Entanglement of disjoint intervals in 2D CFT and Riemann surfaces

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Based on various papers in collaboration with:

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Andrea Coser, Cristiano De Nobili, Luca Tagliacozzo

Tau Functions of Integrable Systems and Their Applications

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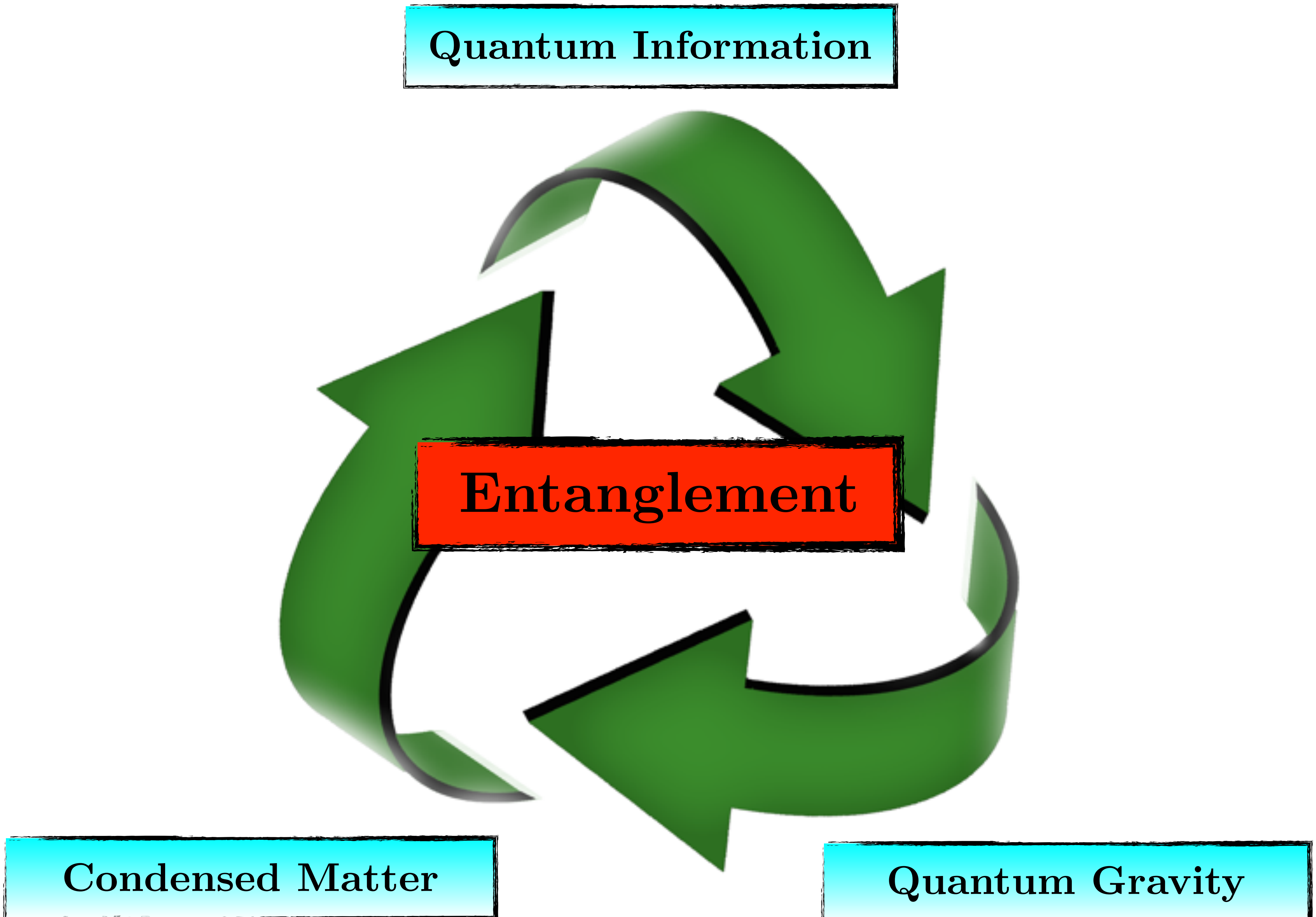
Entanglement: a crossroad of interests



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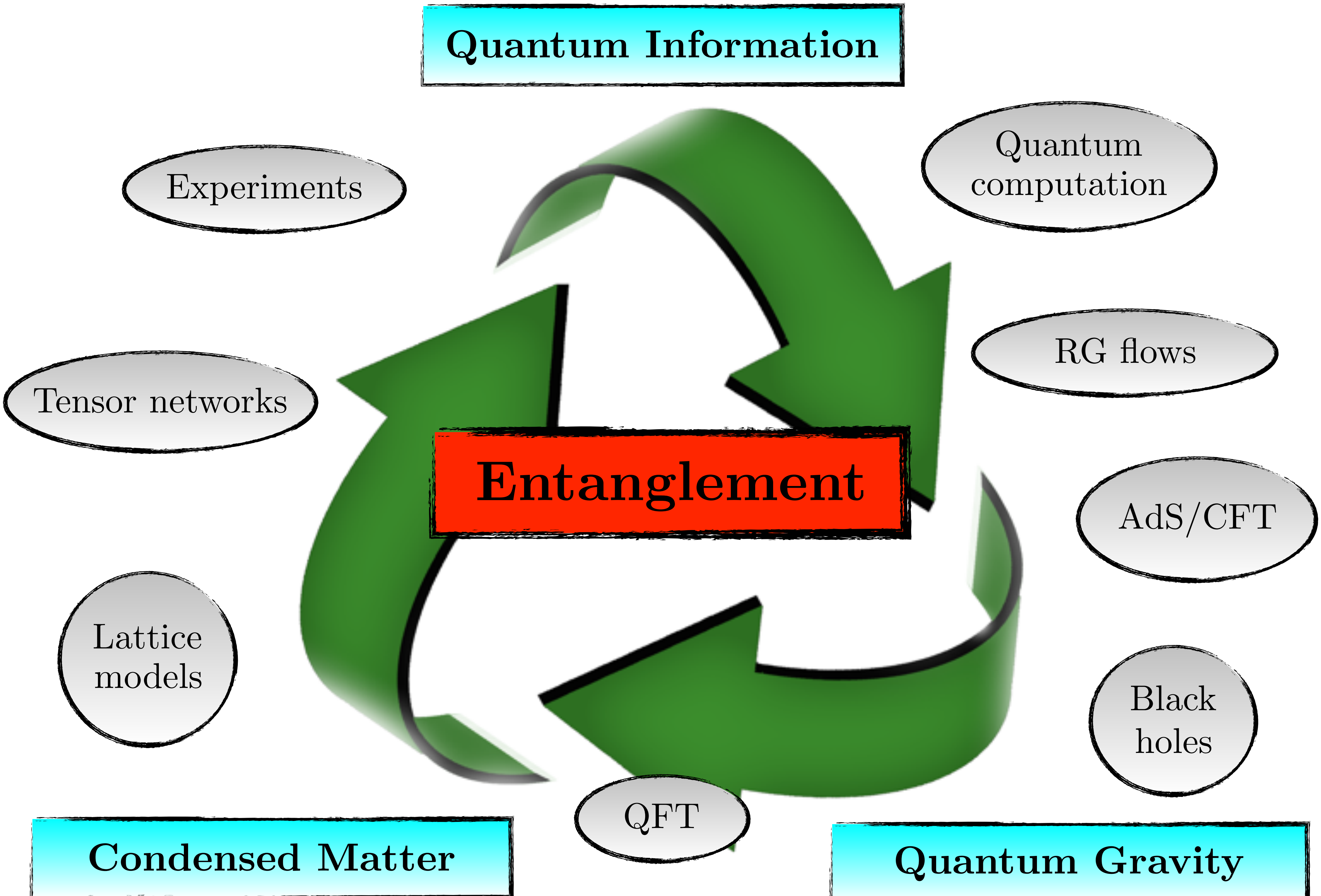
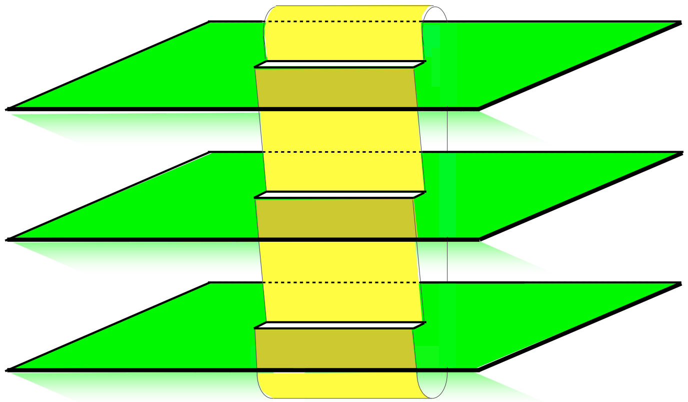


Outline

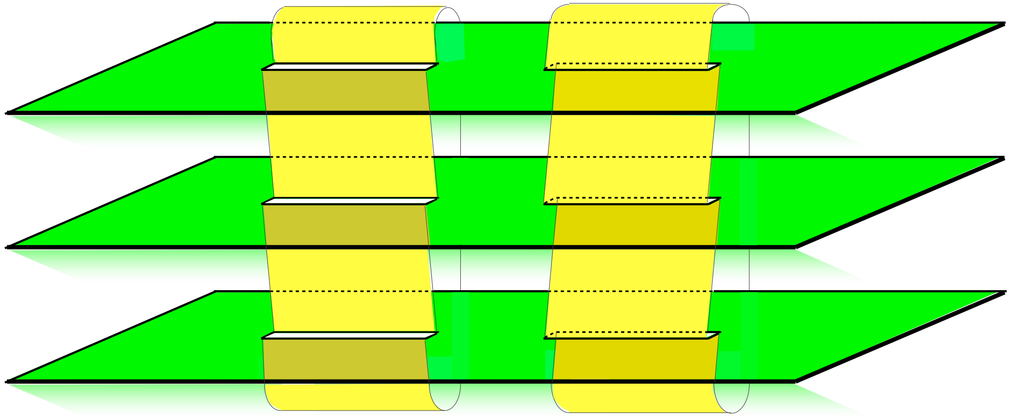
➜ **Entanglement entropy** in 2D CFT

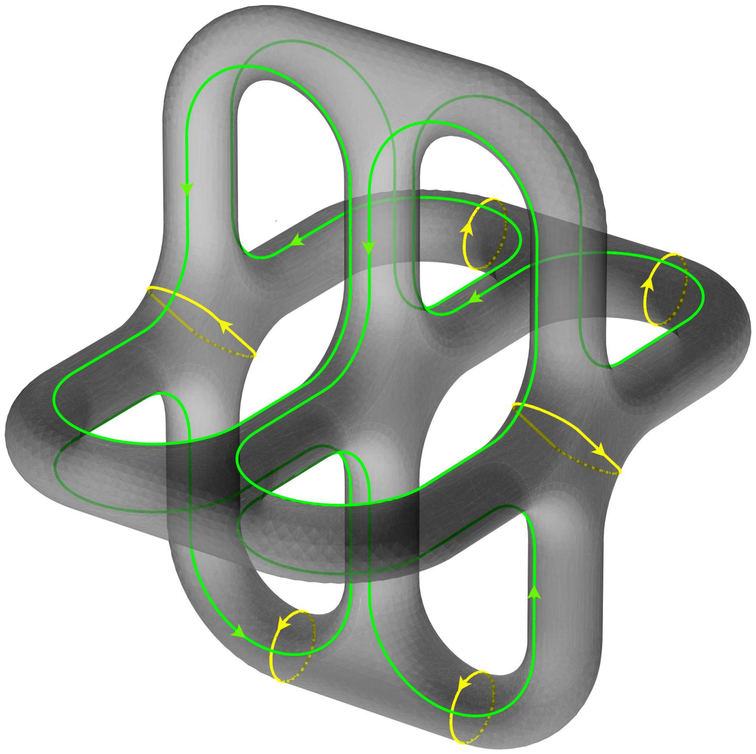
- Single interval
- Disjoint intervals



➜ **Entanglement negativity** in 2D CFT

- Two adjacent intervals
- Two disjoint intervals

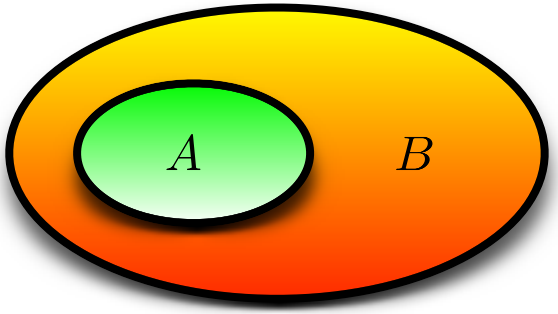




Entanglement entropies: definition

- Quantum system ($\mathcal{H}$) in the ground state $|\Psi\rangle$
  Density matrix $\rho = |\Psi\rangle\langle\Psi| \quad \Longrightarrow \quad \mathrm{Tr}\rho^n = 1$

- Hilbert space $\boxed{\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B}$



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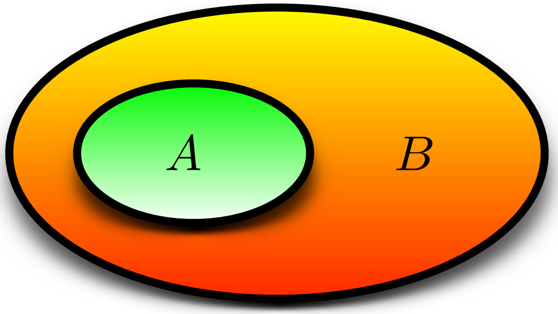

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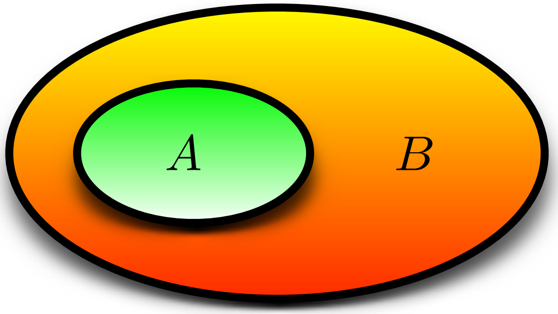

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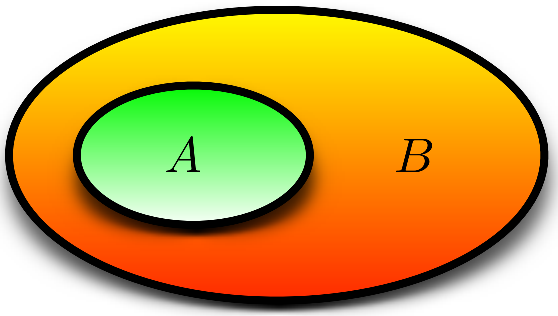

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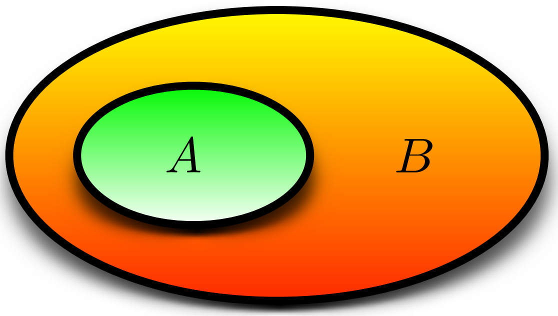

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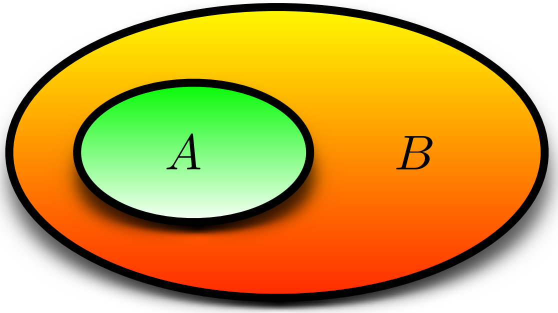

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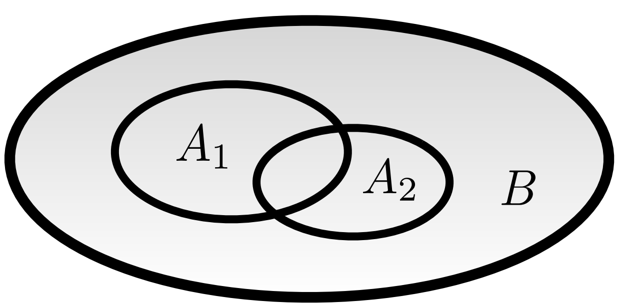
- Subadditivity $\qquad A_1 \cap A_2 = \emptyset \qquad S_{A_1} + S_{A_2} \geqslant S_{A_1 \cup A_2}$

- Strong Subadditivity

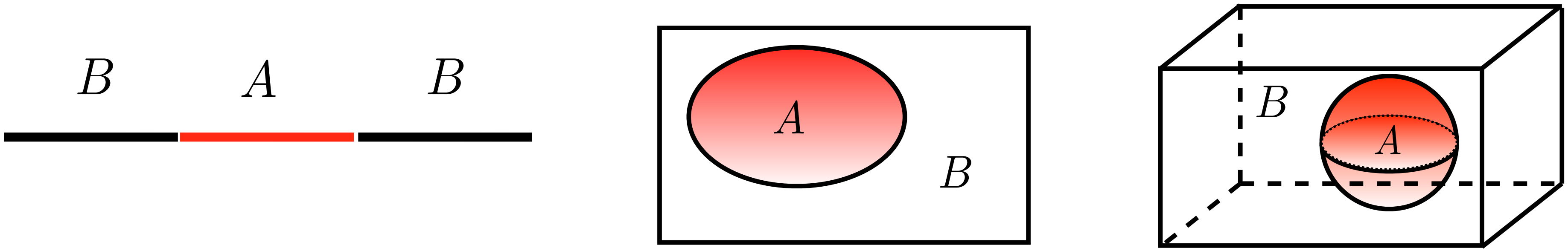


$$S_{A_1} + S_{A_2} \geqslant S_{A_1\cup A_2} + S_{A_1\cap A_2}$$

$$S_{A_1} + S_{A_2} \geqslant S_{A_1\setminus A_2} + S_{A_2\setminus A_1}$$

Geometric entropy: area law

- Assume that A and B correspond to a *spatial bipartition* of the system



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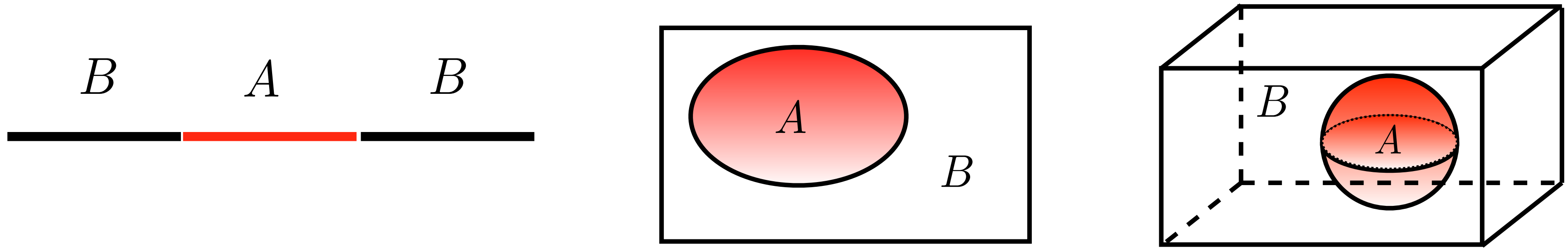

- *Area law:* In d spatial dimensions when $\rho = |\Psi\rangle\langle\Psi|$ $(S_A = S_{A^c})$

$$\boxed{S_A \propto \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + \ldots}$$

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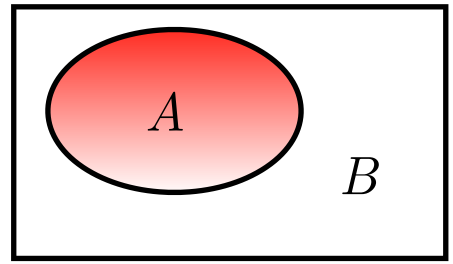

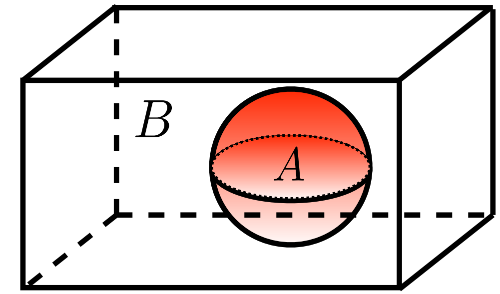

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➔ In $1+1$ CFTs at $T=0$

[Holzhey, Larsen, Wilczek, (1994)]
[Calabrese, Cardy, (2004)]

$$\boxed{S_A = \frac{c}{3}\log\frac{\ell}{a} + \text{const}}$$

➔ In $2+1$ CFTs for a circle

$$\boxed{S_A = \gamma\,\frac{2\pi R}{a} - f}$$

- Area law violated in presence of Fermi surfaces: $S_A \sim L^{d-1}\log L$

[Wolf, (2005)] [Gioev, Klich, (2005)]

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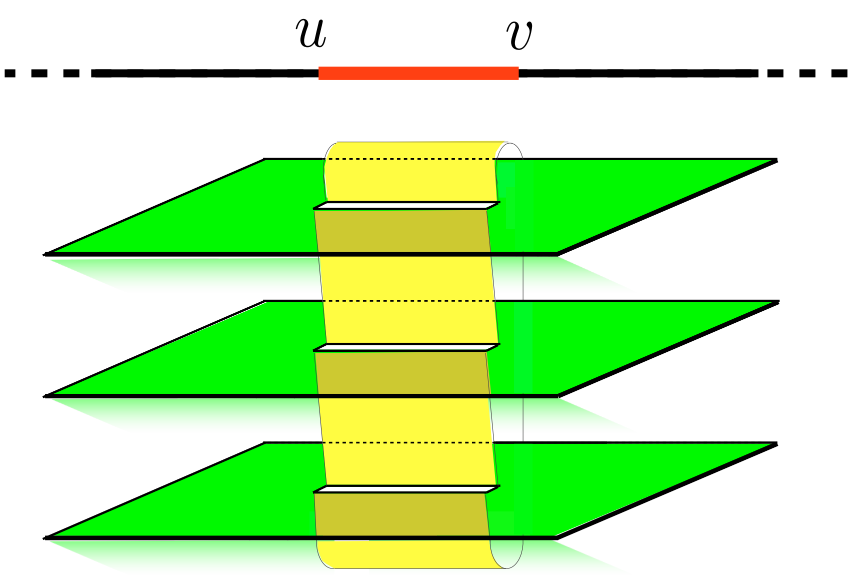

$\mathcal{R}_{1,n}$

$\mathrm{Tr}\rho_A^n$

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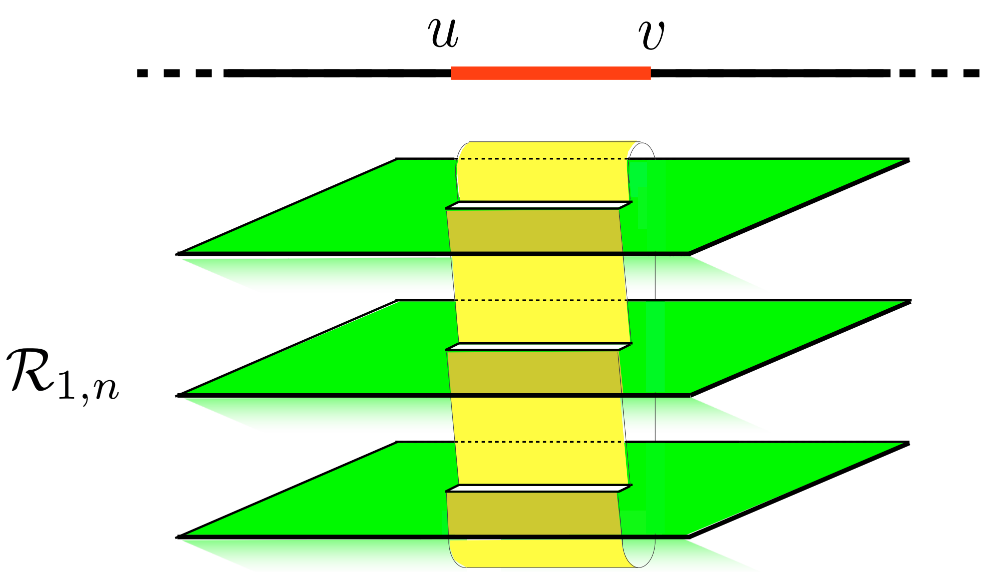
- Single interval ($N = 1$): the Rényi entropies can be written as a two point function of *twist fields* on the sphere [Calabrese, Cardy, (2004)]



