Factorization and Asymptotics of Block Toeplitz Matrices

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We begin with a matrix-valued function ϕ defined on the unit circle $\mathbb{T}$ with Fourier coefficients

$$\phi_k = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})\, e^{-ik\theta}\; d\theta,$$

$$\phi(e^{i\theta}) = \sum_{-\infty}^{\infty} \phi_k\, e^{i\,k\,\theta} = \sum_{-\infty}^{\infty} \phi_k\, z^k.$$

and consider the matrix

$$T_n(\phi) \;=\; (\phi_{j-k})_{j,\,k\,=\,0,\,\cdots\,,\,n-1}$$

We refer to ϕ as the symbol of the matrix.

This matrix has the form

$$\begin{bmatrix} \phi_0 & \phi_{-1} & \phi_{-2} & \cdots & \phi_{-(n-1)} \\ \phi_1 & \phi_0 & \phi_{-1} & \cdots & \phi_{-(n-2)} \\ \phi_2 & \phi_1 & \phi_0 & \cdots & \phi_{-(n-3)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \phi_{n-1} & \phi_{n-2} & \phi_{n-3} & \cdots & \phi_0 \end{bmatrix}$$

In the matrix-valued case, each entry is itself a matrix of fixed size.

The Szegö - Widom Limit Theorem states that if the matrix valued symbol ϕ defined on the unit circle $\mathbb{T}$ has a sufficiently well-behaved logarithm then the determinant of the block Toeplitz matrix

$$T_n(\phi) = (\phi_{j-k})_{\,j,k=0,\cdots,n-1}$$

has the asymptotic behavior

$$D_n(\phi) = \det T_n(\phi) \;\sim G(\phi)^n\, E(\phi) \quad \text{as } n \to \infty.$$

Here are the constants:

$$G(\phi) = e^{(\log \det \phi)_0}$$

and

$$E(\phi) = \det\left(T(\phi)T(\phi^{-1})\right)$$

where

$$T(\phi) = (\phi_{j-k}) \qquad 0 \le j,k < \infty$$

is the Toeplitz operator defined on H^2 (the Hardy space) of the circle.

To make sense of the term $\det\big(T(\phi)T(\phi^{-1})\big)$ we should note that we can always define the determinant of an operator of the form

$$I + T$$

where T is a trace class operator.

Such operators T are compact with discrete eigenvalues λ_i that satisfy

$$\sum_{i=0}^{\infty} |\lambda_i| < \infty$$

and thus

$$\det(I + T) = \prod_{i=0}^{\infty} (1 + \lambda_i)$$

is well defined.

More precisely, let $\mathcal{B}$ stand for the set of all function ϕ such that the Fourier coefficients satisfy

$$\|\phi\|_{\mathcal{B}} := \sum_{k=-\infty}^{\infty} |\phi_k| + \Big(\sum_{k=-\infty}^{\infty} |k| \cdot |\phi_k|^2 \Big)^{1/2} < \infty.$$

With the norm, and pointwise defined algebraic operations on $\mathbb{T}$, the set $\mathcal{B}$ becomes a Banach algebra of continuous functions on the unit circle.

The Szegö - Widom Limit Theorem holds providing $\phi \in \mathcal{B}$ and the function $\det \phi$ does not vanish on $\mathbb{T}$ and has winding number zero.

The most direct way to prove the Szegö-Widom theorem is to prove an identity for the determinants, an identity called the Borodin-Okounkov-Case-Geronimo (BOCG) identity.

To state the identity, in addition to the Toeplitz operator, we also define a Hankel operator

$$\begin{aligned} T(\phi) &= (\phi_{j-k}), \qquad 0 \leq j,k < \infty, \\ H(\phi) &= (\phi_{j+k+1}), \quad 0 \leq j,k < \infty. \end{aligned}$$

For $\phi, \psi \in L^\infty(\mathbb{T})^{N\times N}$ the identities

$$\begin{aligned} T(\phi\psi) &= T(\phi)T(\psi) + H(\phi)H(\tilde{\psi}) \\ H(\phi\psi) &= T(\phi)H(\psi) + H(\phi)T(\tilde{\psi}) \end{aligned}$$

are well-known. Here $\tilde{\phi}(e^{i\theta}) = \phi(e^{-i\theta})$.

