsymbol{t})$$

with

$$F_g(x,\boldsymbol{t}) = \sum_k \sum_{i_1,\dots,i_k} a_g(i_1,\dots,i_k) t_{i_1}\dots t_{i_k} x^h$$

with $h = 2-2g-k+|i|/2$ and $i = i_1+\cdots+i_k$ and

$$a_g(i_1,\dots,i_k) = \frac{1}{k!}\#\{\text{connected oriented ribbon graph of genus } g \text{ with } k \text{ vertices of valencies } i_1,\dots,i_k\}$$

# Enumerative geometry and Random Matrices

$$\log \frac{Z_n(\boldsymbol{t};\epsilon)}{Z_n(\mathbf{0};\epsilon)} = \epsilon^{-2}[6x^3t_3^2+2x^3t_4+216x^4t_3^2t_4+18x^4t_4^2+288x^5t_4^3+45x^4t_3t_5$$
$$+2160x^5t_3t_4t_5+90x^5t_5^2+5400x^6t_4t_5^2+5x^4t_6+1080x^5t_3^2t_6$$
$$+144x^5t_4t_6+4320x^6t_4^2t_6+108000x^6t_3t_5t_6+270000x^7t_5^2t_6$$
$$300x^6t_6^2+21600x^7t_4t_6^2+36000x^8t_6^3]$$
$$+\frac{3}{2}xt_3^2+xt_4+234x^2t_3^2t_4+30x^2t_4^2+1056x^3t_4^3+60x^2t_3t_5+6480x^3t_3t_4t_5$$
$$+300x^3t_5^2+32400x^4t_4t_5^2+10x^2t_6+3330x^3t_3^2t_6+600x^3t_4t_6$$
$$31680x^4t_4^2t_6+66600x^4t_3t_5t_6+283500xt_5^2t_6+2400x^4t_6^2+270000x^5t_4t_6^2$$
$$+696000x^6t_6^3+O(\epsilon^2)$$

coeff $1/\epsilon^2$: $2x^3t_4 \leftrightarrow a_0(4)$, $18x^4t_4^2 \leftrightarrow a_0(4,4)$,
coeff $1/\epsilon^0$: $xt_4 \leftrightarrow a_1(4)$, $30x^2t_4^2 \leftrightarrow a_1(4,4)$.

# References

- Relation between random matrices expansion and enumerative geometry: E.Brézin, C.Itzykson, G.Parisi, J.B.Zuber 1978, E.Brézin, C.Itzykson, J.B.Zuber 1980.
- Existence of the expansion in even power of $\epsilon^2$: N. Ercolani, K. Mc Laughin 2003, P. Bleher, A. Its 2004.
- Explicit computation of the coefficients $a_g(k_1, \ldots, k_m)$: Harer-Zagier (1986), Morozov-Shakirov (2009), Dubrovin-Di (2016).

# Existence of the expansion in even power of $1/n$

Key ideas ($\epsilon = 1/n$):

- the one point function $\rho_n(\lambda)$ has an asymptotic expansion in even powers of $1/n$:

$$\int_{-\infty}^{\infty} f(\lambda)\rho_n(\lambda)d\lambda = f_0 + \frac{f_1}{n^2} + \frac{f_2}{n^4} + \dots$$

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- $\frac{\partial}{\partial t_k} \log Z_n = -n^2 \mathbb{E}\left(\frac{1}{n}\mathrm{Tr} M^k\right) = -n^2 \int \lambda^k \rho_n(\lambda) d\lambda =$
$n^2 e_0^{(k)}(\boldsymbol{t}) + e_1^{(k)}(\boldsymbol{t}) + \frac{1}{n^2} e_2^{(k)}(\boldsymbol{t}) + \dots .$

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$n^2 e_0^{(k)}(\boldsymbol{t}) + e_1^{(k)}(\boldsymbol{t}) + \frac{1}{n^2} e_2^{(k)}(\boldsymbol{t}) + \dots.$
- The integration with respect to $t_k$ is performed term by term in the above expansion taking as a reference point the GUE, and using the result that the space of 1-cut potentials in the class of polynomials weights, is path connected.