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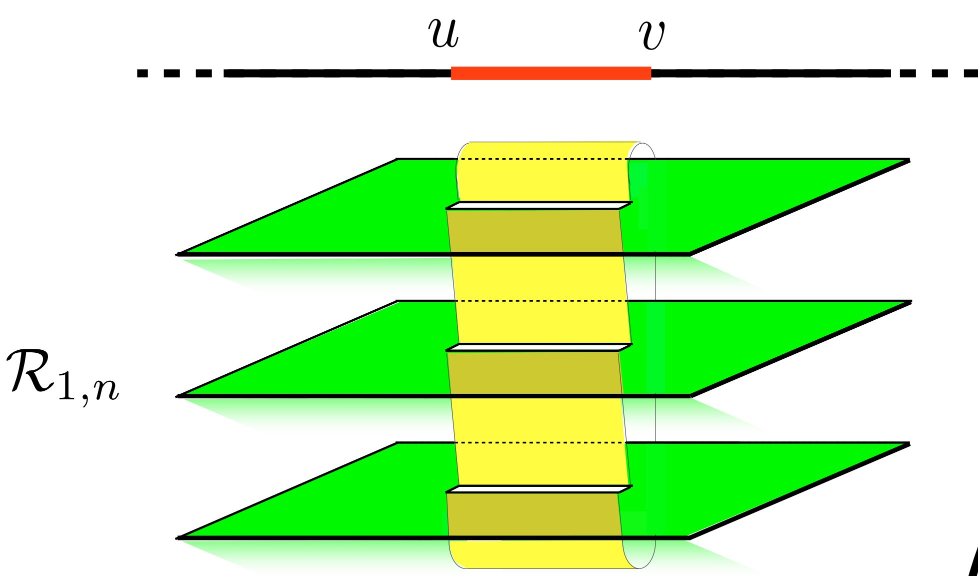

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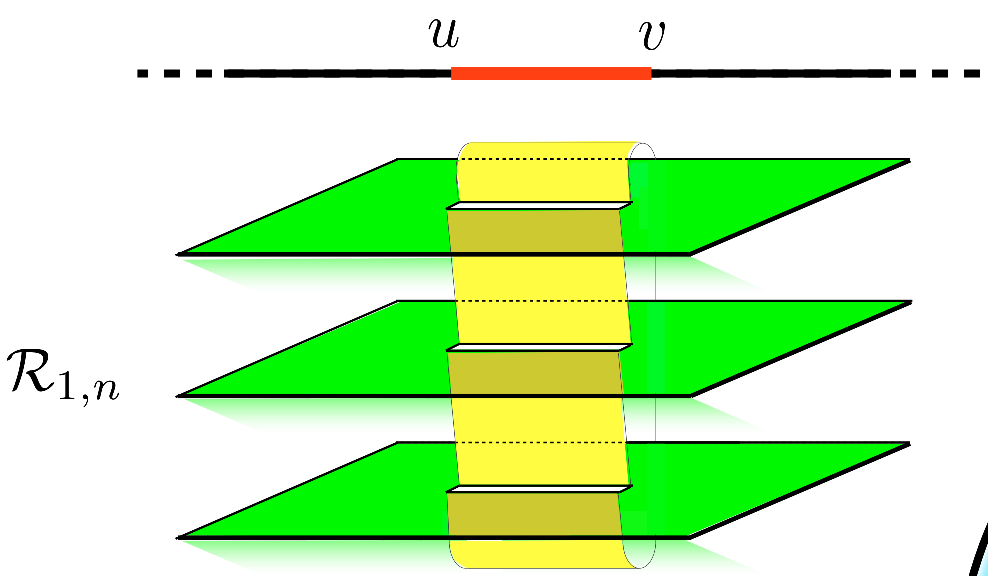

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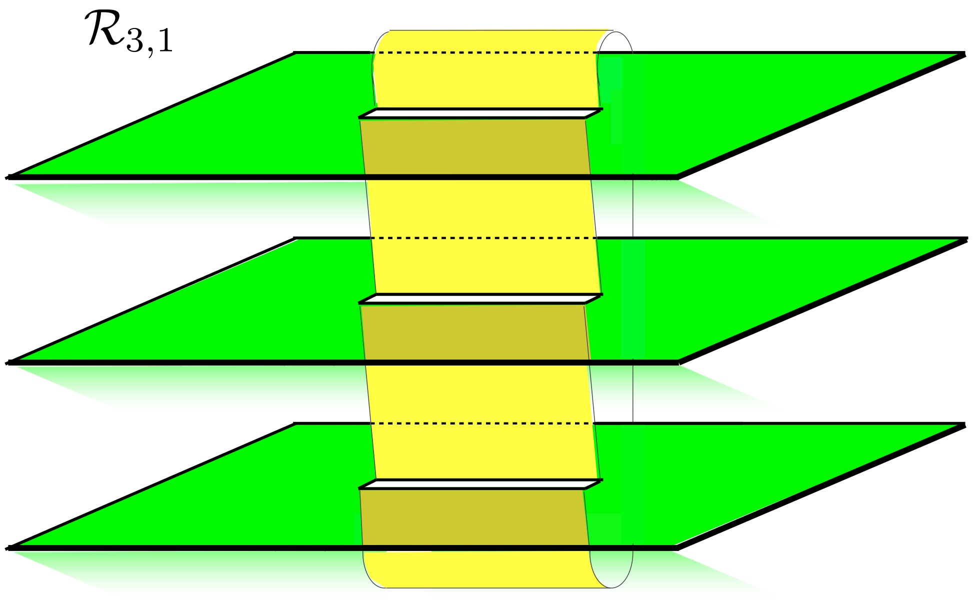
- Twist fields have been largely studied in the 1980s
  [Zamolodchikov, (1987)] [Dixon, Friedan, Martinec, Shenker, (1987)]
  [Knizhnik, (1987)] [Bershadsky, Radul, (1987)]

- Integrable field theories [Casini, Fosco, Huerta, (2005)] [Casini, Huerta, (2005)]
  [Cardy, Castro-Alvaredo, Doyon, (2008)]

Boundary conditions & Twist fields in 2D CFT

- Global symmetry $j \mapsto j+1 \bmod n$
- Boundary conditions:

$$\varphi_j(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \varphi_{j-1}(z,\bar{z})$$



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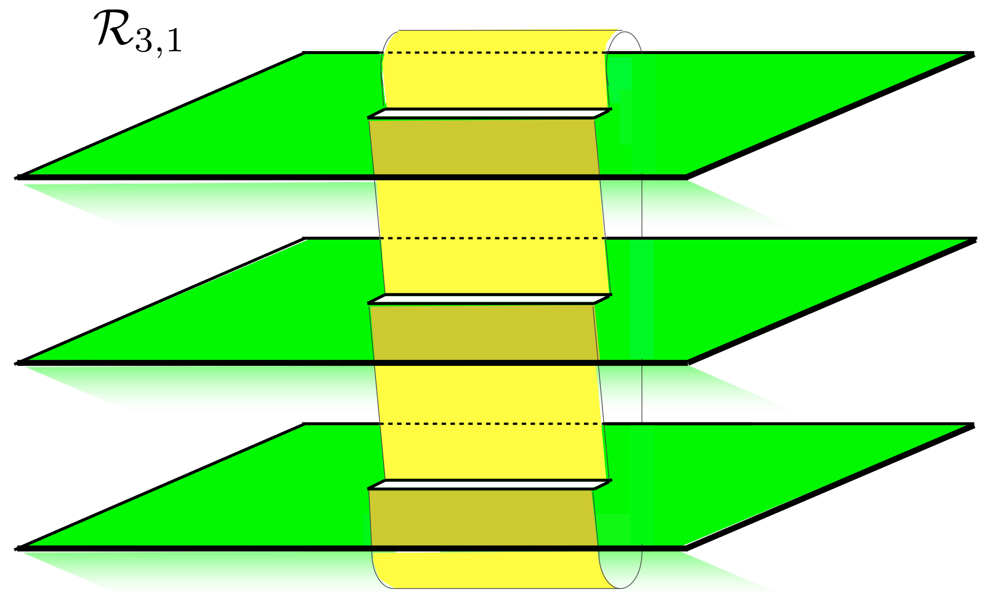

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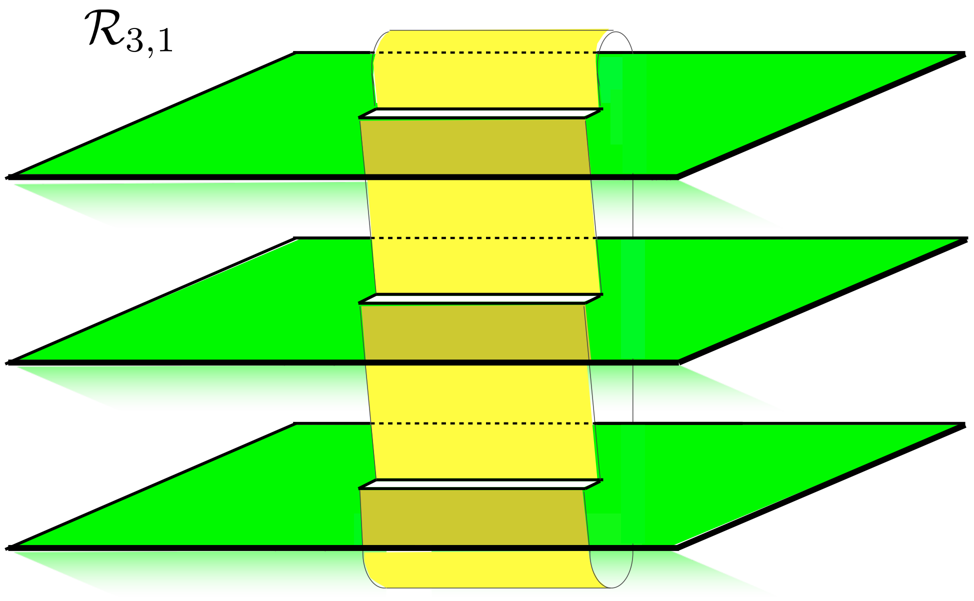

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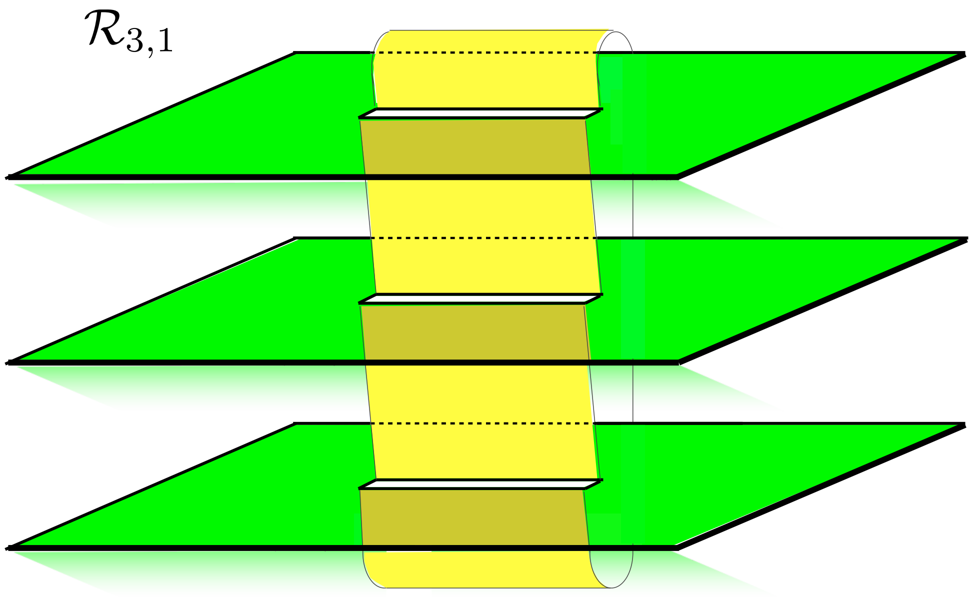

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- Branch-point twist field $\mathcal{T}_{n,k}$ in the origin

[Dixon, Friedan, Martinec, Shenker, NPB (1987)] [Zamolodchikov, NPB (1987)]
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Entanglement entropy of a single interval in CFT

- Two-point function of twist fields for a free complex boson φ

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Entanglement entropy of a single interval in CFT

- Two-point function of twist fields for a free complex boson φ

  [Dixon, Friedan, Martinec, Shenker, NPB (1987)]

$$\langle \mathcal{T}_{k,n}(u) \bar{\mathcal{T}}_{k,n}(v) \rangle \propto \frac{1}{|u-v|^{4\Delta_{k/n}}}$$

$$\boxed{\Delta_{\frac{k}{n}} = \bar{\Delta}_{\frac{k}{n}} = \frac{1}{2} \frac{k}{n}\left(1 - \frac{k}{n}\right)}$$

- Partition function on $\mathcal{R}_{n,1}$ [Calabrese, Cardy, JSTAT (2004)] [Ryu, Takayanagi JHEP (2006)]

$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_n}{\mathcal{Z}_1^n} = \langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) \rangle = \prod_{k=0}^{n-1} \langle \mathcal{T}_{k,n}(u) \bar{\mathcal{T}}_{k,n}(v) \rangle = \frac{c_n}{|u-v|^{\frac{1}{3}\left(n-\frac{1}{n}\right)}}$$

  Entanglement entropy of a single interval for the free real boson

$$c = 1$$

$$\boxed{S_A = -\,\partial_n \mathrm{Tr}\rho_A^n\big|_{n=1} = \frac{1}{3} \log \frac{\ell}{a} + c_1'}$$

  [Holzhey, Larsen, Wilczek, NPB (1994)]

- $\mathcal{R}_{n,1}$ is topologically a *sphere* $\Rightarrow$ it can be uniformized into a sphere

  This allows to find S_A for any CFT

2D CFT: Renyi entropies for two disjoint intervals

- Two disjoint intervals:

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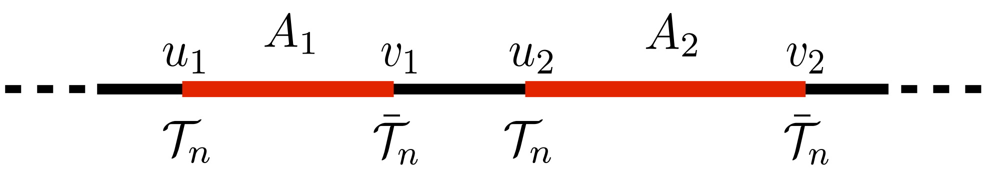
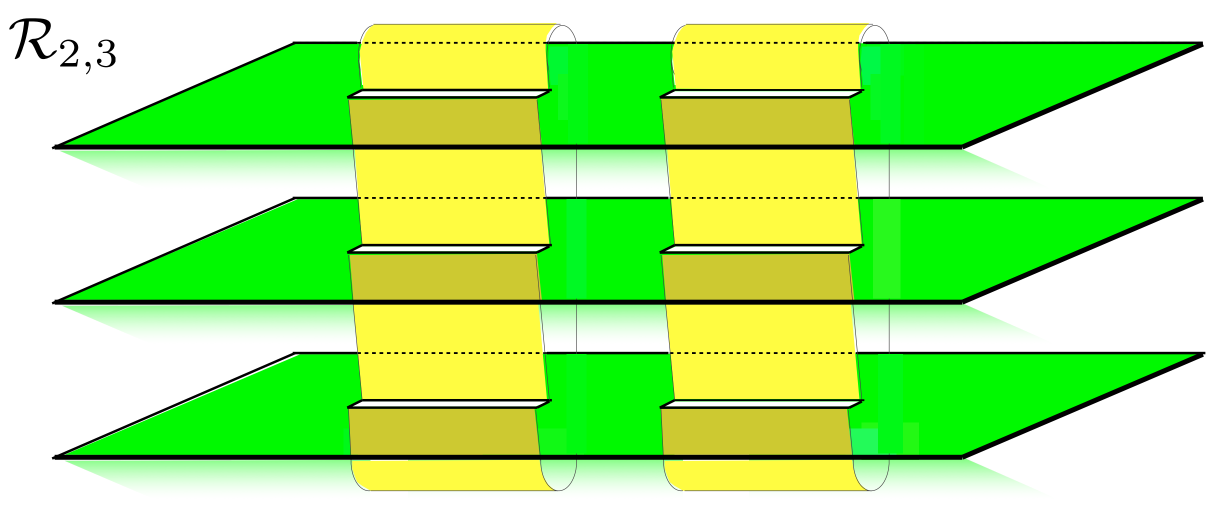


$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{2,n}}{\mathcal{Z}^n} = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$



$\mathcal{Z}_{2,n}$ is the partition function of $\mathcal{R}_{2,n}$, a particular genus $n-1$ Riemann surface obtained through replication

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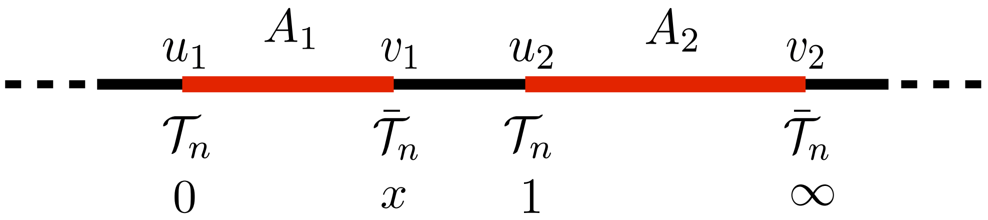

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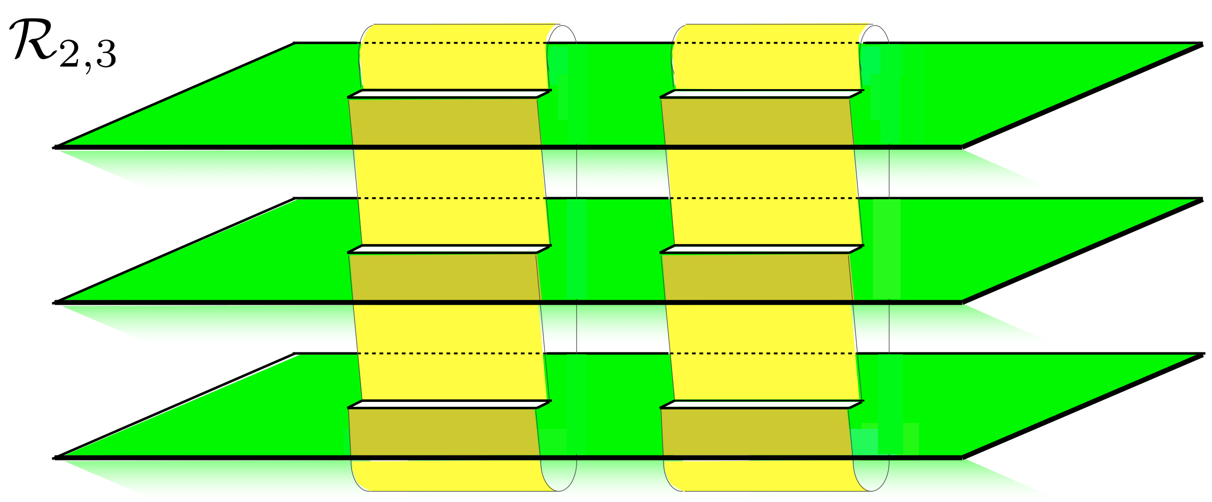

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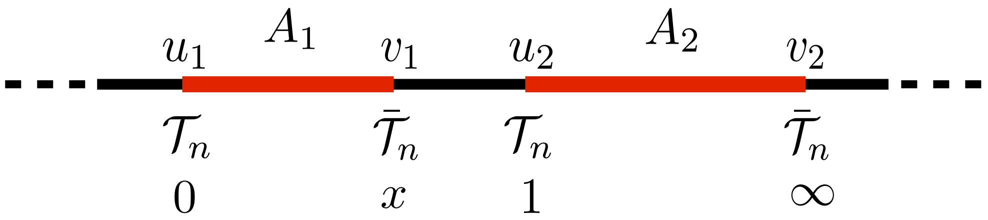

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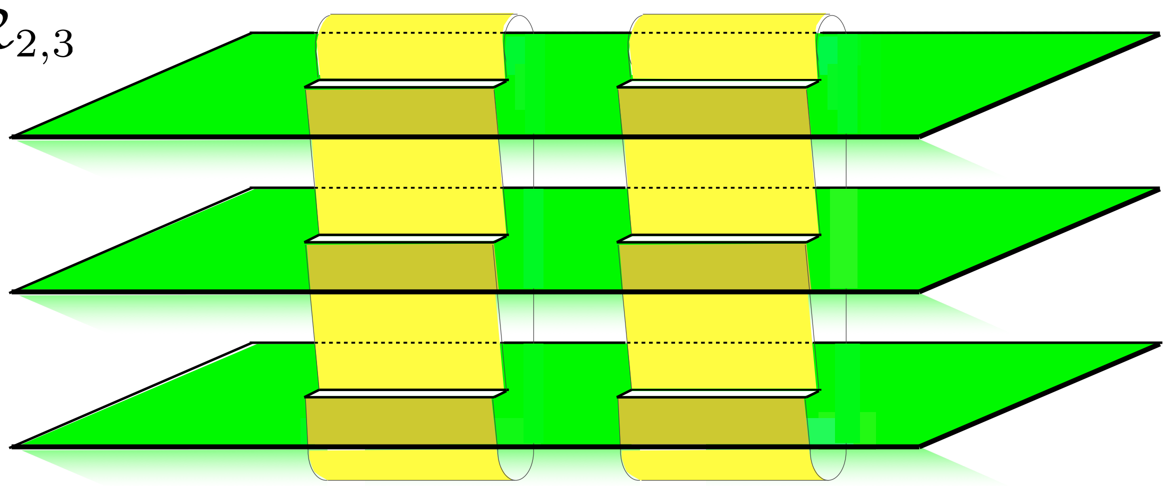

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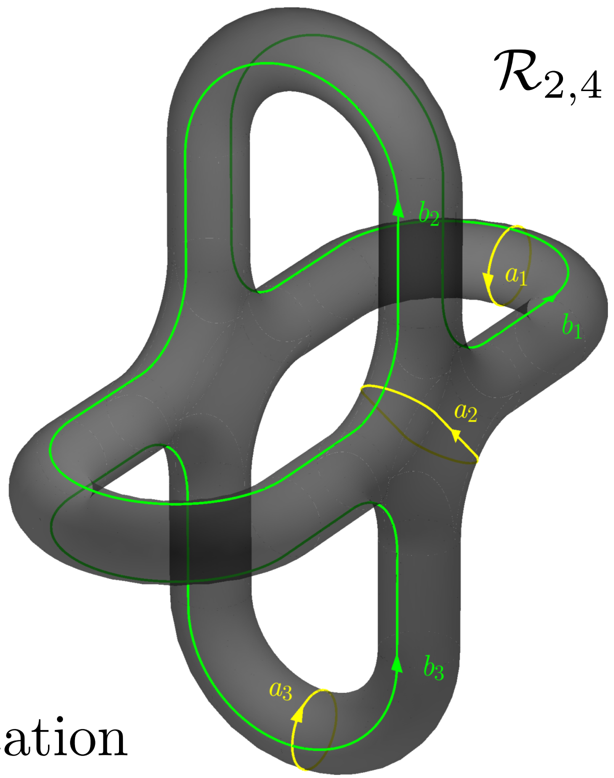

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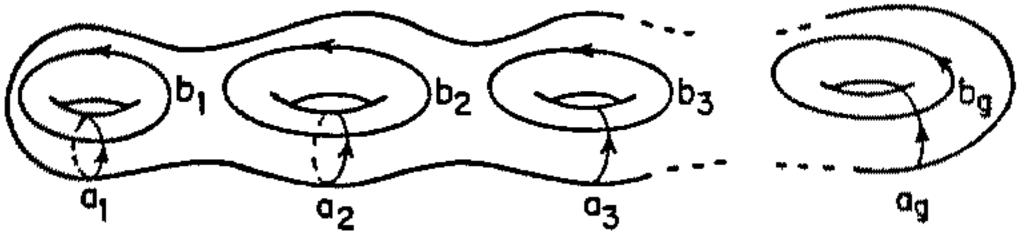
Higher genus Riemann surfaces from replication



A key object is the *period matrix* τ

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad \oint_{b_i} \omega_j = \tau_{ij}$$

τ is $g \times g$, symmetric and $\mathrm{Im}(\tau) > 0$

$3g - 3$ complex moduli for $g \geqslant 2$

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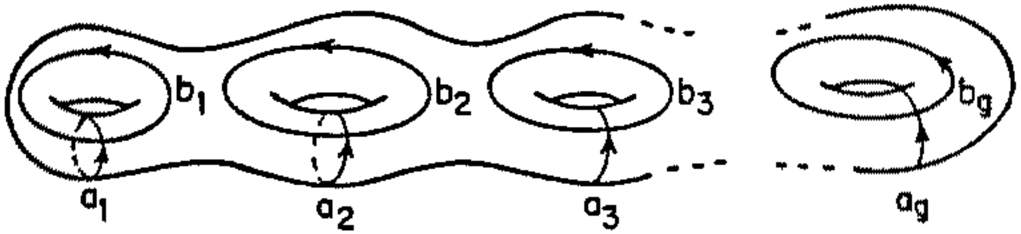