It follows from these identities that if ψ_- and ψ_+ have the property that all their Fourier coefficients vanish for $k > 0$ and $k < 0$, respectively, then

$$T(\psi_-\phi\psi_+) = T(\psi_-)T(\phi)T(\psi_+),$$

$$H(\psi_-\phi\tilde{\psi}_+) = T(\psi_-)H(\phi)T(\psi_+).$$

Here is one form of the statement of the BOCG identity.

If the conditions of the theorem hold and in addition,

$$\phi = u_- u_+ = v_+ v_-$$

(with invertible factors) then

$$\det T_n(\phi) \quad = \quad G(\phi)^n \; E(\phi) \cdot \det\left(I - H(z^{-n} v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+ z^{-n})\right).$$

From this BOCG identity we have an instant proof of the Szegö-Widom theorem, since we can show that given our conditions on ϕ, the operator

$$H(z^{-n}v_-u_+^{-1})H(\tilde{u}_-^{-1}\tilde{v}_+z^{-n}))$$

tends to zero in the trace norm and thus

$$\det(I - H(z^{-n}v_-u_+^{-1})H(\tilde{u}_-^{-1}\tilde{v}_+z^{-n}))$$

tends to one and

$$D_n(\phi) = \det T_n(\phi) \ \sim G(\phi)^n \, E(\phi) \quad \text{as } n \to \infty.$$

In the scalar case, $E(\phi)$ has a nice concrete description.

If we have a Wiener-Hopf factorization for $\phi = \phi_- \phi_+$, then

$$T(\phi)T(\phi^{-1}) = T(\phi_-)T(\phi_+)T^{-1}(\phi_-)T^{-1}(\phi_+)$$

and this is of the form

$$e^A e^B e^{-A} e^{-B}$$

where

$$A = T(\log(\phi_-)), \quad B = T(\log(\phi_+)).$$

From this we can use a formula for determinants of multiplicative commutators of this form,

$$\det\left(e^{A}e^{B}e^{-A}e^{-B}\right) = \exp\left(\operatorname{trace}(AB - BA)\right)$$

and this then becomes the well-known formula

$$\exp\left(\sum_{k=1}^{\infty} k\,(\log\phi)_k\,(\log\phi)_{-k}\right).$$

This does not hold in general in the block case.

A much harder question is how do you compute $E(\phi)$ in the block case?

There is one particular result, also due to Widom, where something can be said about the infinite determinant.

Let $\phi \in \mathcal{B}$ be such that the function $\det \phi$ does not vanish on the unit circle and has winding number zero.

Assume that $\phi_k = 0$ for all $k > m$ or that $\phi_{-k} = 0$ for all $k > m$.

Then

$$E(\phi) = G(\phi)^m \det T_m(\phi^{-1}).$$

The result also follows from a different form of the BOCG identity:

If $\phi \in \mathcal{B}$ then the BOCG identity can be rewritten in the following form.

$$\det T_n(\phi^{-1}) = \frac{E(\phi)}{G(\phi)^n} \cdot \det\left(I - H(z^{-n}\phi)T^{-1}(\tilde{\phi})H(\tilde{\phi}z^{-n})T^{-1}(\phi)\right).$$

The conditions of Widom guarantee that one of the Hankel operators vanishes and thus

$$\det T_m(\phi^{-1})G(\phi)^m = E(\phi).$$

Here is an example:

Let

$$\phi_{\alpha,2}(z) = a(\alpha)\begin{pmatrix} 1 & \alpha\, z^{-2} \\ -\bar{\alpha}\, z^{2} & 1 \end{pmatrix}$$

where $z = e^{i\theta}$, $a(\alpha) = (1 + |\alpha|^2)^{-1/2}$.

Note $\phi^{-1} = \phi^{*}$, $\det\phi = 1$, and thus $\phi \in SU(2)$.

$$E(\phi) = G(\phi)^2 \det T_2(\phi^{-1}) = (1 + |\alpha|^2)^{-2}.$$

To summarize, we know how to compute $G(\phi)$ and we know how to compute $E(\phi)$ in two cases:

1. For scalar ϕ's
2. For matrix valued ϕ's that satisfy Widom's criteria.

Is there a way to put these cases together to compute more complicated examples, especially the ones that seem to arise in statistical mechanics?