# Computation of the coefficients $a_g(k_1, \ldots, k_m)$

Take $\epsilon = 1$ and consider

$$\langle \mathrm{Tr} M^{i_1} \ldots \mathrm{Tr} M^{i_k} \rangle_c = k! \sum_{0 \le g \le \frac{1}{2}(\frac{|i|}{2} - k + 1)} a_g(i_1, \ldots, i_k) N^{2-2g-k+\frac{|i|}{2}}$$

and define

$$C_k(n, \lambda_1, \ldots, \lambda_k) = \sum_{i_1, \ldots, i_k = 1}^{\infty} \frac{\langle \mathrm{Tr} M^{i_1} \ldots \mathrm{Tr} M^{i_k} \rangle_c}{\lambda^{i_1+2} \ldots \lambda^{i_k+1}}$$

- $C_1(n, \lambda_1)$ was obtained by Harer-Zagier (1986),
- $C_2(n, \lambda_1, \lambda_2)$ was obtained by Morozov-Shakirov (2009),
- $C_k(n, \lambda_1, \ldots, \lambda_k)$, $k \ge 1$ was obtained by Dubrovin-Di (2016).

Alternatively $C_k(n, \lambda_1, \ldots, \lambda_k)$ can be obtained using topological recursion formulas.

# Two cuts case

- The support of the equilibrium measure $d\mu_{V_t}$ consists of 2 intervals.
- The recurrence coefficients of the orthogonal polynomials $\gamma_n(\boldsymbol{t}, \epsilon)$ and $\beta_n(\boldsymbol{t}, \epsilon)$ are highly oscillatory and described by Jacobi $\theta$-functions as $n \to \infty$ (Deift Kriecherbauer McLaughlin Venakides Zhou, 1999).

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- The partition function has an oscillatory behaviour described by (Bonnet-David-Eynard 2000, Eynard 2012, Scherbina, Guionnet-Borot 2013)

$$Z_n(\boldsymbol{t}, \epsilon) \propto \theta(nc^*; \tau) e^{-n^2 F_0 - F_1 + \cdots}, \quad \epsilon = \frac{1}{n}$$

where $c^*$ is the fraction of eigenvalues in one of the two intervals and $\theta$ is the Jacobi $\theta$-function.

# Derivation by Bonnet-David-Eynard

Suppose that the potential is 2-cut with minimum values $E_1 < E_2$. Then the eigenvalues of the matrix model are distributed asymptotically in two intervals.

$$Z_n \propto \sum_{j=0}^{n} Z_{n,c_j}, \quad c_j = \frac{j}{n}$$

where $Z_{n,c_j}$ is the matrix model obtained by forcing to have $j$ eigenvalues in $(-\infty, E_0)$ and $n-j$ in $(E_0, +\infty)$, with $E_1 < E_0 < E_2$. Then

$$-\log Z_{n,c_j} = n^2 F_0(c_j) + F_1(c_j) + \frac{1}{n^2} F_2(c_j) + O(n^{-4})$$

Performing a Taylor expansion of $F_0(c_j)$ near the stationary point $c^*$

$$Z_n \propto \sum_{j=0}^{n} Z_{n,c_j} \simeq e^{-n^2 F_0(c^*) - F_1(c^*)} \sum_{j=0}^{n} e^{F_0''(c^*)(j - c^* n)^2} + \dots$$

$$\simeq e^{-n^2 F_0(c^*) - F_1(c^*)} \theta(nc^*; \tau) + \dots, \quad \tau = \frac{2\pi i}{F_0''(c^*)}$$

# Loop equations and determination of $F_j$

- 1-point resolvent

$$W_1(z) := \frac{d}{dV(z)} \frac{1}{N^2} \log Z_N, \quad \frac{d}{dV(z)} = -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \frac{d}{dt_j}.$$

Assuming $W_1(z)$ has a $1/n^2$ expansion $W_1(z) = \sum_{k=0}^{\infty} = \frac{W_1^{(k)}(z)}{n^{2k}}$, then

$$W_1^{(k)}(z) := \frac{d}{dV(z)} F_k.$$

$W_1^{(k)}$ are obtained by solving the loop equation

$$\oint_{\mathcal{C}} \frac{V_{\boldsymbol{t}}'(x) W_1(x)}{z-x} dx = W_1(z)^2 + \frac{1}{N^2} W_2(z,z)$$

iteratively using the topological recursion (Chekhov-Eynard-Orantin).