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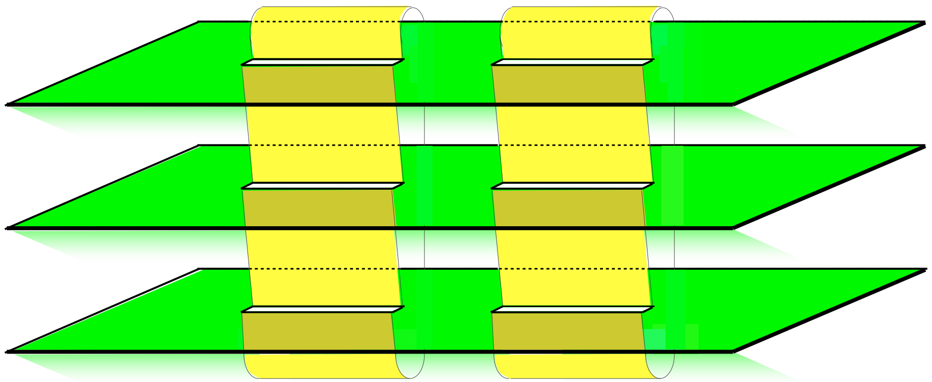

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Indeed $\tau = \tau(\boldsymbol{x})$

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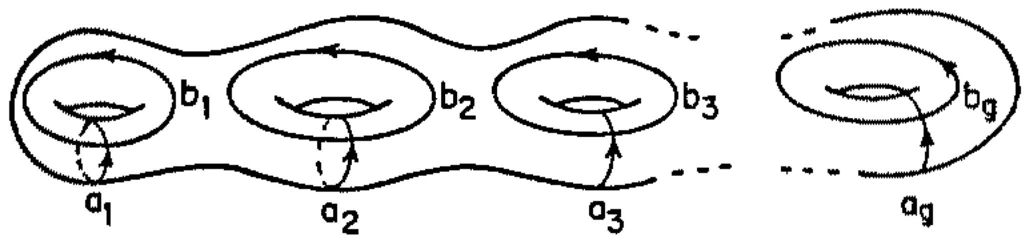

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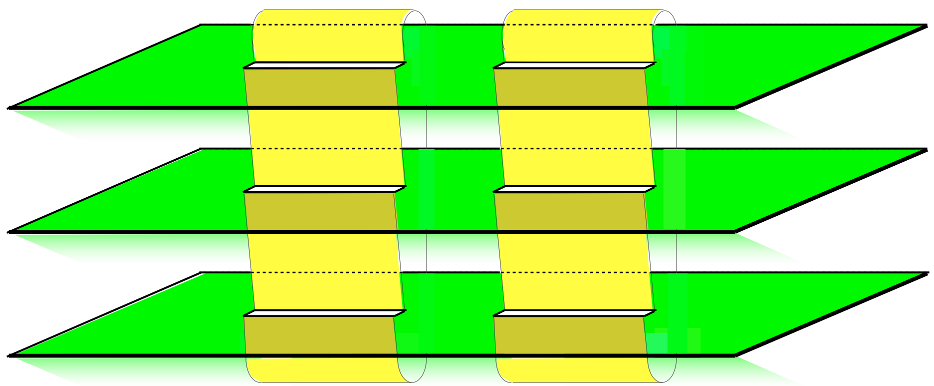

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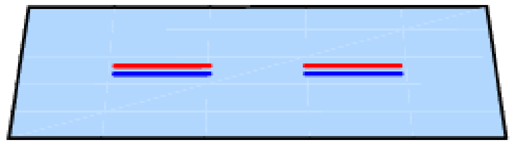

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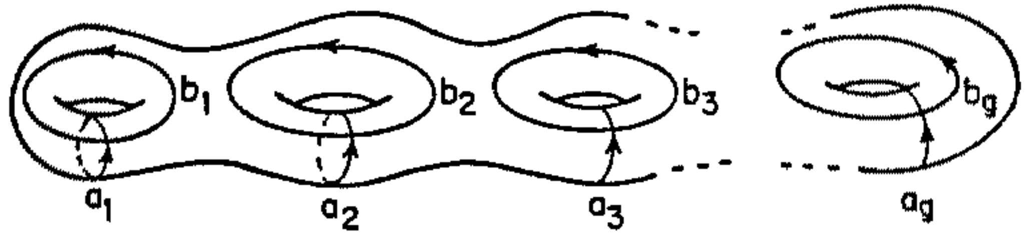

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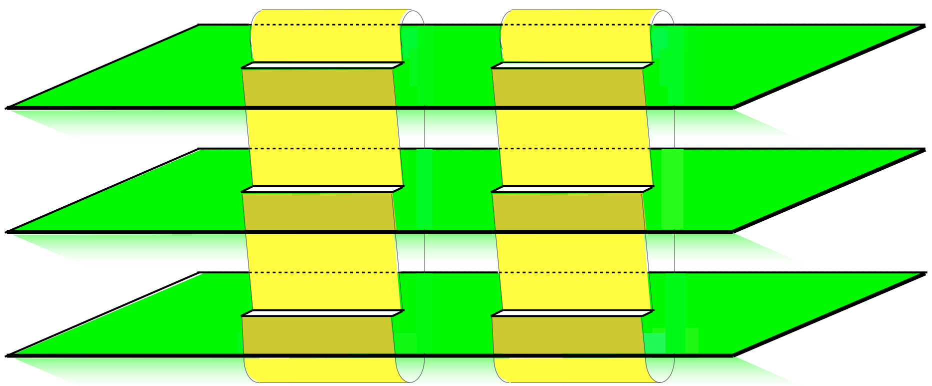

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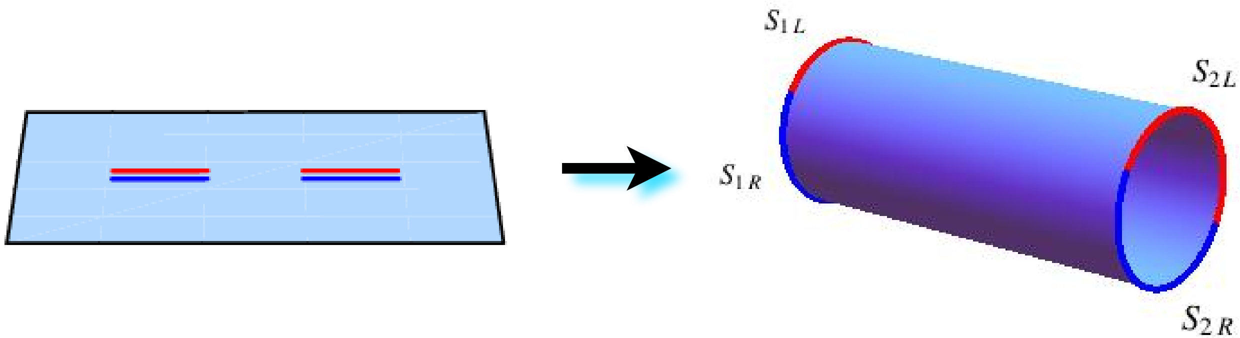

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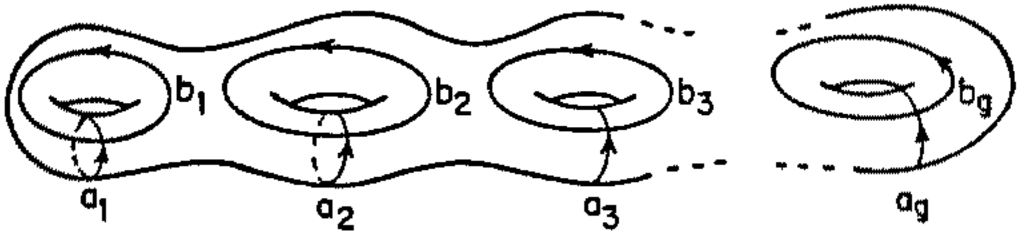

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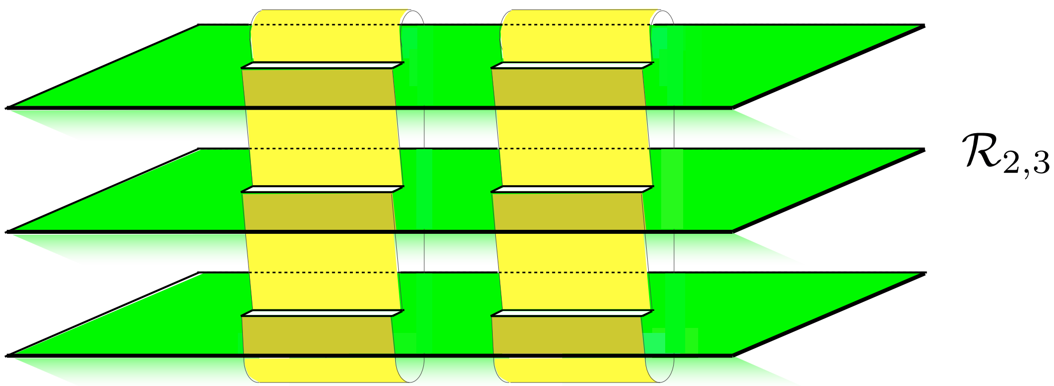

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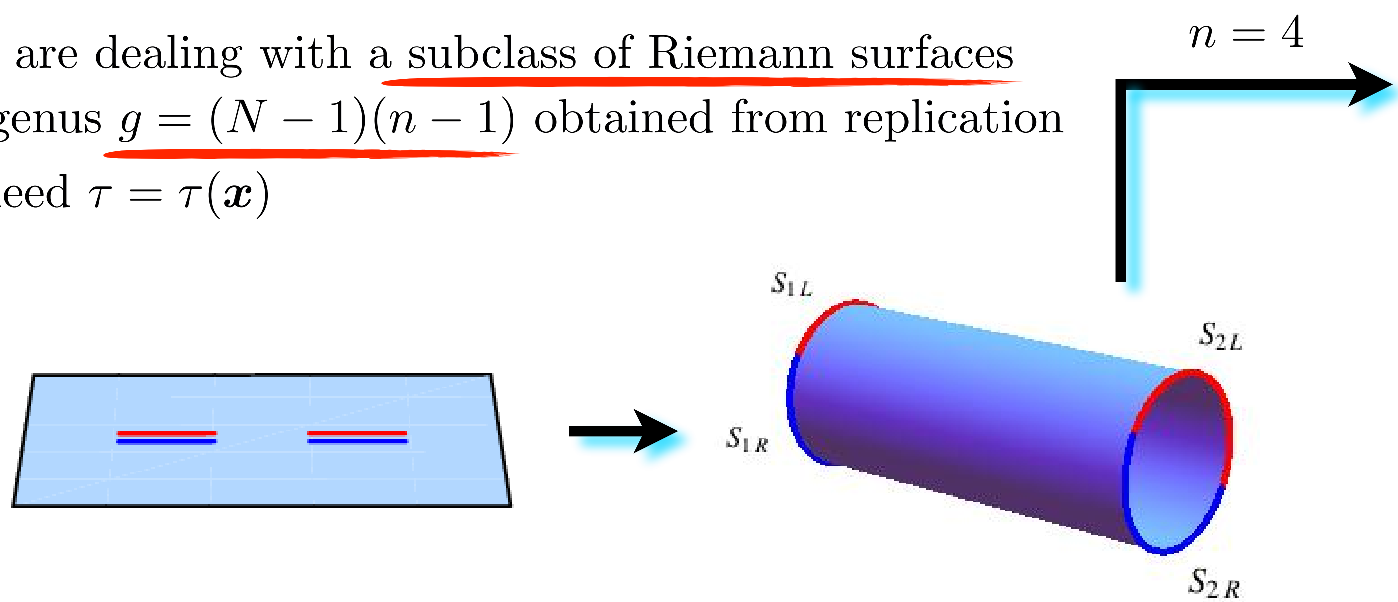

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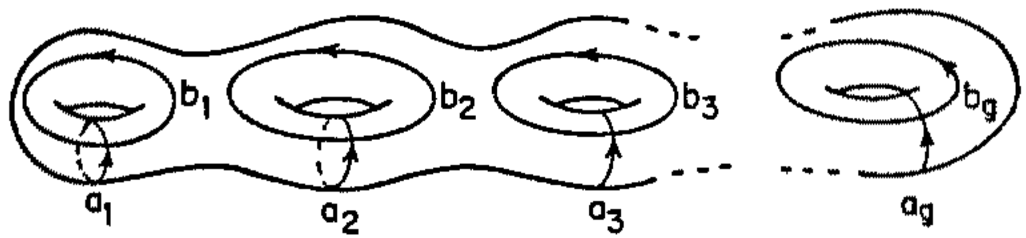

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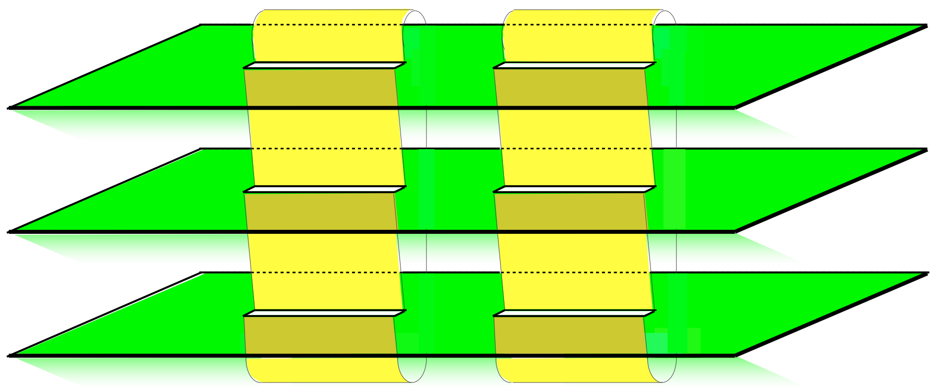

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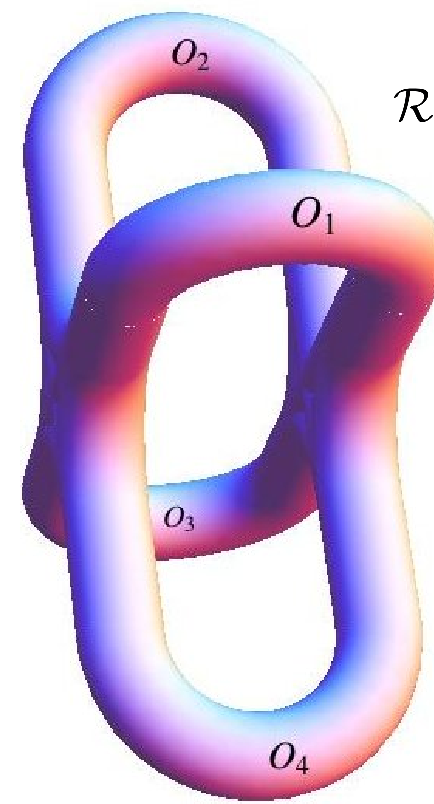

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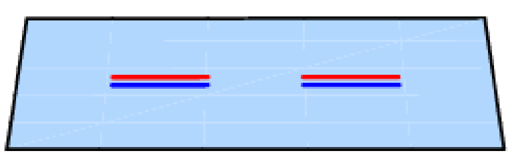

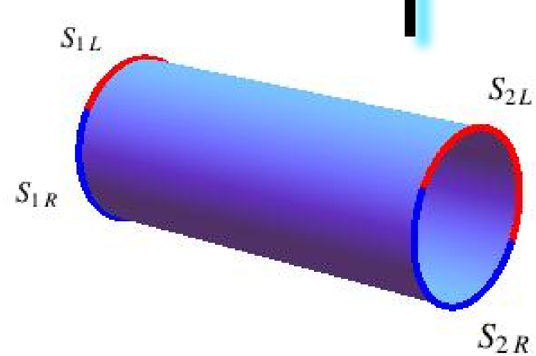

Free compactified boson: Renyi entropies

[Calabrese, Cardy, E.T., JSTAT (2009)]

- *Riemann-Siegel theta function*

$$\Theta(0|\Gamma) \equiv \sum_{m \in \mathbf{Z}^G} \exp\left[\, i\pi \, m^{\mathrm{t}} \cdot \Gamma \cdot m \,\right]$$

$$\boxed{\mathcal{F}_n(x) = \left[\frac{\Theta\big(0|\eta\Gamma\big)\,\Theta\big(0|\Gamma/\eta\big)}{\Theta\big(0|\Gamma\big)^2} \right]^2}$$

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- Previous studies of the period matrix in [Korotkin, (2003)] [Enolski, Grava, (2003)]

Bosonization on higher genus Riemann surfaces

[Alvarez-Gaume, Moore, Vafa; CMP (1986)]
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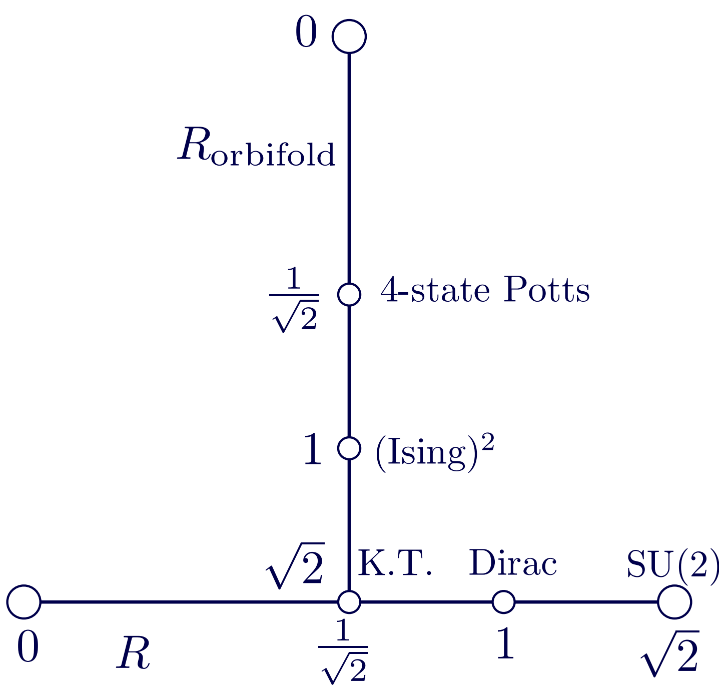
- $c = 1$ theories in 2D
  (compact boson and $\mathbb{Z}_2$ orbifold)



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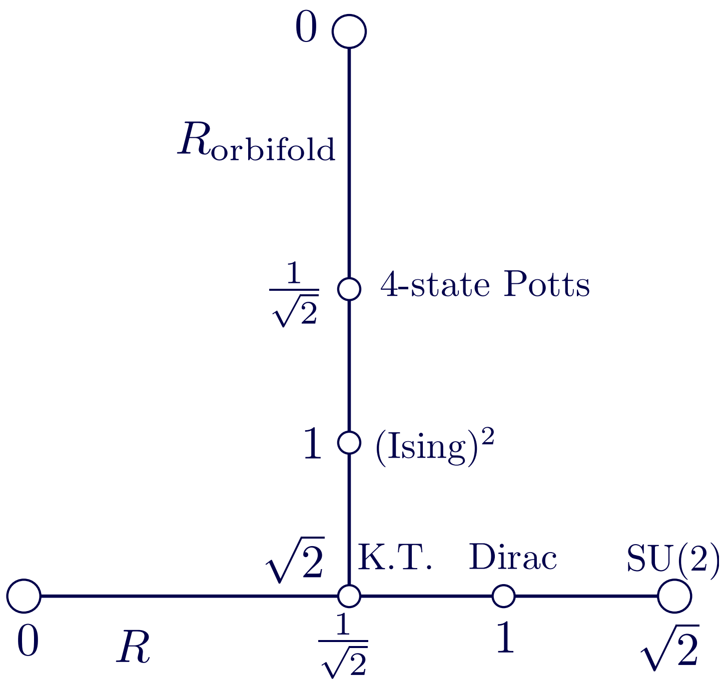

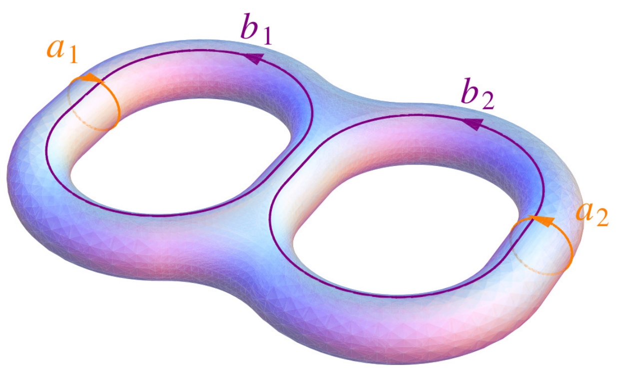

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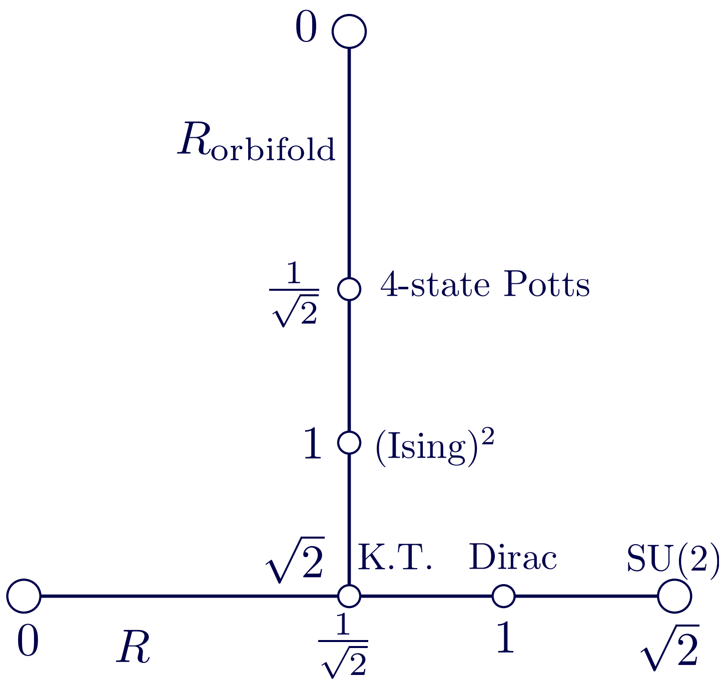

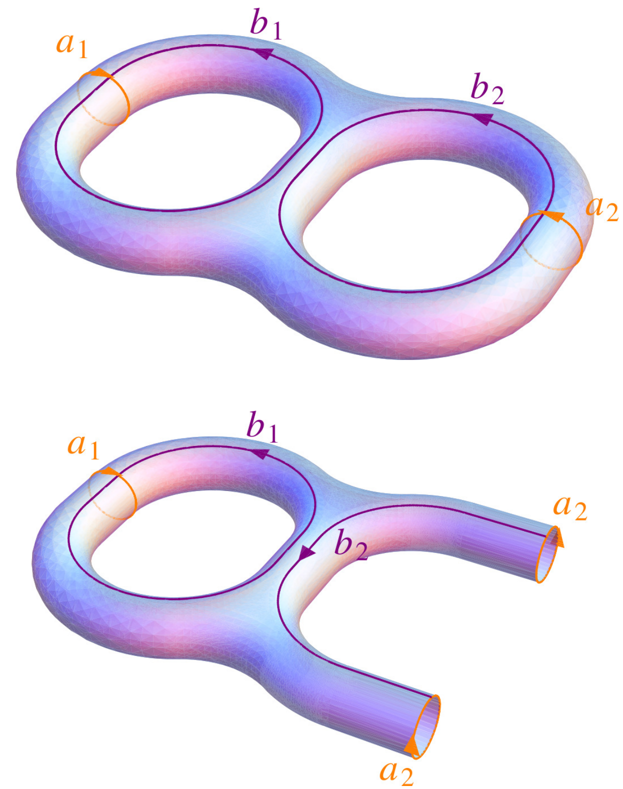

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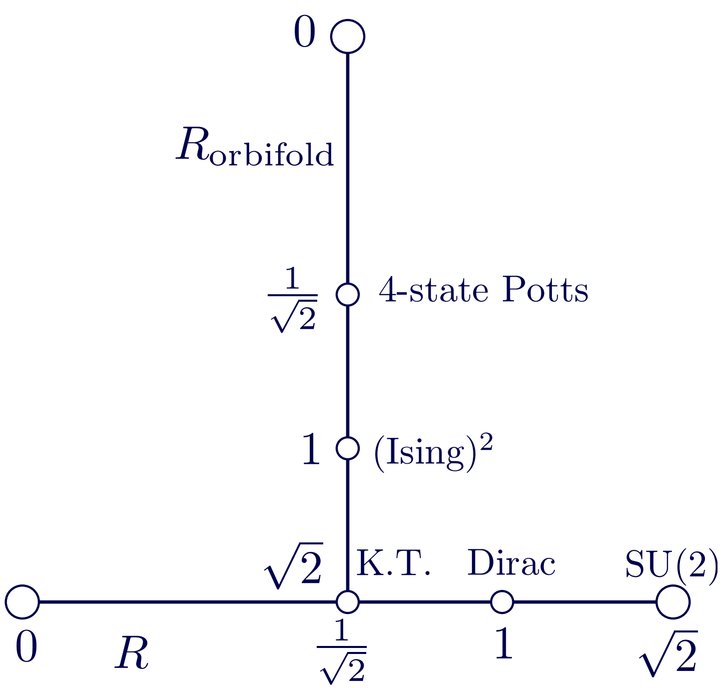