First, two basic properties of $E(\phi)$:

$$E(\phi) = E(\phi^{-1})$$

and

$$E(\phi\,\psi) \;=\; E(\phi)E(\psi) \times M(\phi, \psi)$$

where

$$M(\phi, \psi) = \det T(\phi)^{-1} T(\phi\psi) T(\psi)^{-1} \det T(\widetilde{\phi})^{-1} T(\widetilde{\phi}\widetilde{\psi}) T(\widetilde{\psi})^{-1}.$$

This follows (as almost everything does) from the identity

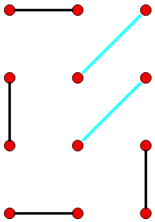
$$T(\phi\,\psi) = T(\phi)T(\psi) + H(\phi)H(\widetilde{\psi}).$$

So whenever we can compute a determinant of the form

$$\det T(\phi)^{-1}T(\phi\psi)T(\psi)^{-1}$$

explicitly we can then build answers from known ones.

An example where this idea proved to be useful is an application to a dimer model and here is a picture to illustrate.



A monomer placed on a lattice site forbids a dimer from being placed at the site.

The monomer-monomer correlator is the ratio of the number of configurations with monomers at sites q and r to the number of configurations without the monomers.

If we assume that one of the sites is at the origin and the other at site in an adjacent row n spacings apart, then it was shown by Fendley, Moessner and Sondi that the correlator can be computed from the determinant of a block Toeplitz matrix.

The symbol of interest was of the form

$$\begin{pmatrix} c & d \\ \\ \tilde{d} & \tilde{c} \end{pmatrix}$$

where

$$c = \frac{(t\cos\theta + \sin^2\theta)(t - e^{i\theta})}{\sqrt{t^2 + \sin^2\theta + \sin^4\theta}(1 - 2t\cos\theta + t^2)}$$

$$d = \frac{\sin\theta}{\sqrt{t^2 + \sin^2\theta + \sin^4\theta}}.$$

Using the ideas just outlined, one can then compute that

$$G(\phi) = 1$$

and that

$$D_n(\phi) \sim E(\phi) = \frac{t}{2t(2+t^2) + (1+2t^2)\sqrt{2+t^2}}.$$

Here t is the weight on the diagonal bonds.

But can we can say more?

We return to our BOCG identity,

$$\det T_n(\phi) = E(\phi) \cdot \det\left(I - H(z^{-n} v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+ z^{-n})\right).$$

As a guess, we hope that

$$\det(I+T) = \exp \operatorname{trace}(\log(I+T)) = \exp(\operatorname{trace}(T+\cdots) = 1+\operatorname{trace} T+\cdots$$

We can prove this here and asymptotically compute the trace of the product of Hankels.

The result is (joint work with Ehrhardt and Bleher), for $0 < t < 1/2$

$$D_n(\phi) = E(\phi)\left[1 - \frac{e^{-n/\xi}}{n}\left(C_1 + C_2(-1)^n + \mathcal{O}(n^{-1})\right)\right]$$

and for $1/2 < t < 1$

$$E(\phi)\Big[1 - \frac{e^{-n/\xi}}{n}\left(C_1 \cos(\omega n + \varphi_1) + C_2(-1)^n \cos(\omega n + \varphi_2)\right.$$

$$\left.+ C_3 + C_4(-1)^n + \mathcal{O}(n^{-1})\right)\Big]$$

where $\xi,\ C_1, C_2, C_3, C_4, \omega, \varphi_1$ and φ_2 are explicitly determined and depend on t.

The real issue is how does one compute the factors?

To see a simpler case, consider the matrix

$$\begin{pmatrix} z-2 & -z+1/z \\ -2 & 1+1/2z \end{pmatrix}$$

with determinant $(z+2)(1/2z-1)$.