# Statement of the result

For a polynomial potential $V_{\boldsymbol{t}}(\lambda)$ for which the distribution of eigenvalues is given by the regular measure
$d\mu(\lambda) = h(\lambda)\sqrt{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - a_4)}d\lambda$, with
$\lambda \in [a_1, a_2] \cup [a_3, a_4]$, the partition function has the following expansion

$$\log Z_n(\boldsymbol{t}) = \log C_n - n^2 F_0(\boldsymbol{t}) - F_1(\boldsymbol{t}) + \log\theta(nc^*; \tau(\boldsymbol{t})) + O\left(\frac{1}{n}\right)$$

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where

$$C_n = \frac{n!}{\lfloor \frac{n}{2}\rfloor! \lfloor \frac{n+1}{2}\rfloor!} Z^{GUE}_{\lfloor \frac{n}{2}\rfloor,\sigma^*} Z^{GUE}_{\lfloor \frac{n+1}{2}\rfloor,\sigma^*}, \quad \sigma^* = 4e^{3/2}$$

$$Z^{GUE}_{n,\sigma} = (2\pi)^{n/2}\left(\frac{\sigma}{4n}\right)^{n^2/2}\prod_{j=1}^{n} j!$$

$$F_0(\boldsymbol{t}) = \iint \log\frac{1}{|z-y|}\,d\mu(z)d\mu(y) + \int V_{\boldsymbol{t}}(z)d\mu(z)$$

$$\log Z_n(\boldsymbol{t}) = \log C_n - n^2 F_0(\boldsymbol{t}) - F_1(\boldsymbol{t}) + \log \theta(nc^*(\boldsymbol{t}); \tau(\boldsymbol{t})) + O\left(\frac{1}{n}\right)$$

$$F_1(\boldsymbol{t}) = \frac{1}{24} \log[\mathcal{A}^{12} \prod_{j<k} (a_k - a_j)^4 \prod_{j=1}^{4} h(a_j)]$$ G. Akemann 1996

with $\theta(z;\tau) = \sum_n \in \mathbb{Z} e^{\pi i n^2 \tau + 2\pi i z n}$, $c^*$ the fraction of eigenvalues in $[a_3, a_4]$ and $\mathcal{A}$ the period of the non normalised holomorphic one-form of the elliptic curve $y^2 = \prod_{j=1}^{4} (\lambda - a_j)$ and $\tau$ the elliptic modulus of the curve.

- $F_1$ can also be expressed via the Dedekind $\eta$ function.
- $F_1$ was calculated for hyperelliptic curves by Chekhov and in 2-matrix models by Eynard, Kokotov and Korotkin.

# Strategy to obtain the large $n$ expansion of $\log Z_n$ in the two-interval case.

- We derive an asymptotic expansion in $n$ for the derivatives

$$\frac{\partial}{\partial t_k} \log Z_n(\boldsymbol{t}) = n^2 g_0(\boldsymbol{t}) + n g_1(\boldsymbol{t}, n) + g_2(\boldsymbol{t}, n) + O(1/n), \qquad (1)$$

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- we show that in the space of times $\boldsymbol{t}$ the set $\mathcal{S}$ of points for which the support of eigenvalues consists of two intervals is connected;
- it is possible to integrate term by term the equation (1) from the reference time $\boldsymbol{t}^*$ to any other time $\boldsymbol{t}$ in the set $\mathcal{S}$.

# Reference potential

We consider the potential $V_{r,s}(\lambda) = \frac{1}{s}(\lambda^4 - r\lambda^2)$, with $r > \sqrt{2}s$.
Eigenvalues are distributed on two intervals

$$[-\sqrt{b}, -\sqrt{a}] \cup [\sqrt{a}, \sqrt{b}], \quad a = (r - 2\sqrt{s})/2, \;\; b = (r + 2\sqrt{s})/2.$$

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The corresponding partition function $Z_n(r,s)$ is given by

$$\log Z_{2n}(r,s) = \log(2n)! + \log \hat{Z}_n(-1/2, r, s) + \log \hat{Z}_n(1/2, r, s)$$

$$\log Z_{2n+1}(r,s) = \log(2n+1)! + \log \hat{Z}_n(-1/2, r, s_+) + \log \hat{Z}_n(1/2, r, s_-)$$

with $s_\pm = s(1 \pm 1/(2n+1))$ and

$$\hat{Z}_n(\alpha, r, s) = \frac{1}{n!} \int_{\mathbb{R}_+^n} \prod_{j<i} (\lambda_j - \lambda_i)^2 \prod_{j=1}^{n} \lambda_j^{\alpha} e^{\frac{2n}{s}(\lambda_j^2 - r\lambda_j)} d\lambda_j$$