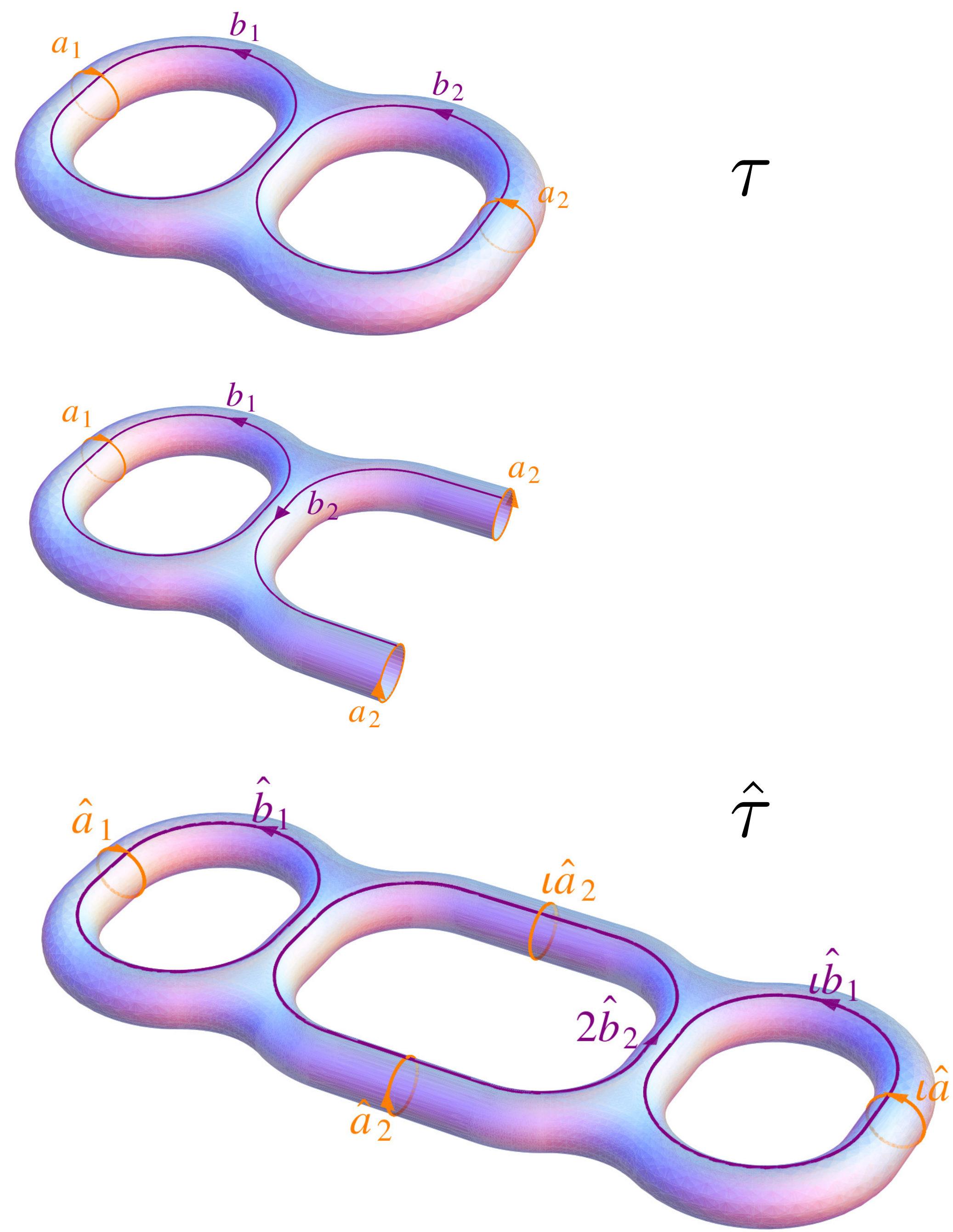

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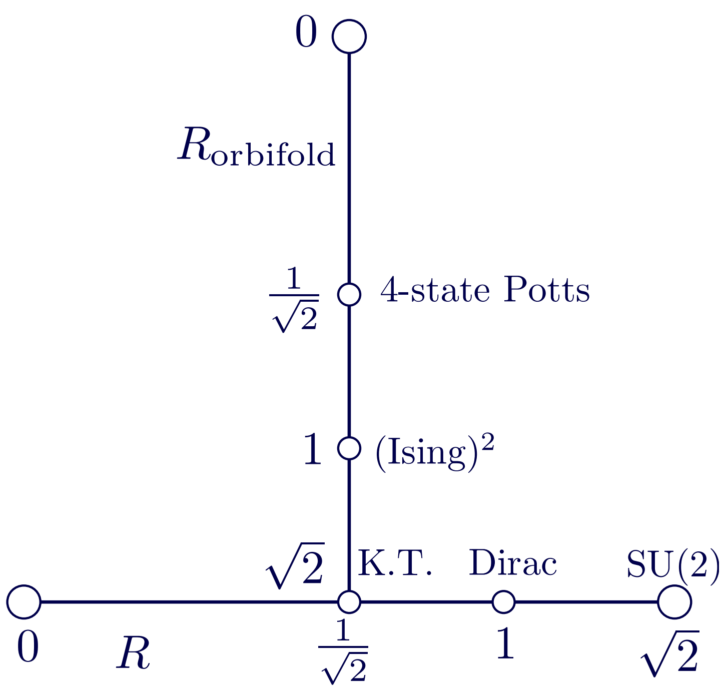
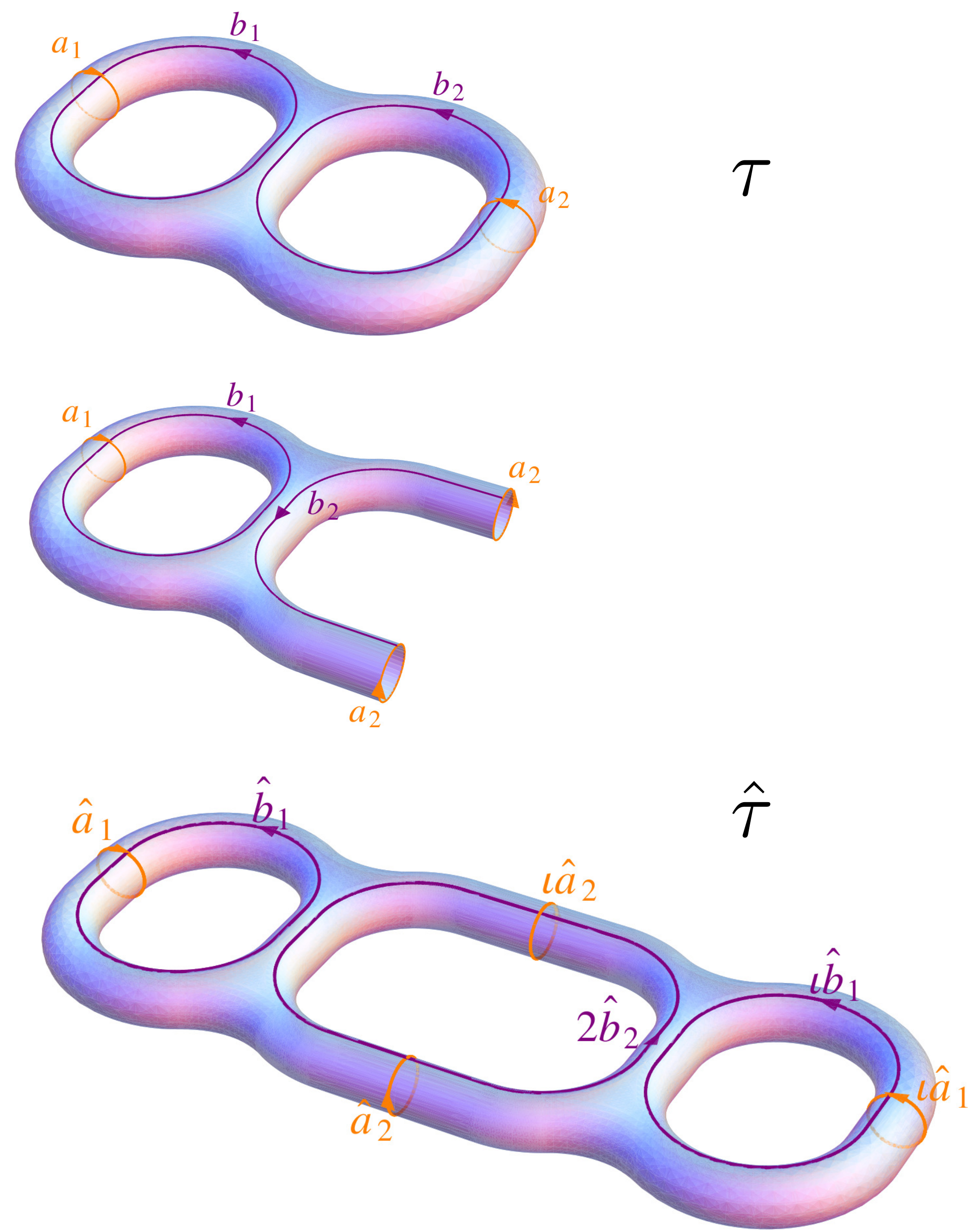


- For the Ising model
  τ is enough to write $\mathcal{Z}$



Ising model: Renyi entropies

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- $$H_{XY} \equiv -\sum_{j=1}^{L} \left(\frac{1+\gamma}{4}\, \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{4}\, \sigma_j^y \sigma_{j+1}^y + \frac{h}{2}\, \sigma_j^z \right)$$

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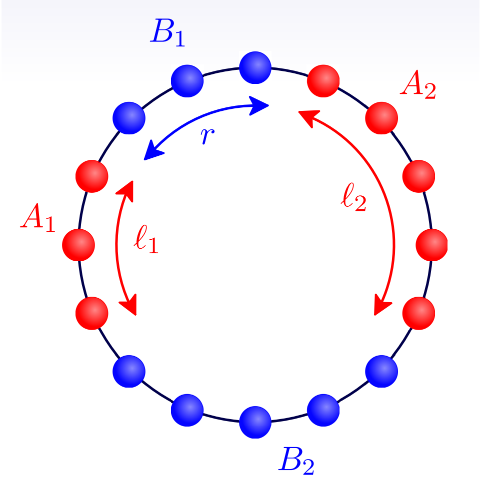


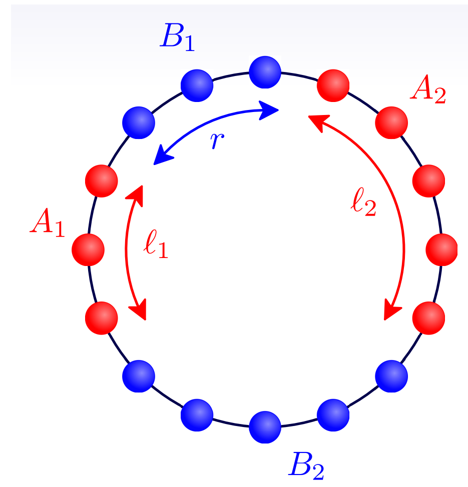
Ising model: Renyi entropies

[Calabrese, Cardy, E.T.; JSTAT (2011)]

- $$H_{XY} \equiv -\sum_{j=1}^{L}\left(\frac{1+\gamma}{4}\,\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{4}\,\sigma_j^y\sigma_{j+1}^y + \frac{h}{2}\,\sigma_j^z\right)$$

  $\gamma =$ anisotropy $\begin{cases} 1 & \text{Ising model} \\ 0 & \text{XX model} \end{cases}$

  $h =$ magnetic field



- Continuum limit: CFT.

  Bosonization on higher genus Riemann surfaces

$$\boxed{\mathcal{F}_n(x) = \frac{1}{2^{n-1}\Theta(0|\Gamma)}\sum_{\varepsilon,\delta}\left|\Theta\left[\begin{array}{c}\varepsilon\\ \delta\end{array}\right](0|\Gamma)\right|}$$

- *Riemann-Siegel theta function with characteristic*

  ε and δ are vectors with $n-1$ elements $\in \{0, 1/2\}$

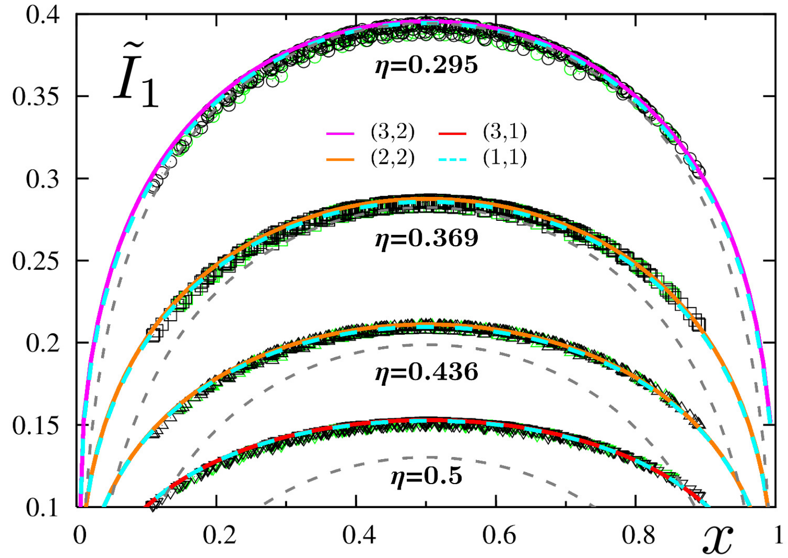
$$\Theta\left[\begin{array}{c}\varepsilon\\ \delta\end{array}\right](z|\Gamma) \equiv \sum_{m\,\in\,\mathbf{Z}^G} \exp\left[\, i\pi\,(m+\varepsilon)^{\mathrm{t}}\cdot\Gamma\cdot(m+\varepsilon) + 2\pi i\,(m+\varepsilon)^{\mathrm{t}}\cdot(z+\delta)\,\right]$$

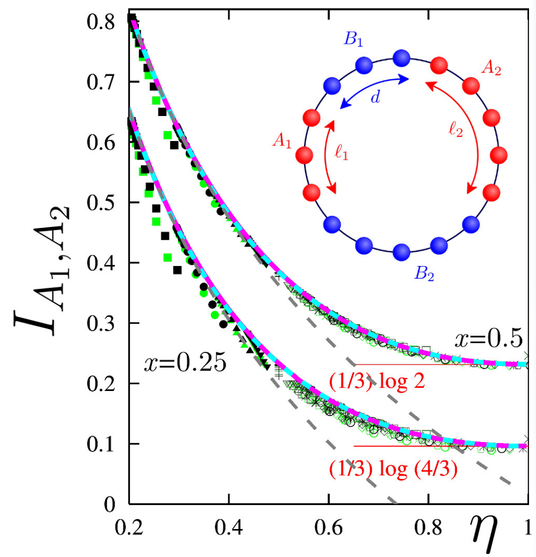
- $\mathcal{F}_n(x)$ is invariant under $x \leftrightarrow 1-x \qquad (S_A = S_B)$

Two disjoint intervals: comparison with numerics

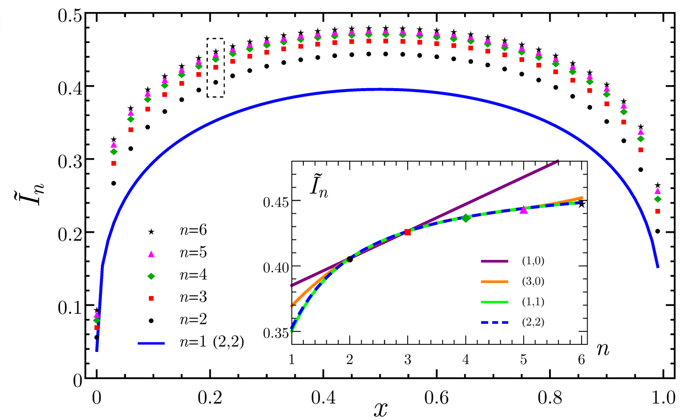
Mutual information in XXZ model

(exact diagonalization) [Furukawa, Pasquier, Shiraishi, (2009)]



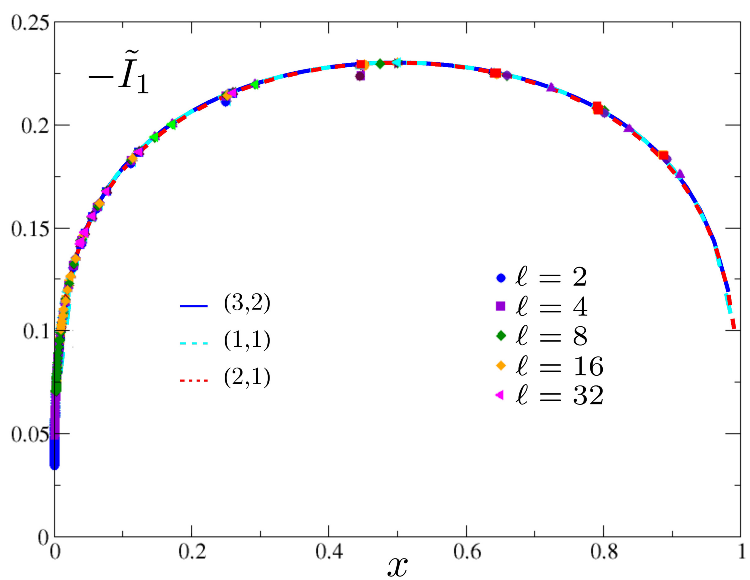


Rational interpolation:
an example



Mutual information in critical Ising chain

(Tree Tensor Network) [Alba, Tagliacozzo, Calabrese, (2010)]



Rational interpolation:

[De Nobili, Coser, E.T., (2015)]

$$W^{(n)}_{(p,q)}(x) \equiv \frac{a_0(x) + a_1(x)n + \cdots + a_p(x)n^p}{b_0(x) + b_1(x)n + \cdots + b_q(x)n^q}$$

Method first employed for Riemann theta functions in $2+1$ dimensions [Agón, Headrick, Jafferis, Kasko, (2014)]

Why disjoint intervals?

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- One interval on the infinite line at $T = 0$

  [Holzhey, Larsen, Wilczek, (1994)]
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$$\boxed{S_A = \frac{c}{3}\log\frac{\ell}{a} + \text{const}}$$

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- Two intervals A_1 and A_2: $\mathrm{Tr}\rho^n_{A_1 \cup A_2}$ for small intervals w.r.t. to other characteristc lengths of the system

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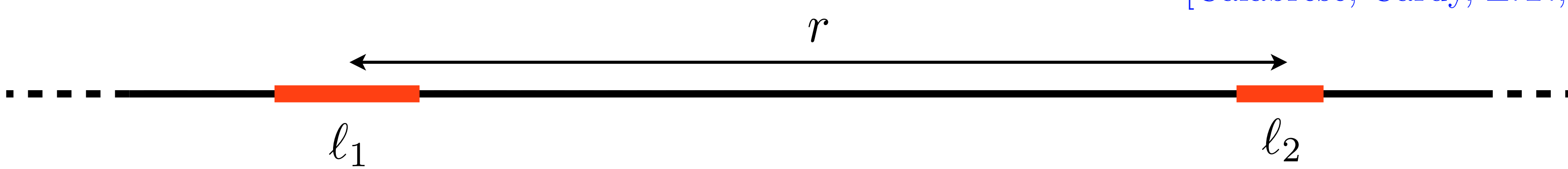

$$\boxed{\mathrm{Tr}\rho_A^n = c_n^2 (\ell_1 \ell_2)^{-c/6(n-1/n)} \sum_{\{k_j\}} \left(\frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \bar{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j} \left(e^{2\pi i j/n} \right) \rangle_{\mathbf{C}}^2}$$

- Generalization to higher dimensions [Cardy, (2013)]

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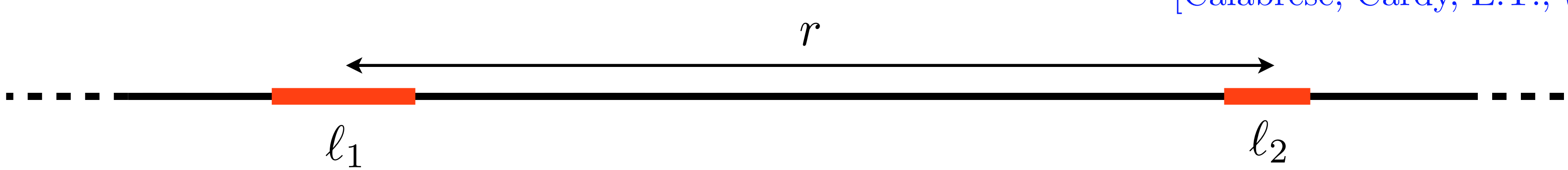

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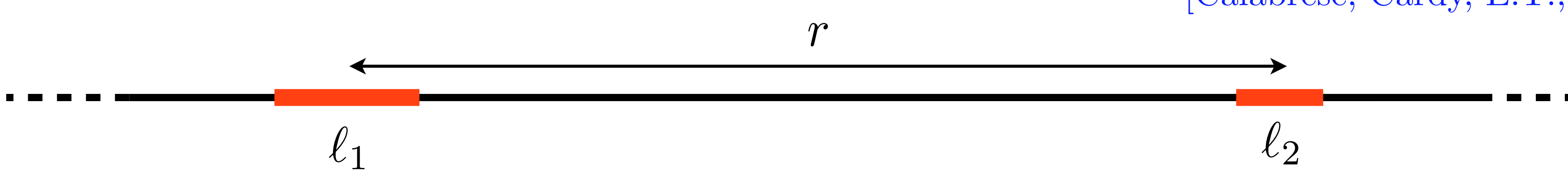

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The vacuum is not empty!