The factors of the determinant are the key.

$$\begin{pmatrix} z-2 & -z+1/z \\ -2 & 1+1/2z \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z-2 & -z+1/z \\ -2 & 1+1/2z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z+2 & -(z+2) \\ -2 & 1+1/2z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z+2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 1+1/2z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z+2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-1/2z \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Multiplying these last matrices we have our factorization

$$\begin{pmatrix} z-2 & -z+1/z \\ -2 & 1+1/2z \end{pmatrix} = \begin{pmatrix} z-2 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1-1/2z \end{pmatrix}.$$

This of course will not work if the determinant is not of sufficiently high degree.

Consider something of the form in $SU(2)$

$$\phi = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}$$

where a, b are in H^∞ and in $\mathcal{B}$. Then we know

$$\phi = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$$

where the right matrix has entries in H^2 and the left in the conjugate of H^2, and we can assume each matrix has determinant 1.

This means that

$$h_3k_1 + h_4k_3 = -b, \quad h_3k_2 + h_4k_4 = a$$

and

$$h_3k_1k_4 + h_4k_3k_4 = -bk_4, \quad h_3k_2k_3 + h_4k_4k_3 = ak_3$$

Subtracting and using the fact that $k_1k_4 - k_2k_3 = 1$,

we have that

$$h_3 = -bk_4 - ak_3.$$

But this says that h_3 is in both H^2 and its conjugate and hence must be a constant. The same argument also says that h_4 is a constant.

With a little more effort one can show that the factorization is of the form (and computable)

$$\phi = \begin{pmatrix} 1 & h_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}.$$

For our previous example

$$\phi_{\alpha,2}(z) = a(\alpha)\begin{pmatrix} 1 & \alpha\, z^{-2} \\ -\bar{\alpha}\, z^{2} & 1 \end{pmatrix}$$

and this is

$$a(\alpha)\begin{pmatrix} 1 & \alpha z^{-2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+|\alpha|^2 & 0 \\ -\bar{\alpha} z^{2} & 1 \end{pmatrix}.$$

But much more can be said about symbols of the form

$$\phi = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}.$$

Not only can they be easily factored, they have an alternate useful factorization for computing determinants.

To give a hint of this, let us return to the $SU(2)$ example and make it more complicated, once again using the idea that we can build our answers from products.

Consider the product:

$$\phi_{\alpha,m}\,\phi_{\beta,n}$$

or

$$a(\alpha)\begin{pmatrix} 1 & \alpha\, z^{-m} \\ -\bar{\alpha}\, z^{m} & 1 \end{pmatrix} a(\beta)\begin{pmatrix} 1 & \beta\, z^{-n} \\ -\bar{\beta}\, z^{n} & 1 \end{pmatrix}.$$

Before using the formula

$$\det T(\phi)^{-1} T(\phi\psi) T(\psi)^{-1}$$

note that

$$T(\phi\psi) = T(\phi)T(\psi) + H(\phi)H(\widetilde{\psi})$$

and thus the above becomes

$$\det(I + T(\phi)^{-1} H(\phi)H(\widetilde{\psi}))T(\psi)^{-1})$$

or

$$\det(I + T(\psi)^{-1} T(\phi)^{-1} H(\phi)H(\widetilde{\psi})).$$

For $\phi_{\alpha,m}$ we have, except for constants,

$$H(\phi_{\alpha,m}) = H(\begin{pmatrix} 0 & 0 \\ -\bar{\alpha}\, z^m & 0 \end{pmatrix})$$

and for $\widetilde{\phi_{\beta,n}}$ this becomes

$$H(\phi_{\beta,n}) = H(\begin{pmatrix} 0 & \beta z^n \\ 0 & 0 \end{pmatrix}).$$

This produces a determinant of the form

$$I + A$$

where A is trace class and has zeros in many columns and many rows.