**Theorem.** The partition function $Z_n(r,s)$ associated to the potential $V_{r,s}(\lambda) = \frac{1}{s}(\lambda^4 - r\lambda^2)$ has an asymptotic expansion

$$\log Z_n(r,s) = \log C_n - n^2 F_0(r,s) - F_1(r,s) + \log\theta(n/2;\tau(r,s)) + O\left(\frac{1}{n}\right)$$

where

$$C_n = \frac{n!}{\lfloor \frac{n}{2} \rfloor! \lfloor \frac{n+1}{2} \rfloor!} Z^{GUE}_{\lfloor \frac{n}{2} \rfloor, \sigma^*} Z^{GUE}_{\lfloor \frac{n+1}{2} \rfloor, \sigma^*}, \quad \sigma^* = 4e^{3/2}$$

As a function of $n$, the oscillatory term assumes only two values.

# Determination of $\frac{\partial}{\partial t_k} Z_n$ as $n \to \infty$ for general potential

The following relation is satisfied (Jimbo-Miwa, Bertola)

$$\frac{\partial}{\partial t_k} \log Z_n(\boldsymbol{t}) = -\frac{n}{2} \operatorname*{Res}_{\lambda=\infty} (\mathrm{Tr}(X_n^{-1}(\lambda) X_n'(\lambda) \sigma_3 \lambda^k d\lambda),$$

where $X_n(\lambda)$ is a $2 \times 2$ matrix (A. Fokas, A. Its, A. Kitaev)

$$X_n(\lambda) = \begin{pmatrix} \gamma_n^{-1} p_n(\lambda) & \frac{\gamma_n^{-1}}{2\pi i} \int_{\mathbb{R}} p_n(s) \frac{e^{-nV_{\boldsymbol{t}}(s)} ds}{s-\lambda} \\ -2\pi i \gamma_{n-1} p_{n-1}(\lambda) & -\gamma_{n-1} \int_{\mathbb{R}} p_{n-1}(s) \frac{e^{-nV_{\boldsymbol{t}}(s)} ds}{s-\lambda} \end{pmatrix}$$

with

$$\int_{-\infty}^{+\infty} p_j(\lambda) p_m(\lambda) e^{-nV_{\boldsymbol{t}}(\lambda)} d\lambda = \delta_{jm},$$

and $\gamma_n$'s are recurrence coefficients for the orthogonal polynomials.

Remark.

• the leading term of the asymptotic expansion of $X_n(\lambda)$ as $n \to \infty$ was a obtained by Deift et all (1999) where they also show that $X_n(\lambda)$ has an asymptotic expansion in the form

$$X_n(\lambda) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k(\lambda, n)}{n^k},$$

where the matrix $\mathcal{P}_k(\lambda, n)$ is uniformly bounded in $n$. For our purpose we obtain the first subleading term.

• The non trivial part of our analysis is to identify the terms of the asymptotic expansion as $n \to \infty$ of r.h.s. of

$$-\frac{\partial}{\partial t_k} \log Z_n = \frac{n}{2} \operatorname*{Res}_{\lambda=\infty} (\mathrm{Tr}(X_n^{-1}(\lambda) X_n'(\lambda) \sigma_3 \lambda^k d\lambda)$$

as an anti-derivative with respect to the times $t_k$.

# Explicit computation

Let us introduce the Szegö kernel

$$S\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(z_0,z_1)=\frac{\theta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}\left(\int_{z_1}^{z_0}du;\tau\right)}{E(z_0,z_1)\theta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(0;\tau)}$$

and the 1-form

$$\Phi_{q_0,p_0}(z):=-S[{}^{0}_{nc^*}](z,p_0)S[{}^{0}_{nc^*}](q_0,z).$$

Notice that

$$\Phi_{q,q}(z):=-B(z,q)-(\log\theta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(0;\tau))''du(z)du(q),$$

where $B(z,q)$ is the so called canonical symmetric bi-differential or Bergman kernel.