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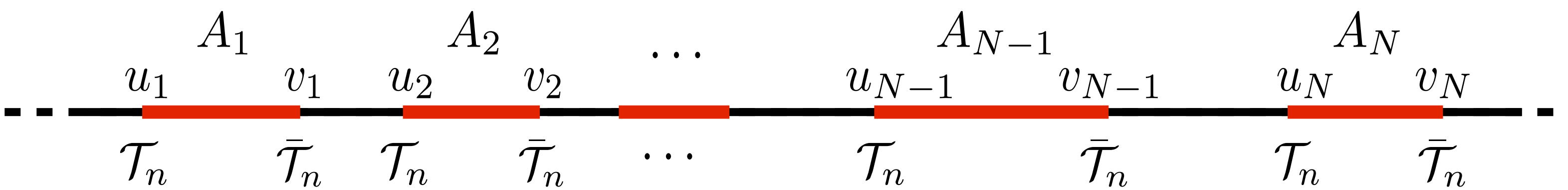
2D CFT: Renyi entropies for many disjoint intervals

- N disjoint intervals $\Longrightarrow$ $2N$ point function of twist fields



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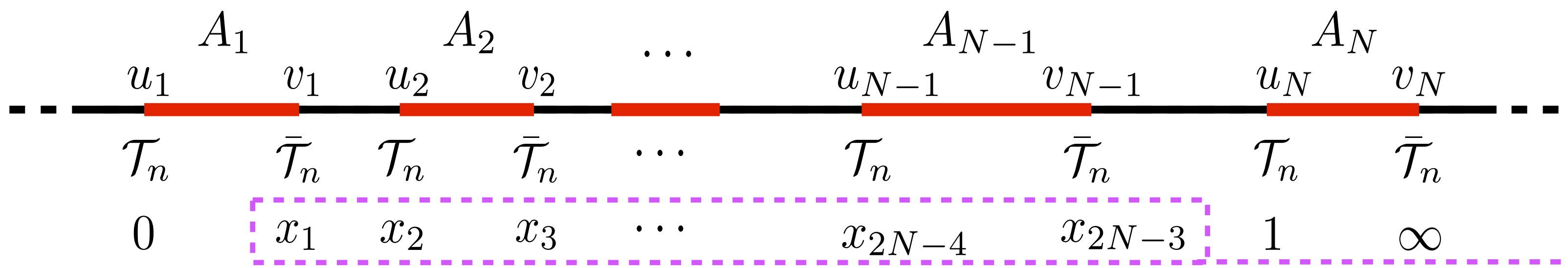

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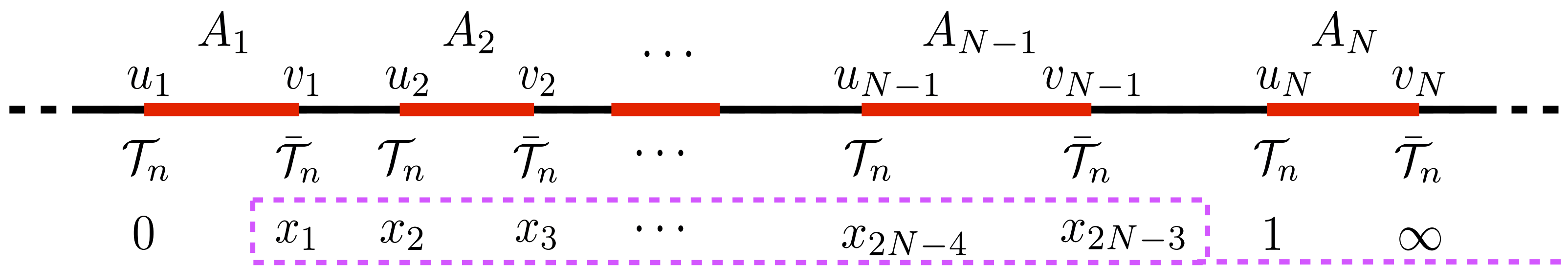

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- $\mathcal{Z}_{N,n}$ partition function of $\mathcal{R}_{N,n}$, a particular Riemann surface of genus $g=(N-1)(n-1)$ obtained through replication

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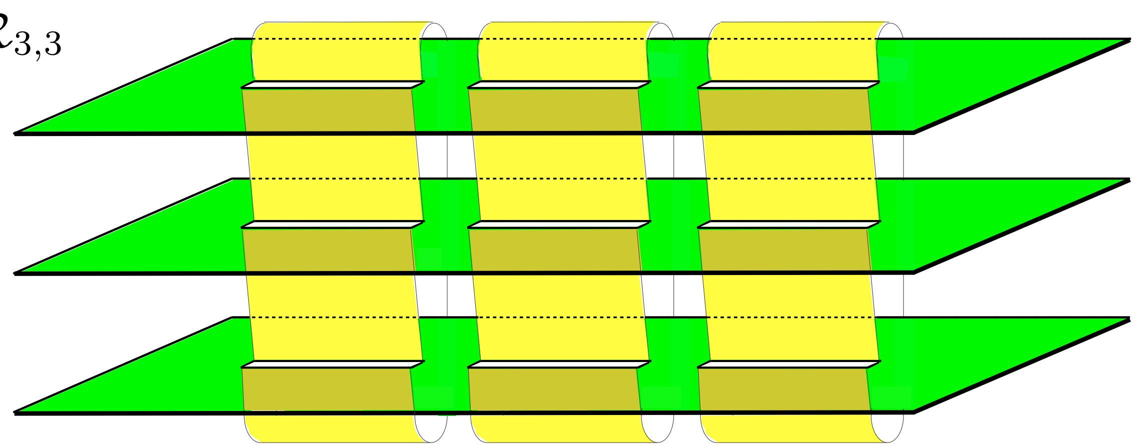

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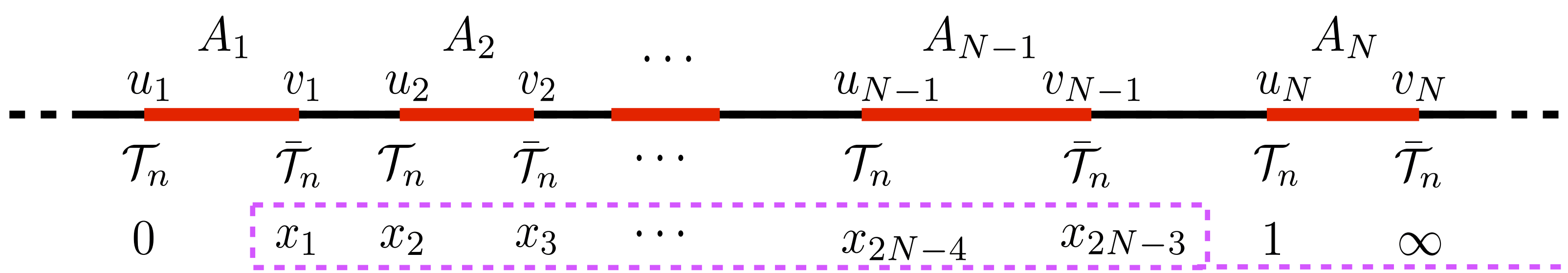

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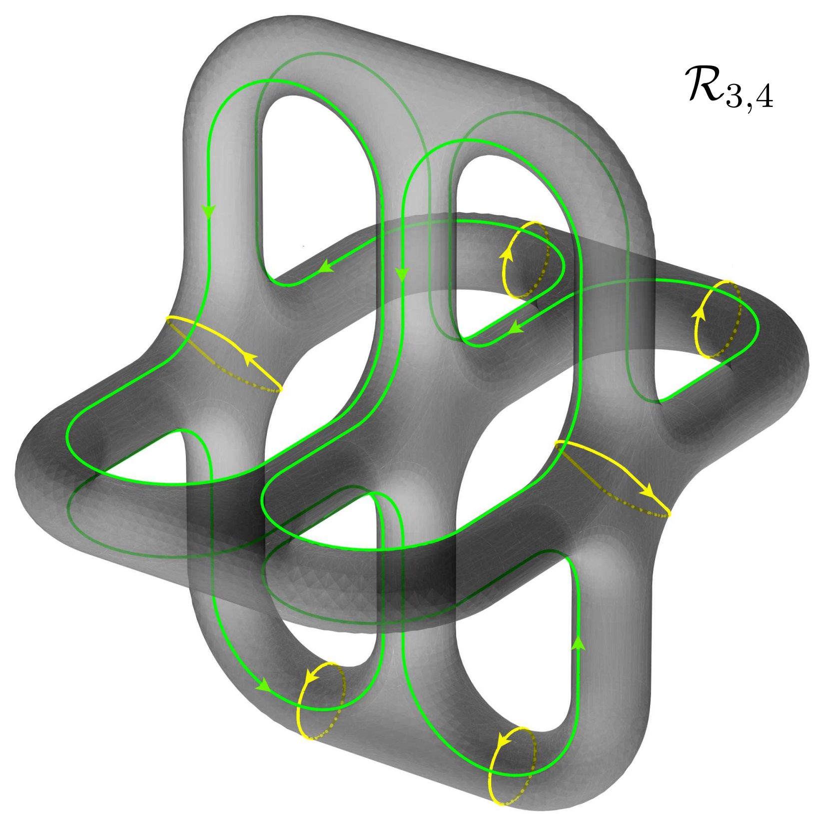

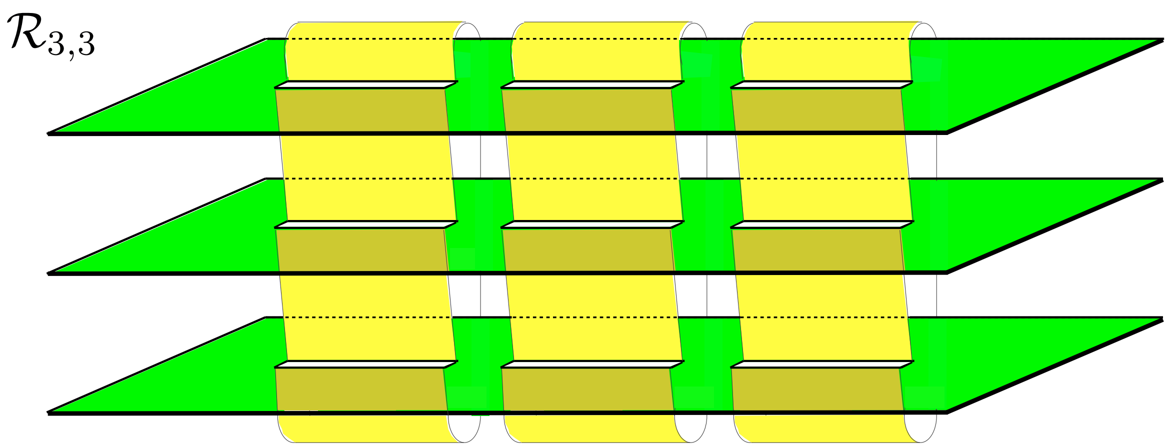

The period matrix

$$y^n = \prod_{\gamma=1}^{N} (z - x_{2\gamma-2}) \left[\prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \right]^{n-1} \qquad g = (N-1)(n-1)$$

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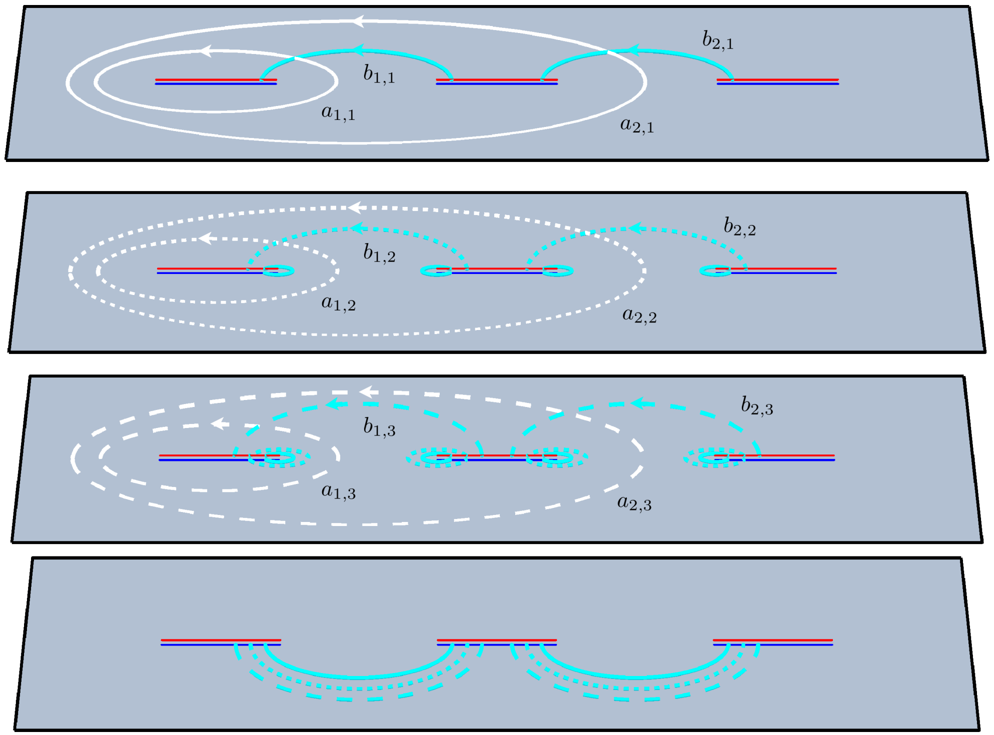

- canonical homology basis $\{a_{\alpha,j}, b_{\alpha,j}\}$

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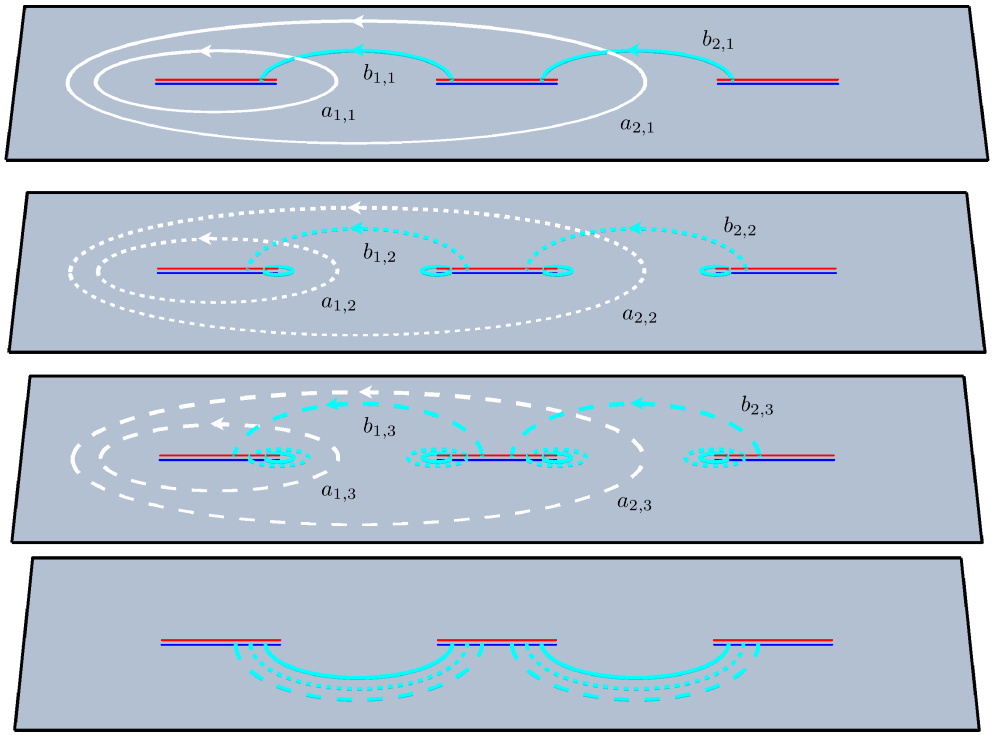

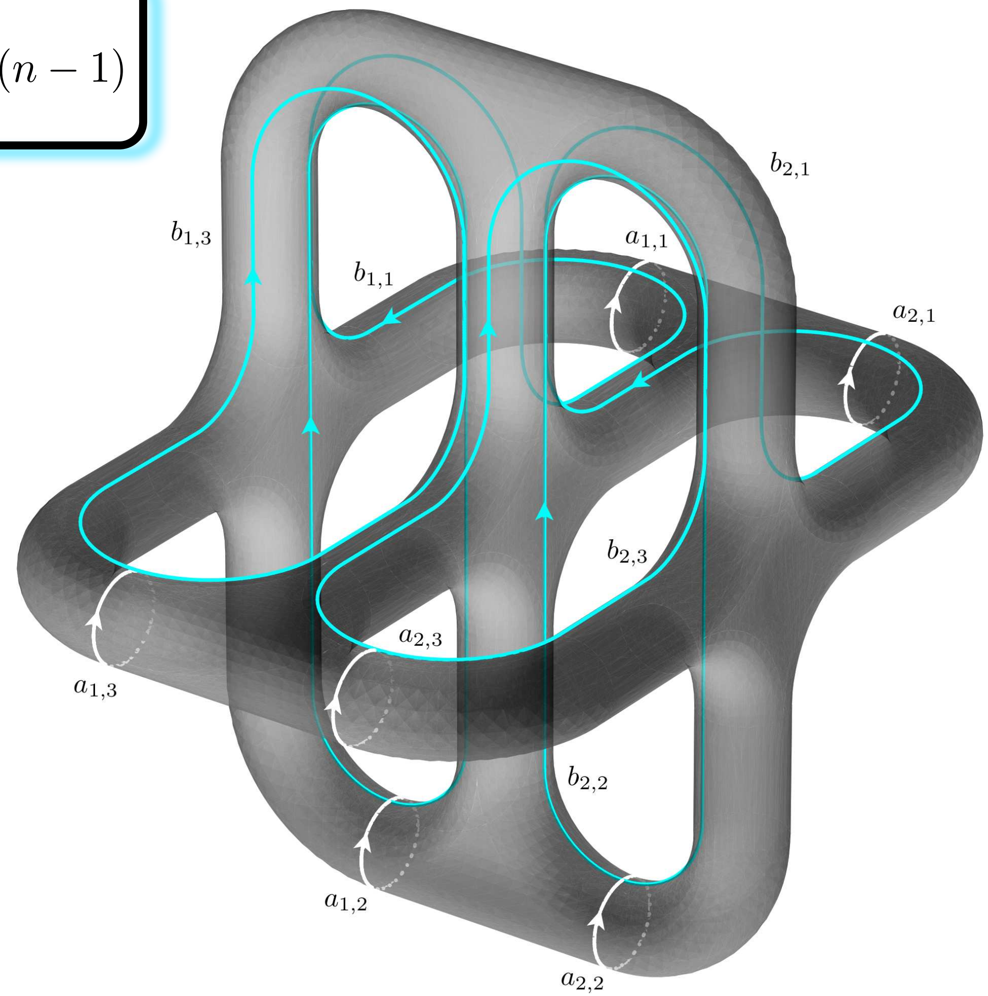

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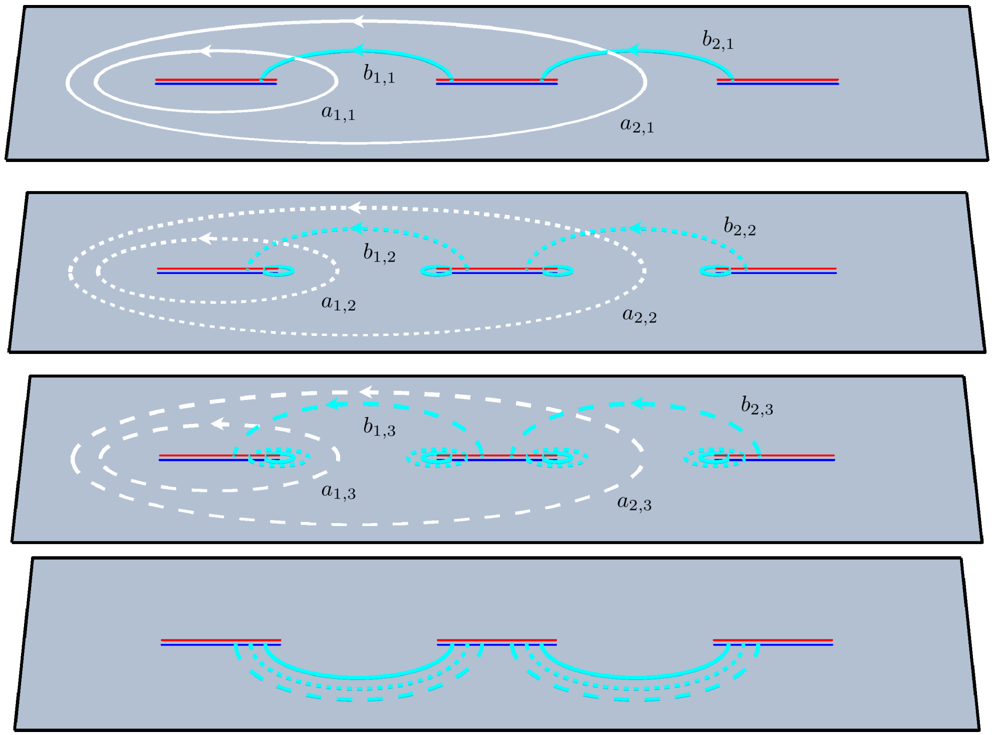

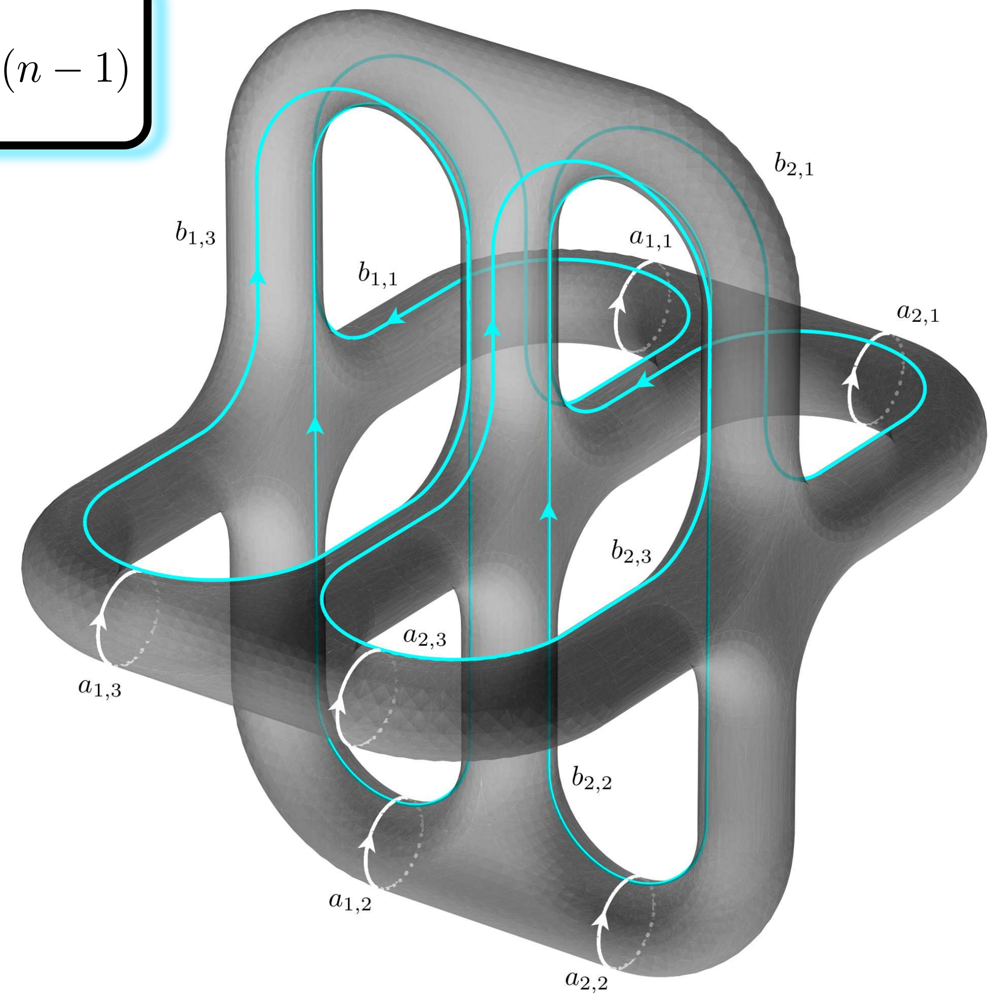

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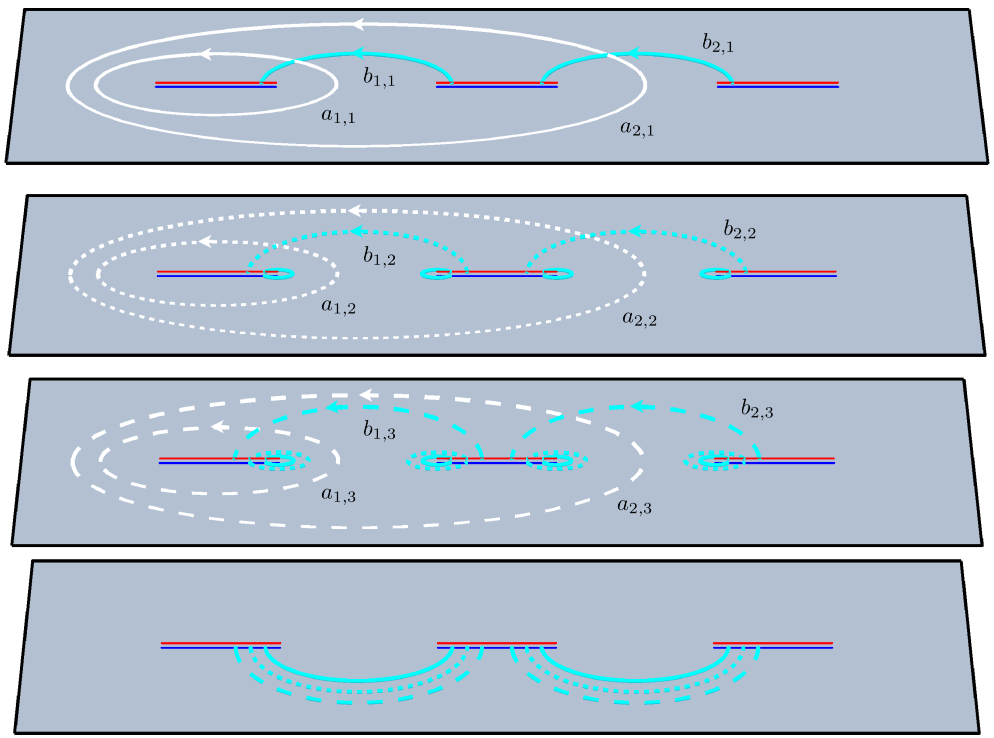