From this, one can show

$$\det T(\phi)^{-1} T(\phi\psi) T(\psi)^{-1} = 1$$

A similar computation shows

$$\det T(\widetilde{\phi})^{-1} T(\widetilde{\phi}\widetilde{\psi}) T(\widetilde{\psi})^{-1} = 1.$$

And thus we see that $E(\phi)$ for this product completely factors, a result not expected in scalar cases.

$$E(\phi) = E(\phi_{\alpha,m}\,\phi_{\beta,n}) = E(\phi_{\alpha,m})E(\phi_{\beta,n})$$

$$= (1 + |\alpha|^2|)^{-m}(1 + |\beta|^2)^{-n}.$$

This result can be extended to show that for any finite product

$$\phi = a(\eta_m)\begin{pmatrix} 1 & -\eta_m z^{-m} \\ -\bar{\eta_m} z^m & 1 \end{pmatrix} \cdots a(\eta_1)\begin{pmatrix} 1 & \eta_1 z^{-1} \\ -\bar{\eta_1} z^1 & 1 \end{pmatrix},$$

$$E(\phi) = \prod_{i=1}^{m}(1+|\eta_i|^2)^{-i}.$$

This holds for an infinite product as well as long as the sequence $\{\eta_i\}$ is rapidly decreasing.

Returning to

$$\phi = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}$$

where a, b are in H^∞ and in $\mathcal{B}$.

One can show, assuming some additional smoothness assumptions on a and b, is that ϕ can be factored as above, that is,

$$\phi = \lim_{n\to\infty} a(\eta_n) \begin{pmatrix} 1 & \eta_n z^{-n} \\ -\bar{\eta}_n z^n & 1 \end{pmatrix} \cdots a(\eta_1) \begin{pmatrix} 1 & \eta_1 z^{-1} \\ -\bar{\eta}_1 z^1 & 1 \end{pmatrix}.$$

and thus $E(\phi) = \prod_{i=1}^{\infty}(1 + |\eta_i|^2)^{-i}$

A similar result holds for something of the form

$$\psi = \begin{pmatrix} c & d \\ -d^* & c^* \end{pmatrix}$$

which can be factored as

$$\lim_{n\to\infty} a(\alpha_n) \begin{pmatrix} 1 & -\bar{\alpha}_n z^n \\ \alpha_n z^{-n} & 1 \end{pmatrix} \cdots a(\alpha_0) \begin{pmatrix} 1 & -\bar{\alpha}_0 \\ \alpha_0 & 1 \end{pmatrix}.$$

Finally, we consider

$$\psi^* \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix} \phi$$

where ϕ, ψ (as before) are where χ is real valued,

Note this a product where all three factors are in $SU(2)$.

It turns out that E splits into three known pieces here.

The simplest case is when $\chi = 0$. Then it is clear that

$$E(\psi^* \phi) = E(\psi^*)\, E(\phi).$$

This follows from the fact that $H(\psi^*)$ is

$$H(\begin{pmatrix} c & d \\ -d^* & c^* \end{pmatrix}^*) = H(\begin{pmatrix} c^* & -d \\ d* & c \end{pmatrix}) = H(\begin{pmatrix} 0 & -d \\ 0 & c \end{pmatrix}).$$

For $H(\widetilde{\phi})$ we have

$$H(\begin{pmatrix} \widetilde{a^*} & \widetilde{b^*} \\ \widetilde{-b} & \widetilde{a} \end{pmatrix}) = H(\begin{pmatrix} \widetilde{a^*} & -\widetilde{b^*} \\ 0 & 0 \end{pmatrix})$$

so that

$$H(\psi^*)H(\widetilde{\phi}) = H(\begin{pmatrix} 0 & -d \\ 0 & c \end{pmatrix})H(\begin{pmatrix} \widetilde{a^*} & -\widetilde{b^*} \\ 0 & 0 \end{pmatrix})$$

and this is the zero operator.

This means using our formula:

$$\det(I + T(\phi)^{-1}T(\psi^*)^{-1}H(\psi^*)H(\widetilde{\phi}))$$

that the above is simply the determinant of the identity operator.

A similar computation can be done with all three factors and the end result (joint work with Doug Pickrell) is that the determinant constant is

$$\prod_{i=1}^{\infty}(1+|\eta_i|^2)^{-i} \times \prod_{i=1}^{\infty}(1+|\alpha_i|^2)^{-i} \times \exp\sum_{k=1}^{\infty} 2\,k\chi_k\chi_{-k}.$$

Some of this can be extended to $SL(2)$ symbols, some to higher dimension, and some to higher genus surfaces, but many open questions remain.