$$\Phi_{q_0,p_0}(z) := -S[{}^{0}_{nc^*}](z,p_0)S[{}^{0}_{nc^*}](q_0,z).$$

Then from steepest decent analysis of the Riemann-Hilbert problem for $X_n(\lambda)$

$$-\frac{d}{dV(z)}\log Z_n = n^2\frac{dF_0}{dV(z)} + n(\log\vartheta(nc^*,\tau))'du(z)/dz$$

$$+\frac{1}{8}\sum_{j=1}^{4}\operatorname*{Res}_{\lambda=a_j}\frac{\Phi_{\bar\lambda,\lambda}(z)-\Phi_{\lambda,\bar\lambda}(\bar z)}{\frac{dz}{3}\int_{\bar\lambda}^{\lambda}d\mu(\xi)} + \frac{1}{48}\sum_{j=1}^{4}\operatorname*{Res}_{\lambda=a_j}\frac{\Phi_{\lambda,\lambda}(z)-\Phi_{\bar\lambda,\bar\lambda}(z)}{\frac{dz}{3}\int_{\bar\lambda}^{\lambda}d\mu(\xi)} + O(1/n)$$

$$=\frac{d}{dV(z)}\left(n^2F_0+F_1-\log\theta\left(nc^*;\tau\right)-\frac{\theta'(nc^*;\tau)}{\theta(nc^*;\tau)}\frac{F_1^{(1)}}{n}-\frac{\theta'''(nc^*;\tau))}{\theta(nc^*;\tau)}\frac{F_0^{(3)}}{6n}\right)$$
$$+O(1/n)$$

# Solving loop equations?

Assume that $W_1(z) = \sum_{k=0}^{\infty} \frac{\widetilde{W}_1^{(k)}(z)}{n^k}$ and $W_1(z,x) = \sum_{k=0}^{\infty} \frac{\widetilde{W}_2^{(k)}(z,x)}{n^k}$ and define $\mathcal{K}f(z) := \oint_{\mathcal{C}} \frac{V_t'(x)f(x)}{z-x}dx$. Then the loop equations give

$$[\mathcal{K} - 2\widetilde{W}_1^{(0)}(z)]\widetilde{W}_1^{(0)}(z) = 0, \quad [\mathcal{K} - 2\widetilde{W}_1^{(0)}(z)]\widetilde{W}_1^{(1)}(z) = 0,$$

$$[\mathcal{K} - 2\widetilde{W}_1^{(0)}(z)]\widetilde{W}_1^{(2)}(z) = (\widetilde{W}_1^{(1)}(z))^2 + W_2^{(0)}(z,z)$$

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- $\widetilde{W}_1^{(0)}(z) \to$ planar limit
- $\widetilde{W}_1^{(1)}(z)dz = -(\log\theta \begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(0;\tau))'du(z)$,
- $\widetilde{W}_2^{(0)}(z,z)dz^2 = \Phi_{\bar{z},\bar{z}}(z) = -B(z,\bar{z}) - (\log\theta \begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(0;\tau))''du(z)du(\bar{z})$,

$$\widetilde{W}_1^{(2)}(z)dz = \sum_{j=1}^{4} \operatorname*{Res}_{\lambda=a_j}\left[\frac{\Phi_{\bar{\lambda},\lambda}(z) - \Phi_{\lambda,\bar{\lambda}}(\bar{z})}{\frac{8}{3}\int_{\bar{\lambda}}^{\lambda} d\mu(\xi)d\xi} + \frac{\Phi_{\lambda,\lambda}(z) - \Phi_{\bar{\lambda},\bar{\lambda}}(z)}{16\int_{\bar{\lambda}}^{\lambda} d\mu(\xi)}\right]$$

# Integration

Theorem. The space of one-cut regular potential in the parameter space $\boldsymbol{t} \in \mathbb{R}^{2d}$ is connected.

• The expansion of the partition function is obtained by integration in the space of times from the point $(0, t_2, 0, t_4, 0, \ldots, 0)$ to any point $\boldsymbol{t}$ corresponding to a one-cut regular potential.

# Conclusion

We derive the asymptotic expansion of the partition function of Hermitian matrix integral in the two cut case as

$$\log Z_n(\boldsymbol{t}) = \log C_n - n^2 F_0(\boldsymbol{t}) - F_1(\boldsymbol{t}) + \log \theta(nc^*(\boldsymbol{t}); \tau(\boldsymbol{t})) + O\left(\frac{1}{n}\right)$$

where

$$C_n = \frac{n!}{\lfloor \frac{n}{2} \rfloor ! \lfloor \frac{n+1}{2} \rfloor !} Z^{GUE}_{\lfloor \frac{n}{2} \rfloor, \sigma^*} Z^{GUE}_{\lfloor \frac{n+1}{2} \rfloor, \sigma^*}, \quad \sigma^* = 4e^{3/2}$$

• Our main contribution is the derivation of the constant $C_n$. For the remaining terms of the expansion, our analysis confirms earlier results by Bonnet-David-Eynard, Eynard.

• Open problem: obtain the expansion by solving the loop equations.