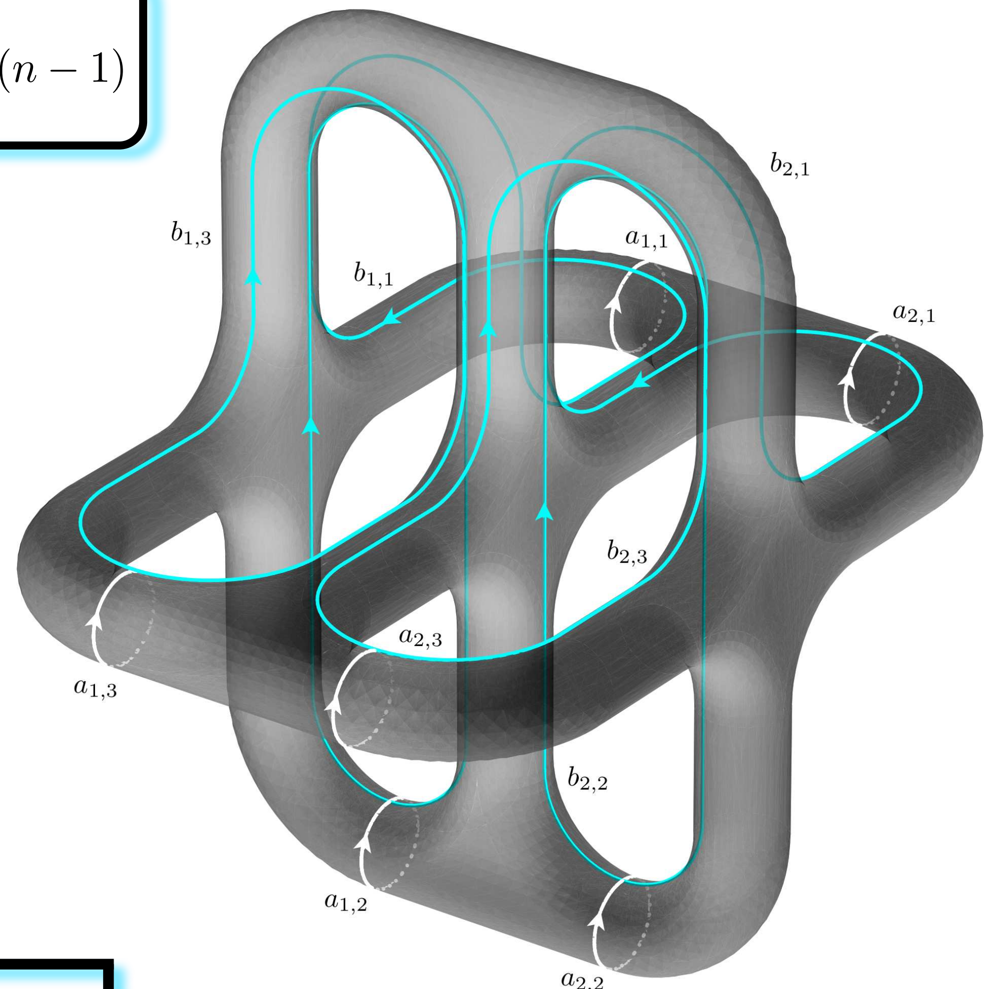

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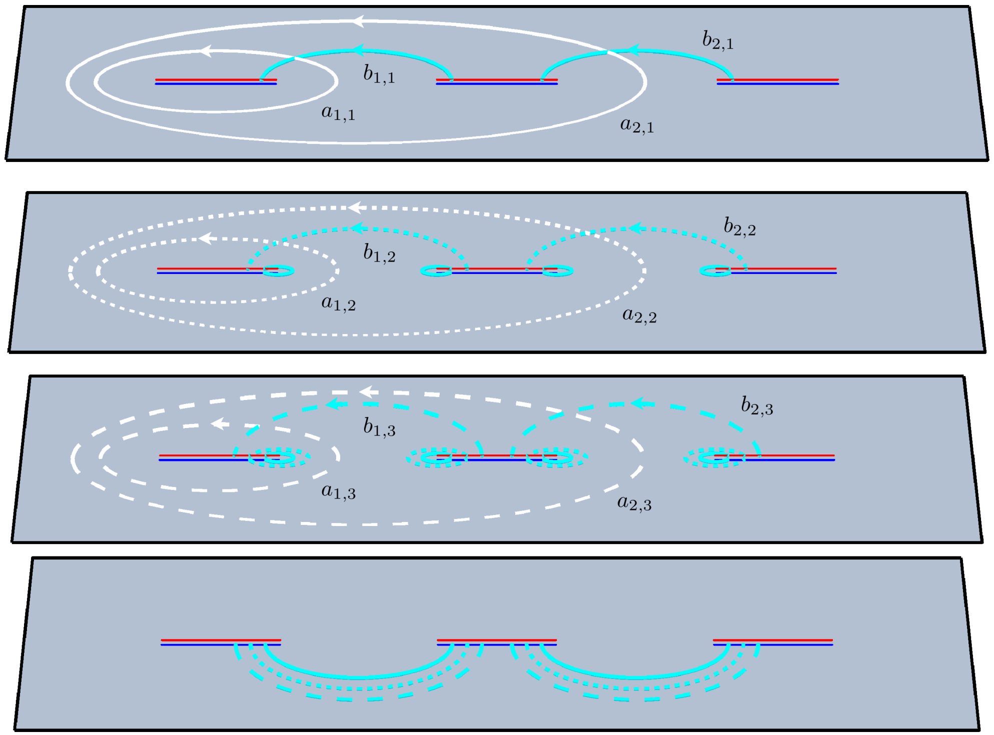

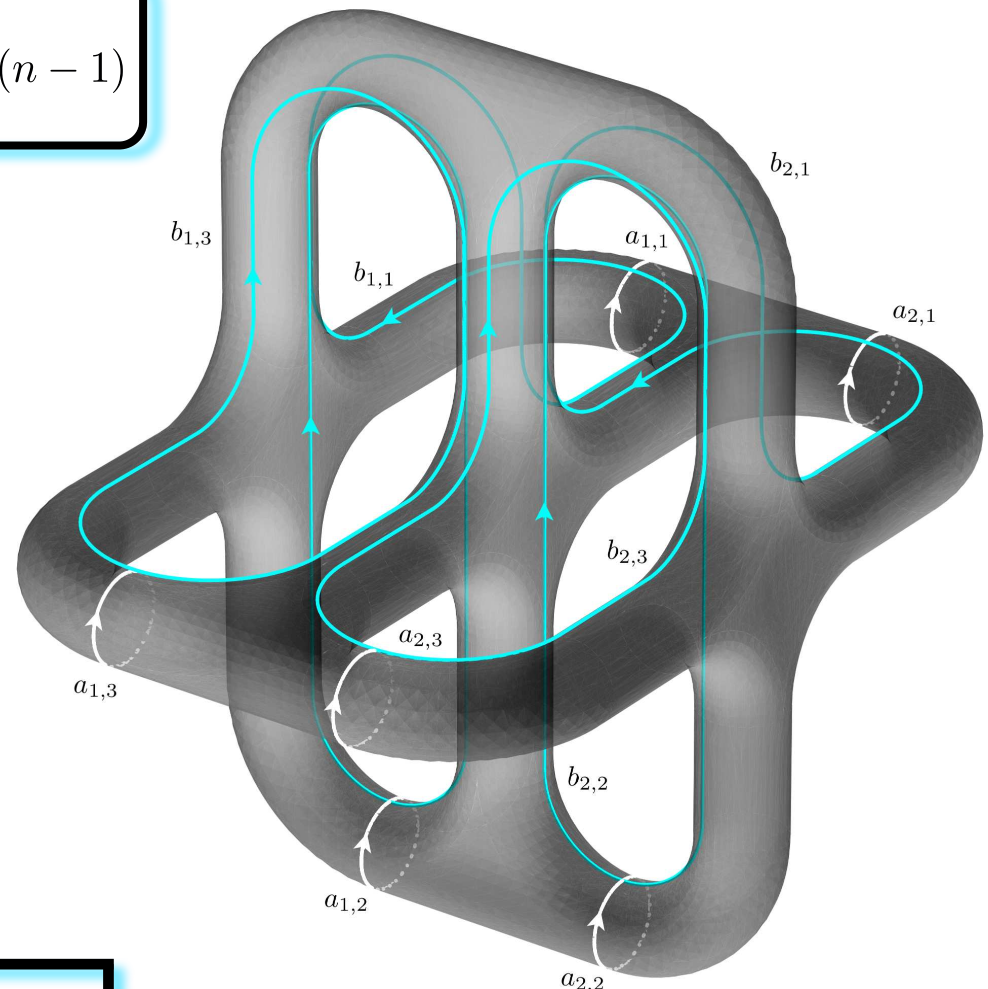

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$$\tau = \tau(\boldsymbol{x})$$

N intervals: free compactified boson & Ising model

- $\mathcal{R}_{N,n}$ is $$\boxed{y^n = \prod_{\gamma=1}^{N}(z - x_{2\gamma-2})\left[\prod_{\gamma=1}^{N-1}(z - x_{2\gamma-1})\right]^{n-1}}$$ $g = (N-1)(n-1)$ [Korotkin, (2003)] [Enolski, Grava, (2003)]

- Partition function for a generic Riemann surface studied long ago in string theory [Zamolodchikov, (1987)] [Alvarez-Gaume, Moore, Vafa, (1986)] [Dijkgraaf, Verlinde, Verlinde, (1988)]

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Nasty n dependence

The periodic harmonic chain

- Periodic chain of harmonic oscillators

$$H = \sum_{n=0}^{L-1} \left(\frac{1}{2M}\, p_n^2 + \frac{M\omega^2}{2}\, q_n^2 + \frac{K}{2} (q_{n+1} - q_n)^2 \right)$$

The massless case in the continuum limit is the $c = 1$ free boson on the line

[Peschel, Chung, JPA (1999)] [Botero, Reznik, PRA (2004)]

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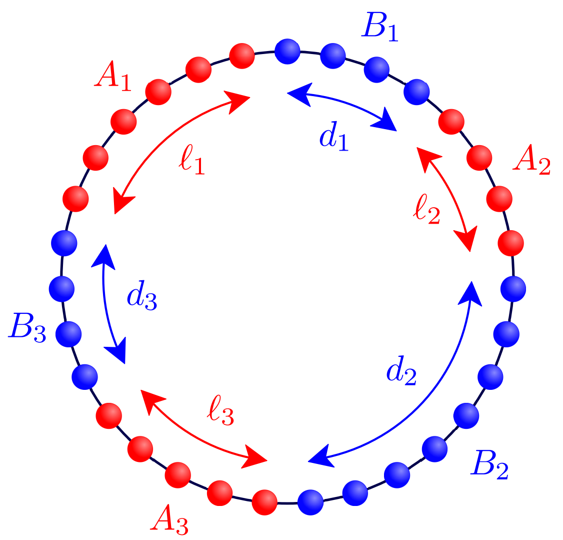

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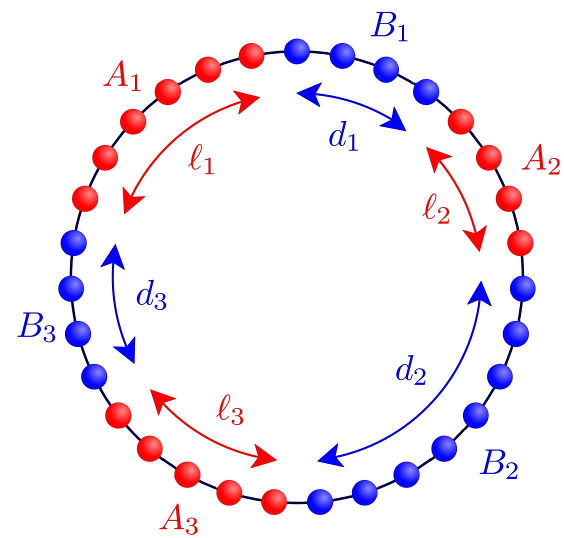


- Decompactification regime (large η)

$$\boxed{\mathcal{F}_{N,n}^{\mathrm{dec}}(\boldsymbol{x}) = \frac{\eta^{g/2}}{\sqrt{\det(\mathcal{I})}\,|\Theta(\mathbf{0}|\tau)|^2}}$$

[Coser, Tagliacozzo, E.T., JSTAT (2014)]

The periodic harmonic chain

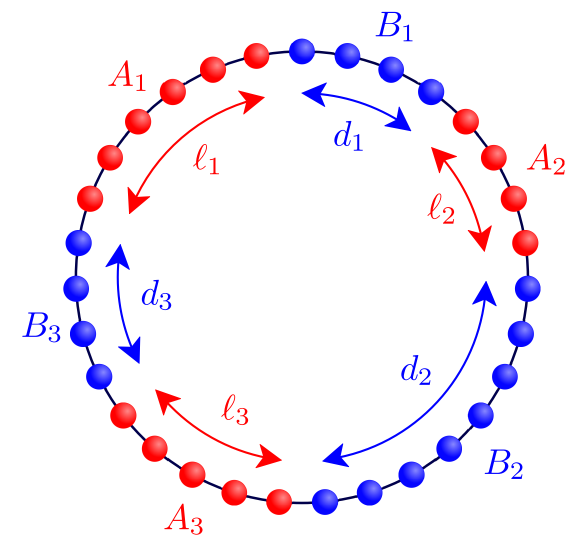
- Periodic chain of harmonic oscillators

$$H = \sum_{n=0}^{L-1} \left(\frac{1}{2M} p_n^2 + \frac{M\omega^2}{2} q_n^2 + \frac{K}{2} (q_{n+1} - q_n)^2 \right)$$

The massless case in the continuum limit is the $c = 1$ free boson on the line

[Peschel, Chung, JPA (1999)] [Botero, Reznik, PRA (2004)]

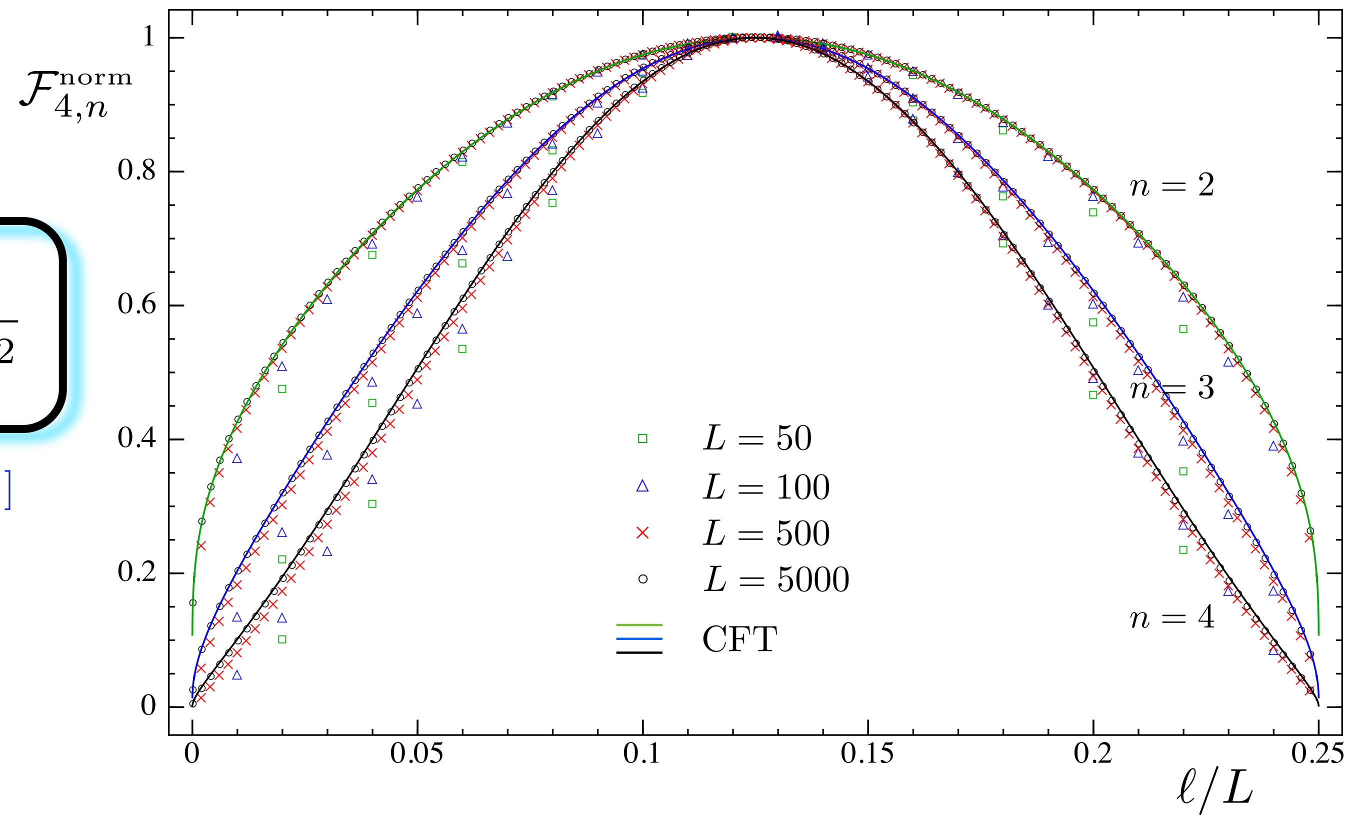
[Audenaert, Eisert, Plenio, Werner, PRA (2002)]



- Decompactification regime (large η)

$$\boxed{\mathcal{F}_{N,n}^{\text{dec}}(\boldsymbol{x}) = \frac{\eta^{g/2}}{\sqrt{\det(\mathcal{I})}\, |\Theta(\mathbf{0}|\tau)|^2}}$$

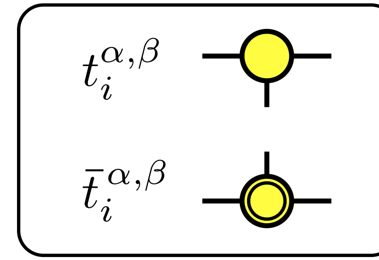
[Coser, Tagliacozzo, E.T., JSTAT (2014)]



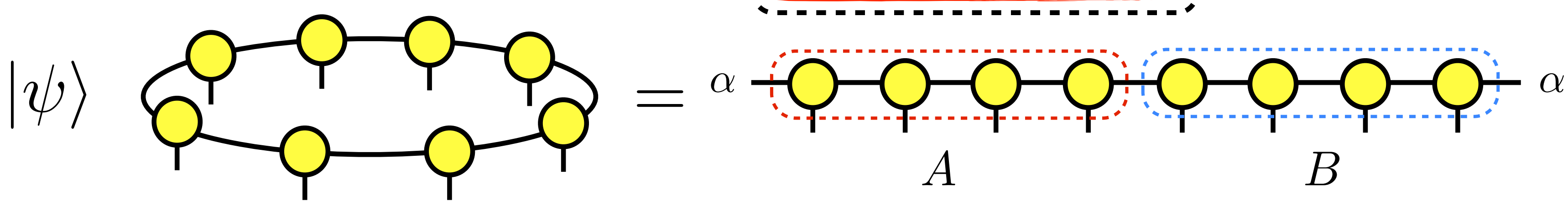
Numerics for the Ising model: Matrix Product States (I)

[White, Noack, PRL (1992)] [Ostuld, Rommer, PRL (1995)] [Vidal, PRL (2003)] [Verstraete, Cirac, PRB (2006)] [Hastings, JSTAT (2007)] $\cdots$

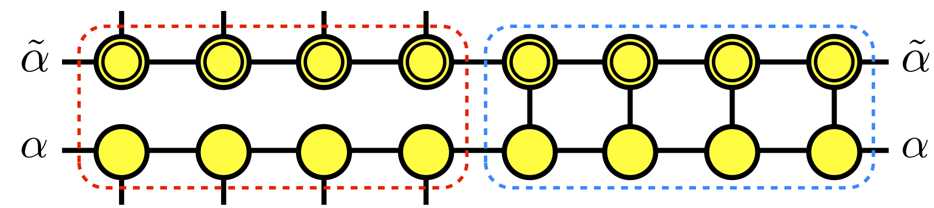
Ground state $\quad |\Psi\rangle = \sum_{i_1,\ldots,i_8=1}^{\delta} \boxed{\sum_{\alpha_1,\ldots,\alpha_8=1}^{\chi} t_{i_1}^{\alpha_1\alpha_2}\, t_{i_2}^{\alpha_2\alpha_3} \cdots t_{i_8}^{\alpha_8\alpha_1}} |i_1\rangle|i_2\rangle\cdots|i_8\rangle$



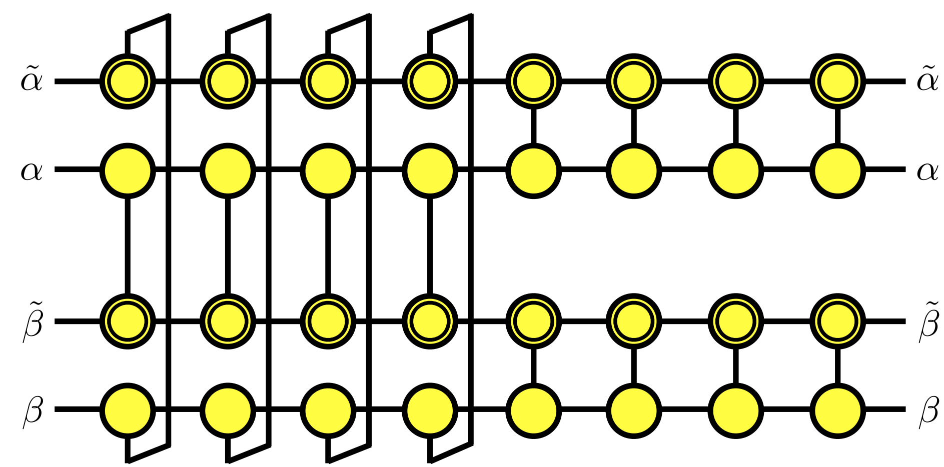
ansatz for $T_{i_1\cdots i_L}$



Reduced density matrix ρ_A



$\mathrm{Tr}\rho_A^2$



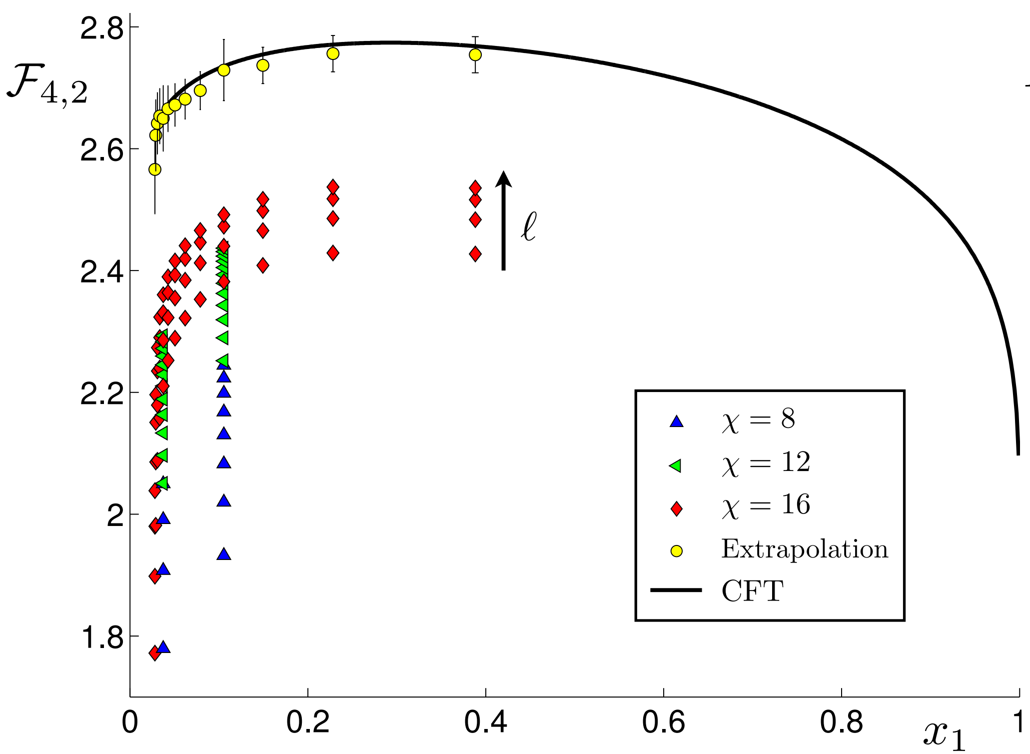
- $t_i^{\alpha\beta}$ are obtained by minimizing $\langle\Psi|H|\Psi\rangle$
- The accuracy of the result depends on the bond dimension χ

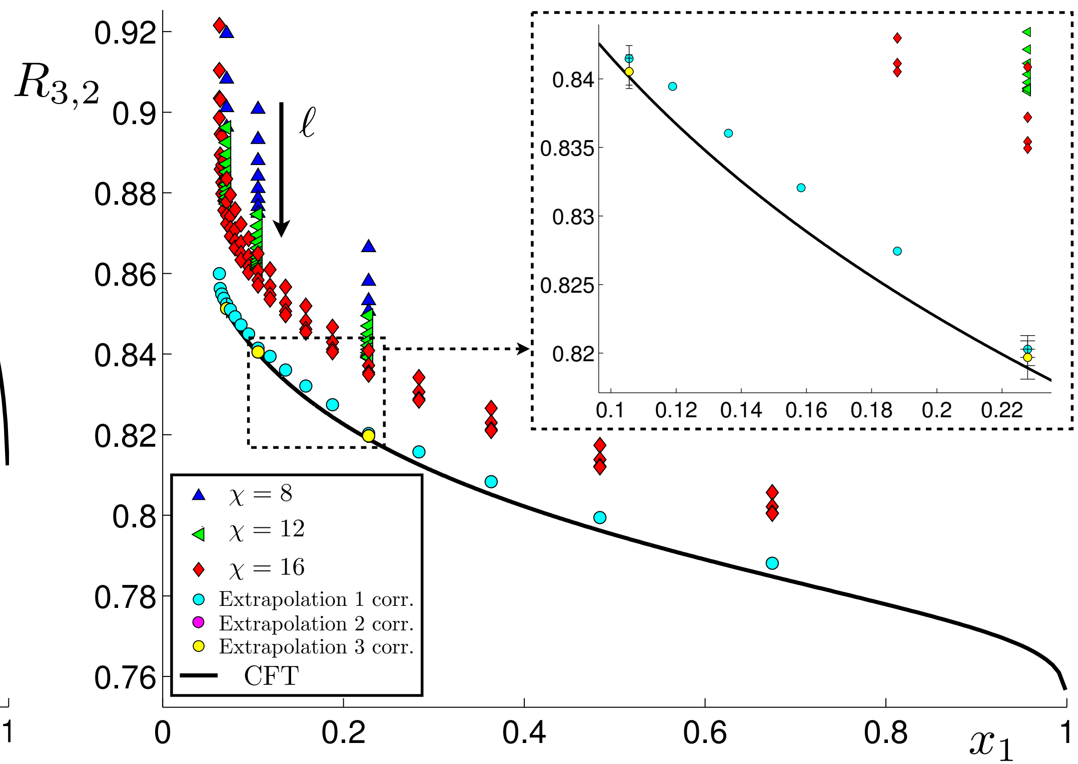
Numerics for the Ising model: Matrix Product States (II)

- The $N = 2$ case has been studied numerically through various methods
  [Caraglio, Gliozzi, JHEP (2008)] [Furukawa, Pasquier, Shiraishi, PRL (2009)]
  [Alba, Tagliacozzo, Calabrese, PRB (2010); JSTAT (2011)]
  [Fagotti, Calabrese, JSTAT (2010)]

- For $N > 2$ we considered Ising chain with $30 \leqslant L \leqslant 500$.
  Variational algorithm of [Pirvu, Verstraete, Vidal, PRB (2011)]
  Finite size corrections must be taken into account

[Coser, Tagliacozzo, E.T., JSTAT (2014)]

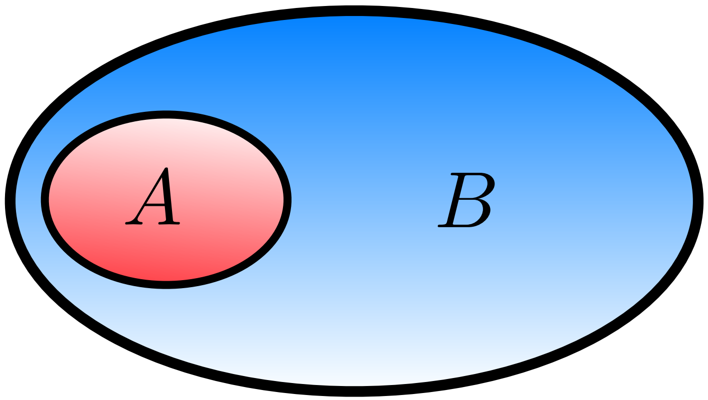




Motivations for Negativity

Motivations for Negativity

- Quantum system ($\mathcal{H}$) in the ground state $|\Psi\rangle$ $\Longrightarrow$ $\rho = |\Psi\rangle\langle\Psi|$ $(\mathrm{Tr}\rho^n = 1)$
- *Bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



reduced density matrix $\boxed{\rho_A = \mathrm{Tr}_B \rho}$

Entanglement entropy $\boxed{S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A)}$

$S_A = S_B$ for pure states

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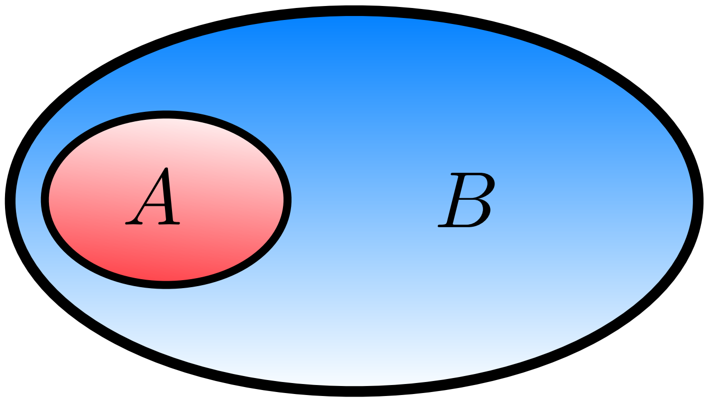

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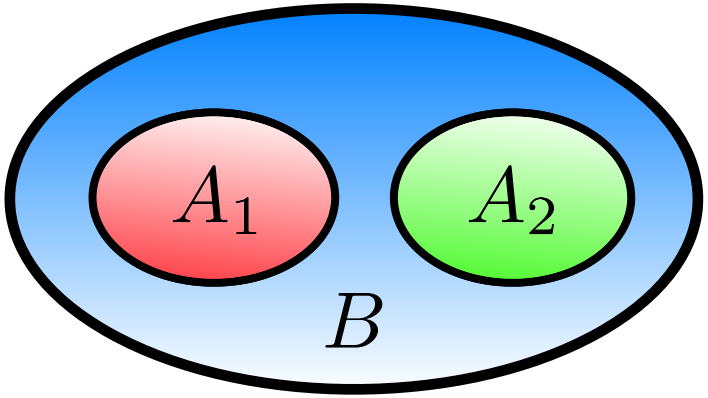

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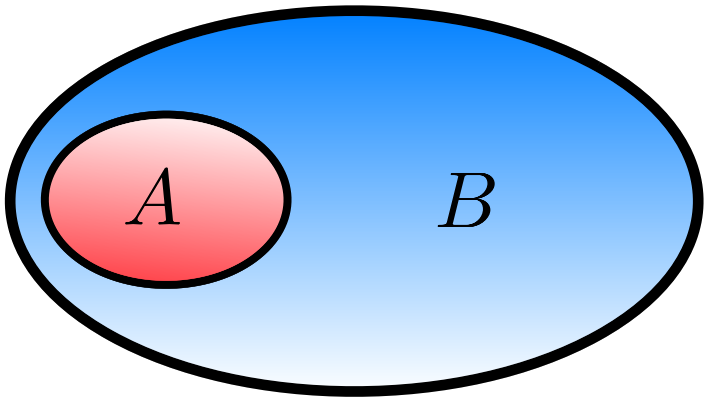

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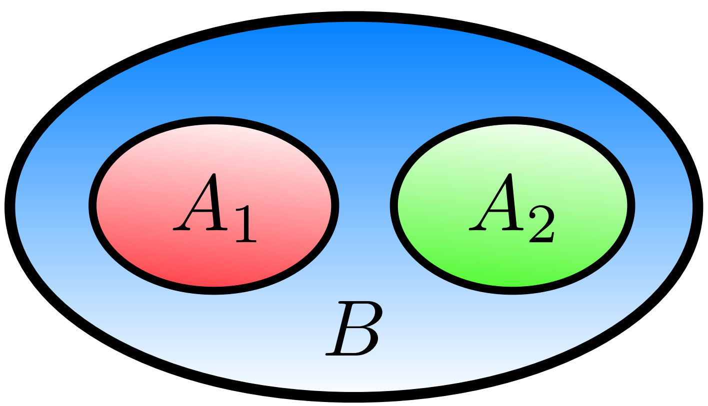

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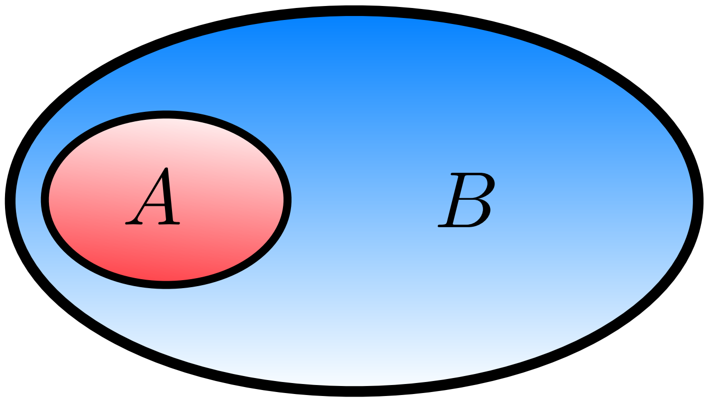

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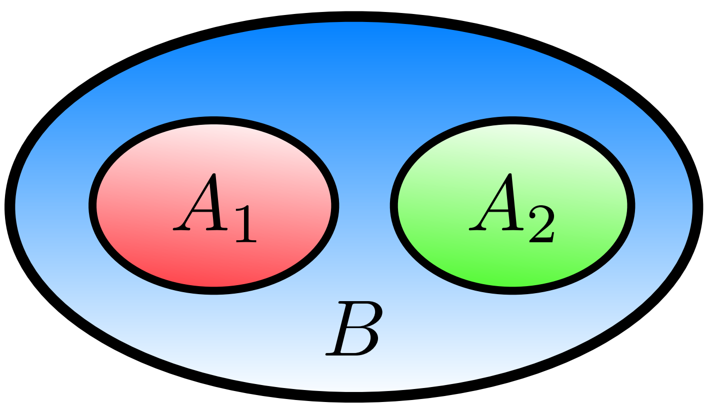
$S_{A_1 \cup A_2}$: entanglement between $A_1 \cup A_2$ and B



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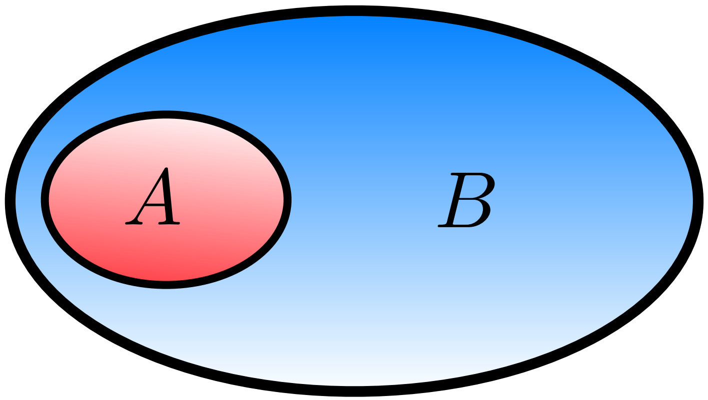

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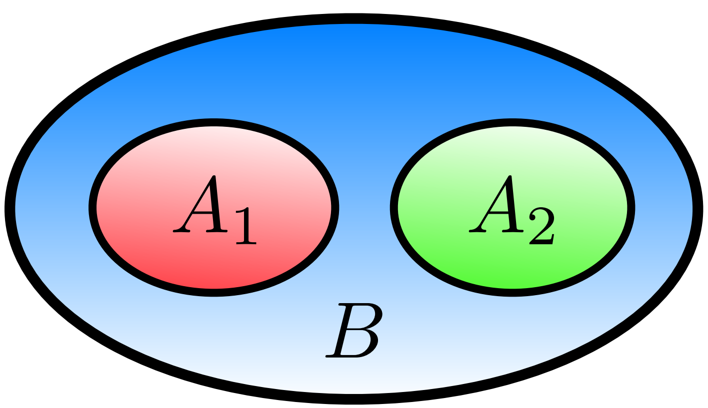

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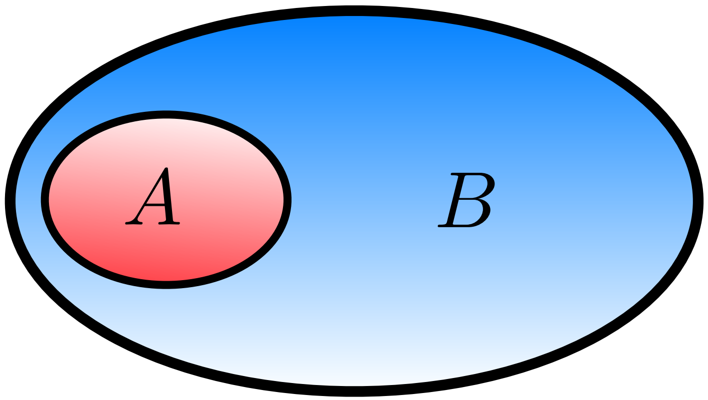

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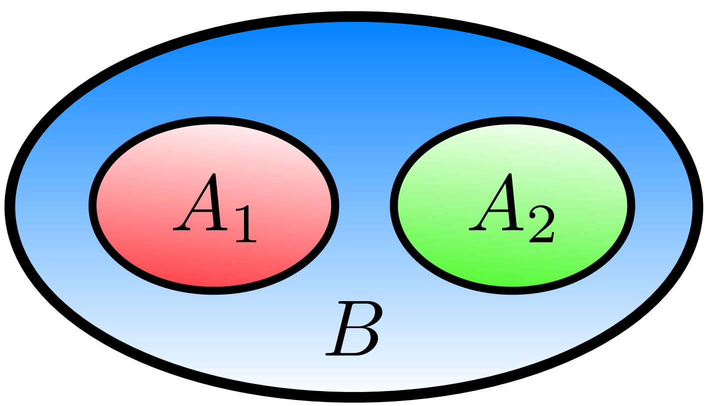

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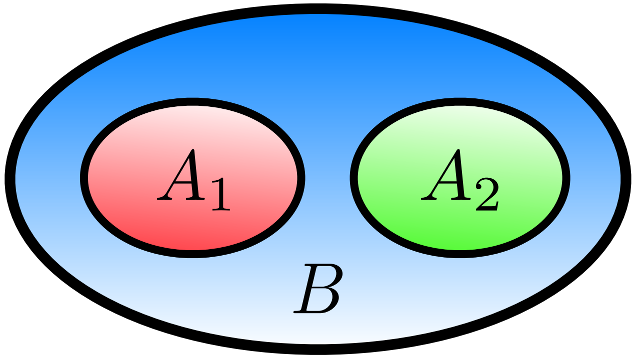
What about the entanglement between A_1 and A_2?

A *computable* measure of the entanglement
is the *logarithmic negativity*

Partial transpose & Negativity: definitions

- $\rho = \rho_{A_1 \cup A_2}$ is a mixed state
  $|e_i^{(1)}\rangle$ and $|e_j^{(2)}\rangle$ bases of $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$



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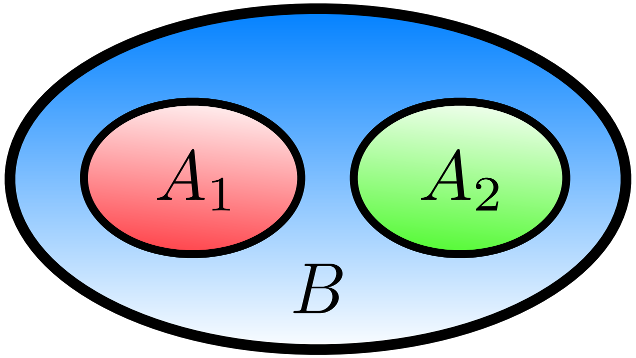

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$$\boxed{\langle e_i^{(1)} e_j^{(2)} |\, \rho^{T_2}\, | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} |\, \rho\, | e_k^{(1)} e_j^{(2)} \rangle}$$

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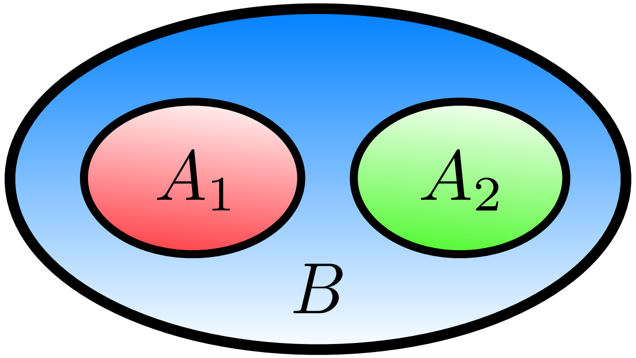

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  λ_j eigenvalues of ρ^{T_2}

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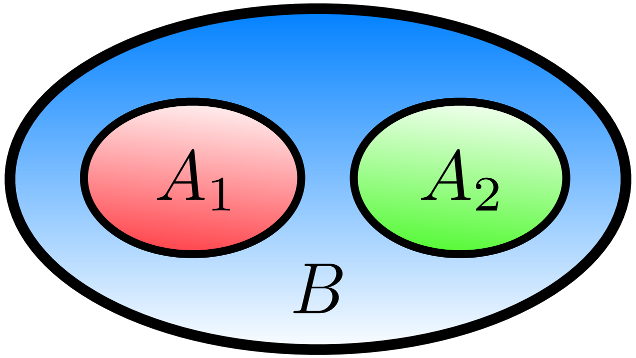

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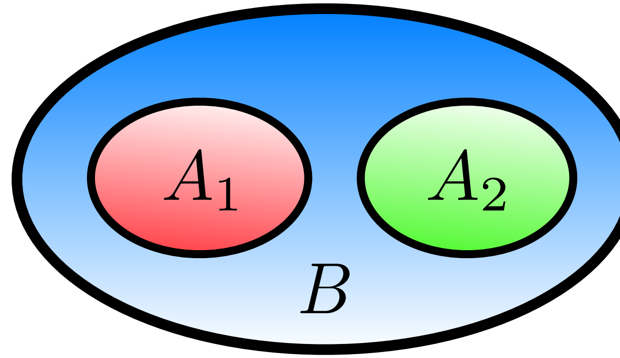

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- Same definition for a bipartite system
  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a *generic* state ρ $\longrightarrow$ $\mathcal{E}_A = \mathcal{E}_B$

Replica approach to Negativity

- $$\boxed{S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \operatorname{Tr} \rho_A^n}$$

  Renyi entropies $S_n = \dfrac{\log \operatorname{Tr} \rho_A^n}{1-n} \xrightarrow[n \to 1]{} S_A$

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$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

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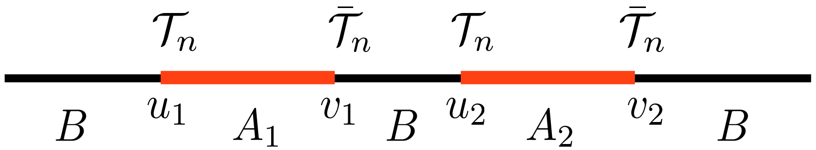
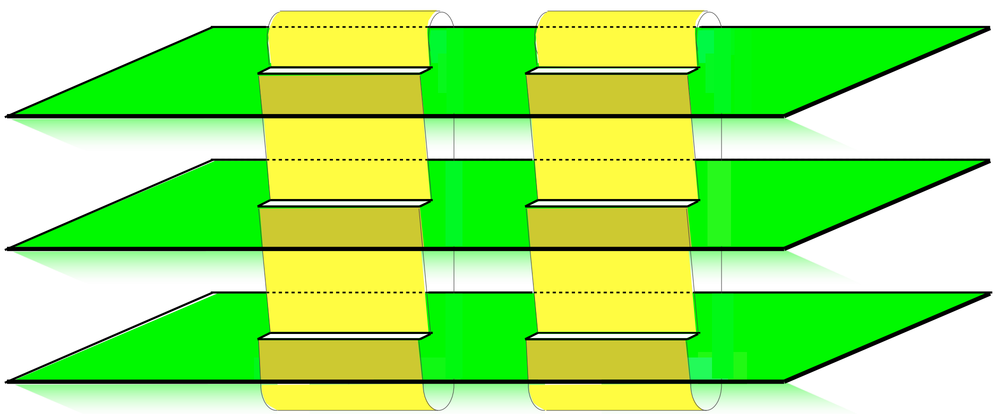
Analytic continuation on the *even* sequence $\mathrm{Tr}(\rho^{T_2})^{n_e}$ (*make 1 an even number*)

Partial transposition: two disjoint intervals

$$\mathrm{Tr}\rho^n_{A_1\cup A_2}$$





$$\mathrm{Tr}\rho^n_A \;=\; \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

[Caraglio, Gliozzi, (2008)]

[Furukawa, Pasquier, Shiraishi, (2009)]

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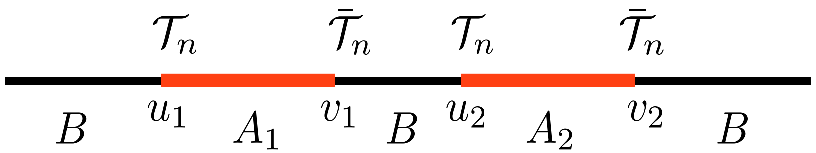

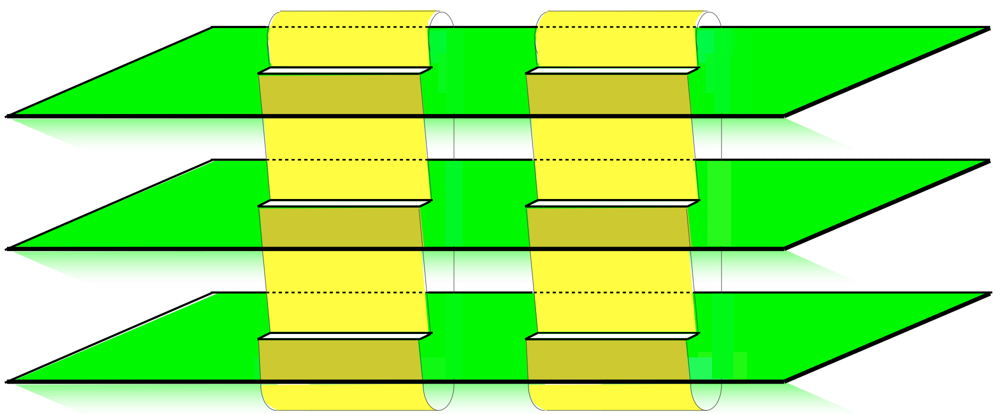

$$\mathrm{Tr}\rho^n_A = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

[Caraglio, Gliozzi, (2008)]
[Furukawa, Pasquier, Shiraishi, (2009)]
[Calabrese, Cardy, E.T., (2009), (2011)]
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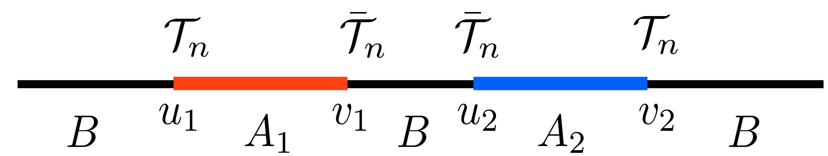
$$\mathrm{Tr}\left(\rho^{T_2}_{A_1 \cup A_2}\right)^n$$



- The partial transposition exchanges $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$

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Partial transposition: two disjoint intervals

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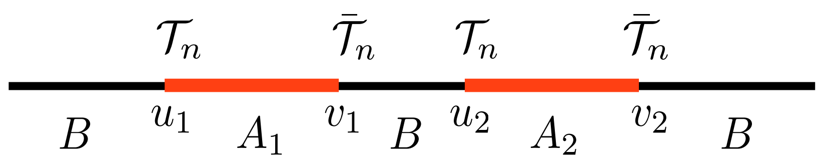

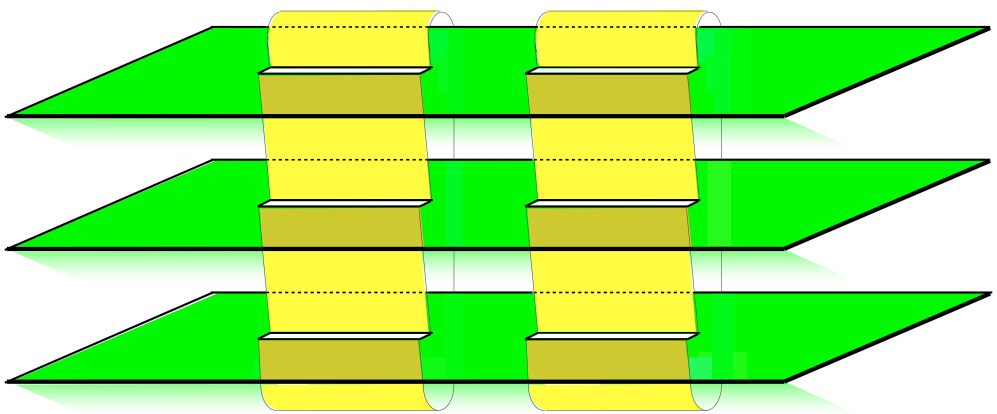

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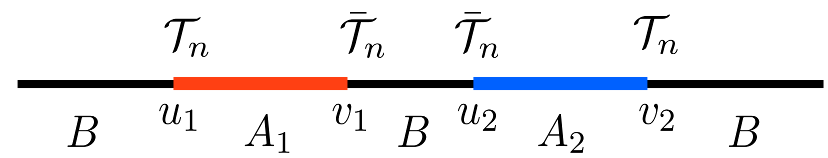

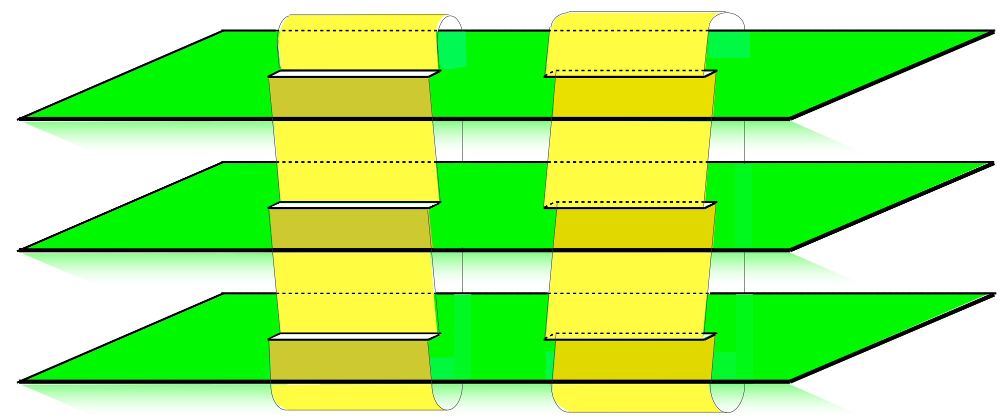

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Renyi entropies vs traces of the Partial Transpose

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Partial Transposition for bipartite systems: pure states (I)

[Calabrese, Cardy and E.T.; PRL (2012), JSTAT (2013)]

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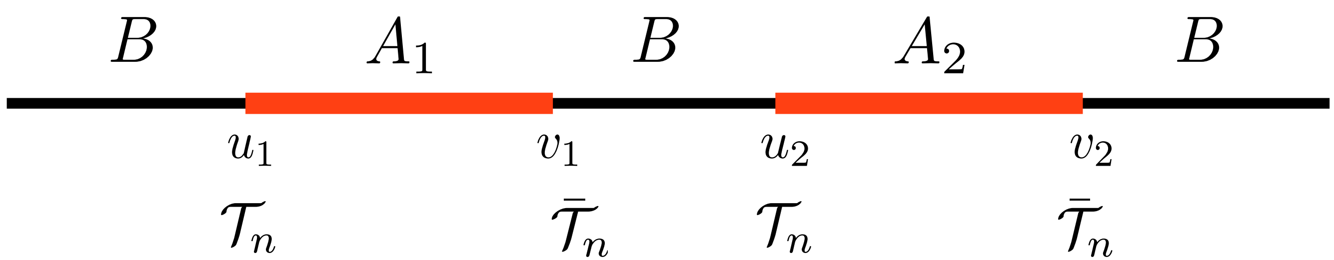

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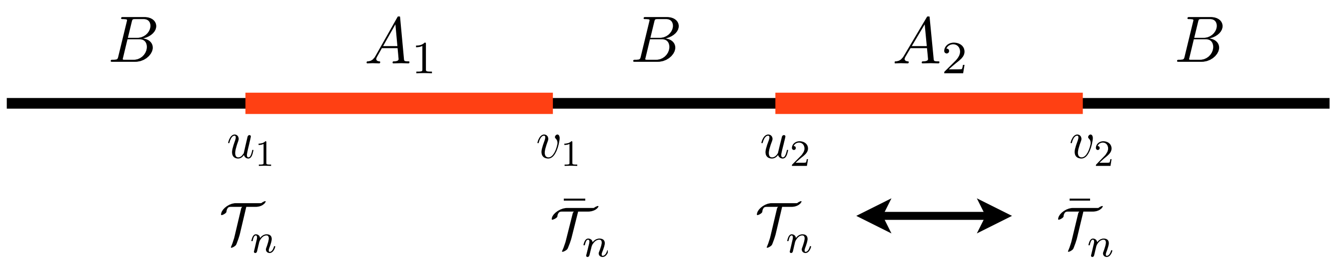

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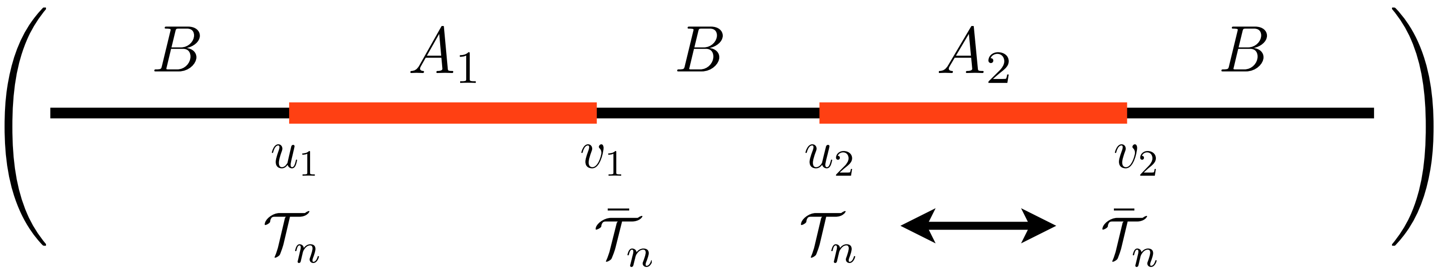

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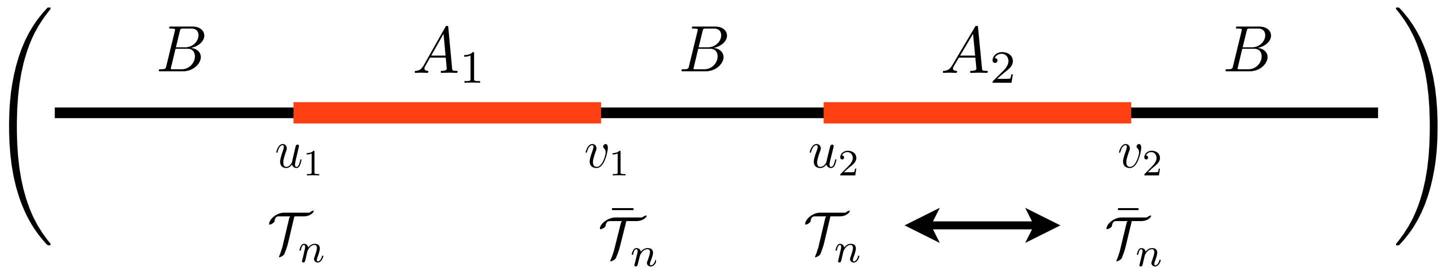

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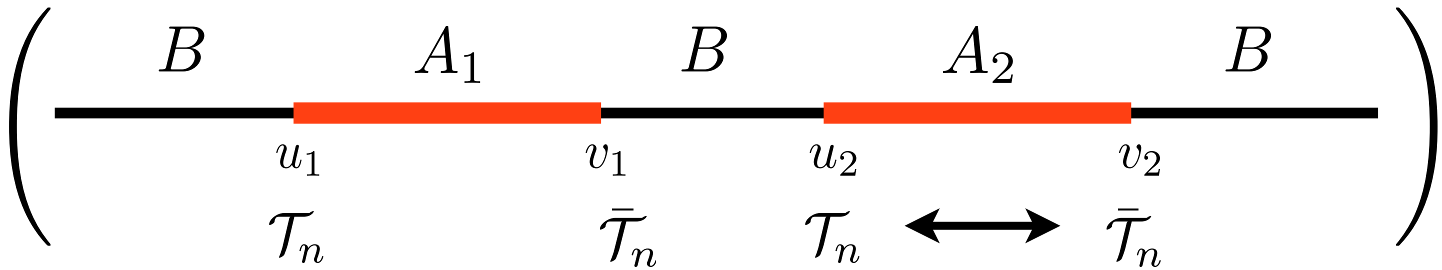

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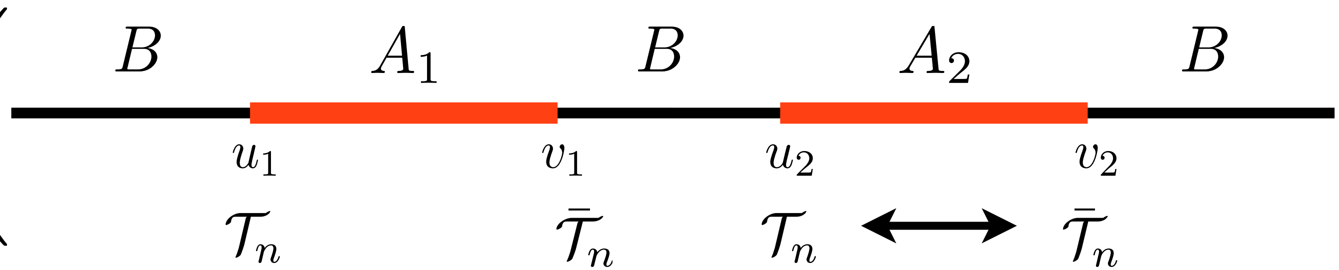

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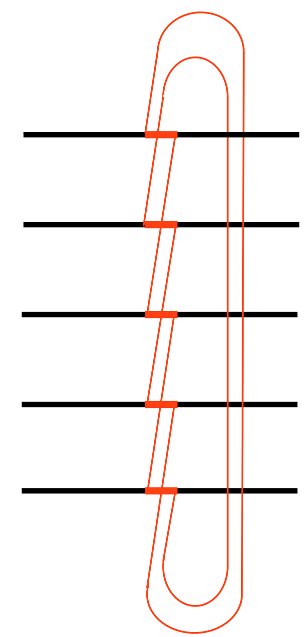

$n = 5$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} = \langle \mathcal{T}_{n_o}(u_2)\bar{\mathcal{T}}_{n_o}(v_2)\rangle = \mathrm{Tr}\,\rho_{A_2}^{n_o}$$

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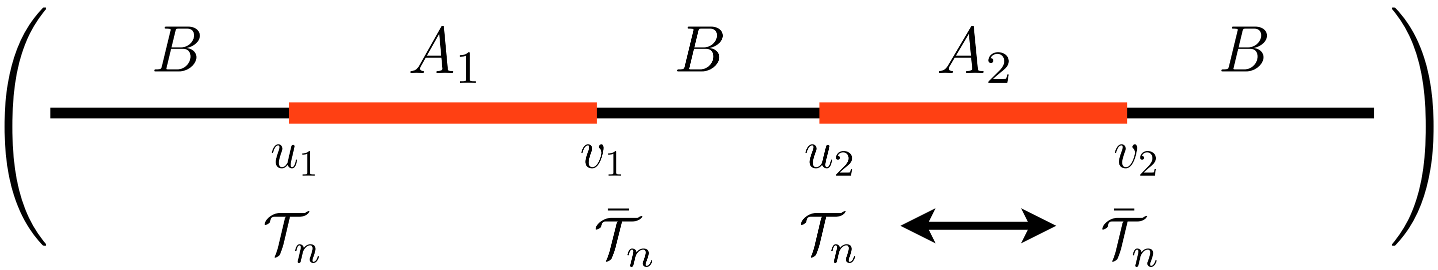

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- Even $n = n_e \implies$ decoupling

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = \left(\langle \mathcal{T}_{n_e/2}(u_2)\bar{\mathcal{T}}_{n_e/2}(v_2)\rangle\right)^2 = \left(\mathrm{Tr}\,\rho_{A_2}^{n_e/2}\right)^2$$

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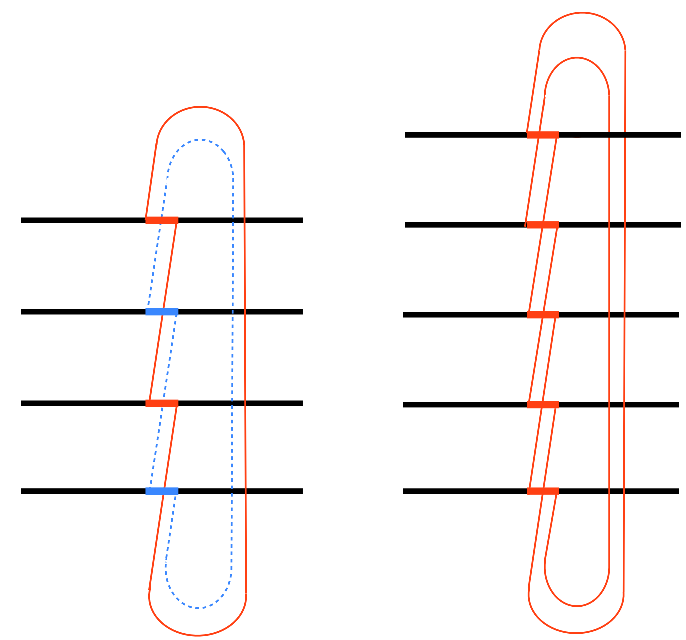

$n=4$ $\qquad$ $n=5$

Partial Transpose: pure states (II)

- $\rho = |\Psi\rangle\langle\Psi| \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

  $\Rightarrow$ *Schmidt decomposition* $|\Psi\rangle = \sum_k c_k\, |\Psi_k\rangle_1 |\Psi_k\rangle_2 \qquad c_k \geqslant 0 \qquad \sum_k c_k^2 = 1$

$$\boxed{\mathrm{Tr}(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\,\rho_2^n & n = n_o \quad \text{odd} \\ \left(\mathrm{Tr}\,\rho_2^{n/2}\right)^2 & n = n_e \quad \text{even} \end{cases}}$$

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$$\Delta_{\mathcal{T}^2_{n_e}} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right) \qquad\qquad \Delta_{\mathcal{T}^2_{n_o}} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right) = \Delta_{\mathcal{T}_{n_o}}$$

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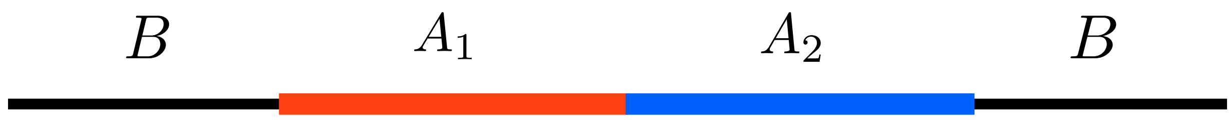
- For $n_e \to 1$ we find $\mathcal{E} = 2 \log \mathrm{Tr} \rho_2^{1/2}$ (Renyi entropy $1/2$)

Partial Transpose in 2D CFT: two adjacent intervals



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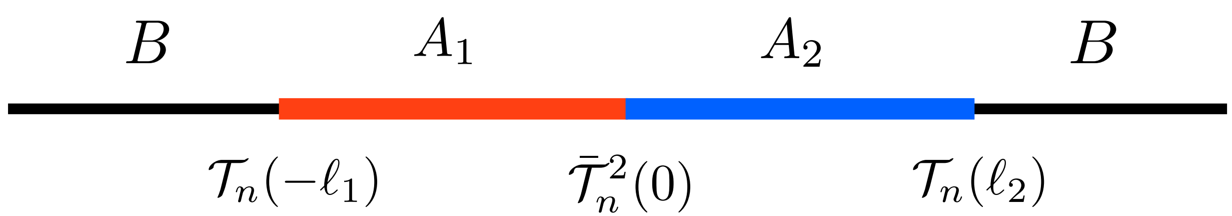

- Three point function

$$\boxed{\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle}$$

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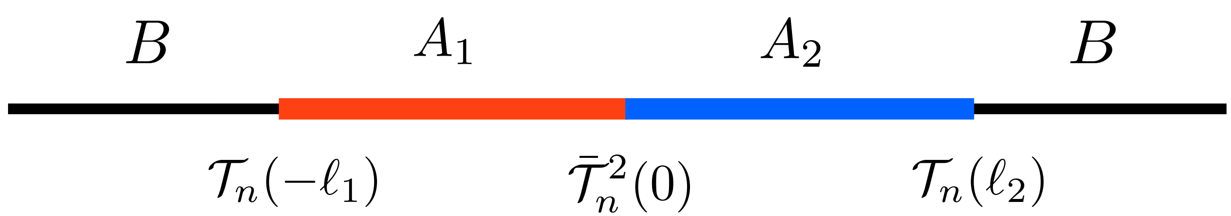

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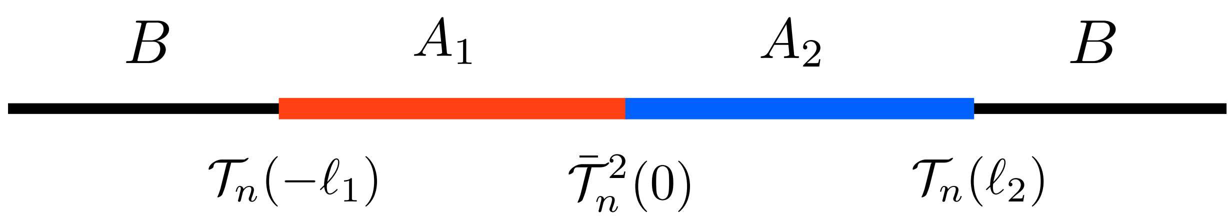

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Partial Transpose in 2D CFT: two disjoint intervals

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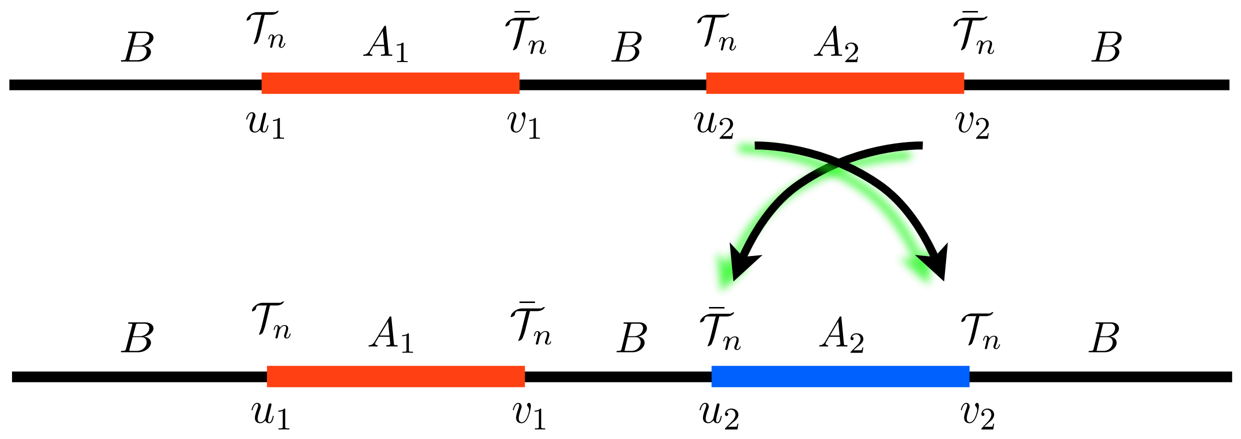

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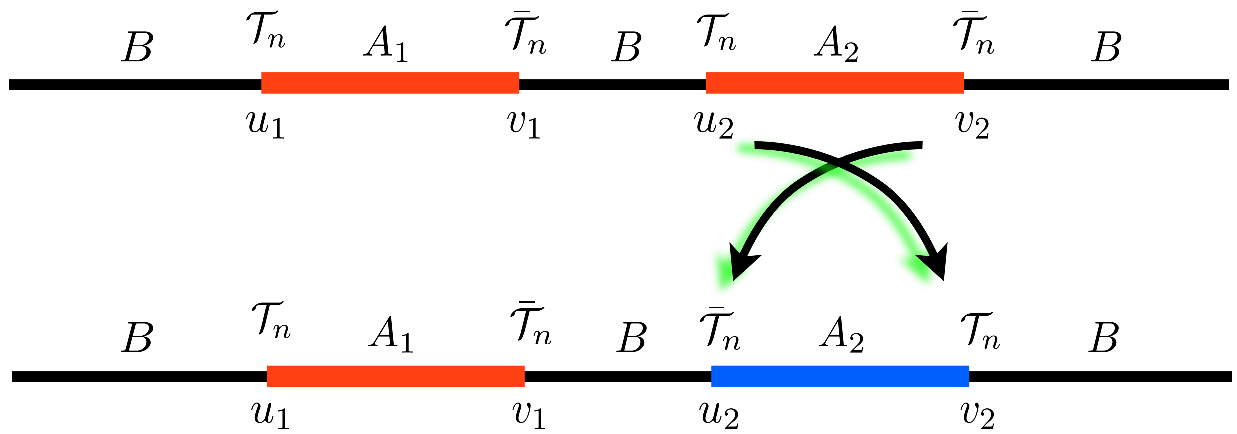

$$\boxed{\mathrm{Tr}\big(\rho^{T_2}_{A_1\cup A_2}\big)^n = c_n^2\big[\ell_1\ell_2(1-y)\big]^{-\frac{c}{6}(n-\frac{1}{n})}\mathcal{G}_n(y)}$$

- $\mathrm{Tr}\big(\rho^{T_2}_{A_1\cup A_2}\big)^n$ is obtained from $\mathrm{Tr}\rho^n_{A_1\cup A_2}$ by exchanging two twist fields

$$\mathcal{G}_n(y) = (1-y)^{\frac{c}{3}(n-\frac{1}{n})}\,\mathcal{F}_n\left(\frac{y}{y-1}\right)$$

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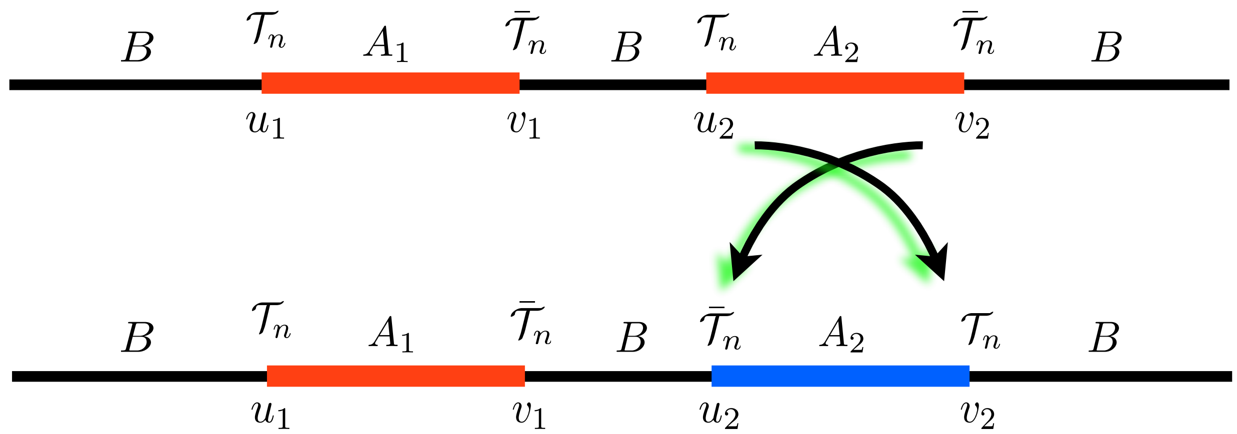

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- $$\boxed{\mathcal{E}(y) = \lim_{n_e\to 1}\mathcal{G}_{n_e}(y) = \lim_{n_e\to 1}\left[\mathcal{F}_n\left(\frac{y}{y-1}\right)\right]}$$

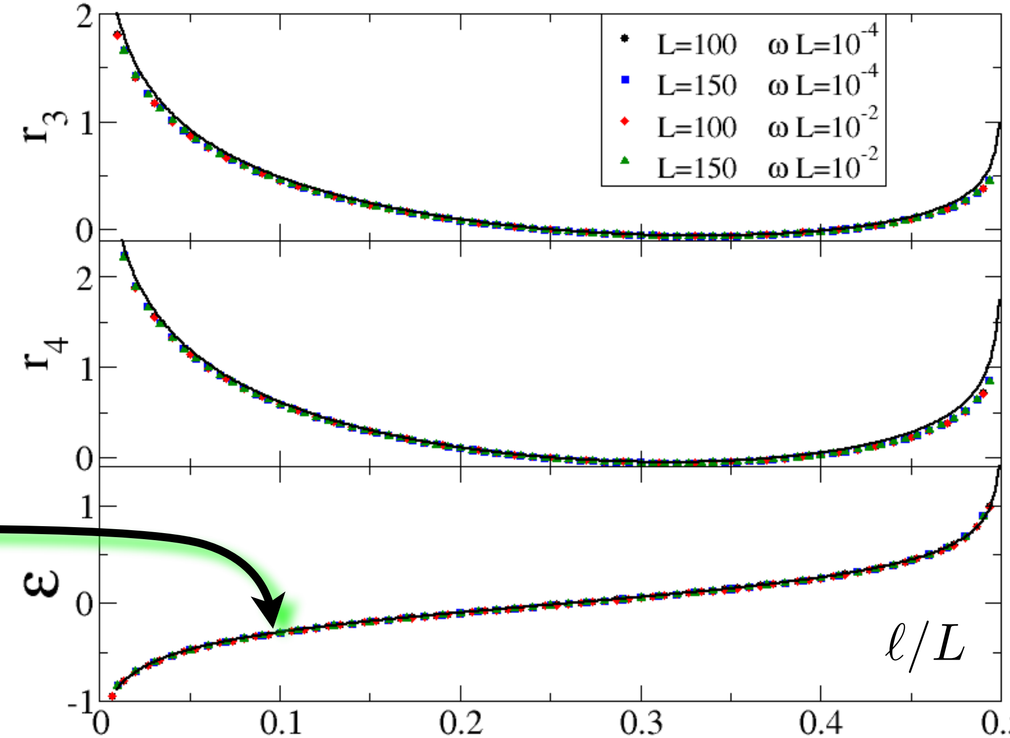
Two adjacent intervals: harmonic chain & Ising model

Critical periodic harmonic chain

Finite system: $\ell \longrightarrow (L/\pi)\sin(\pi\ell/L)$

$$r_n = \ln \frac{\mathrm{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\mathrm{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$

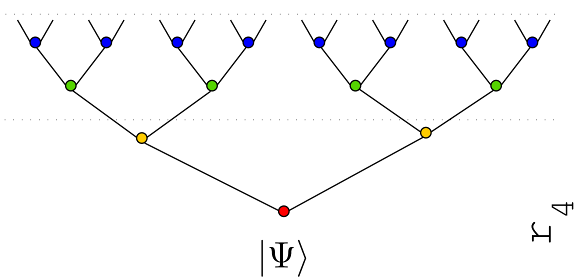
$$\boxed{\frac{1}{4}\log\frac{\sin(\pi\ell_1/L)\sin(\pi\ell_2/L)}{\sin(\pi[\ell_1+\ell_2]/L)} + \mathrm{cnst}}$$



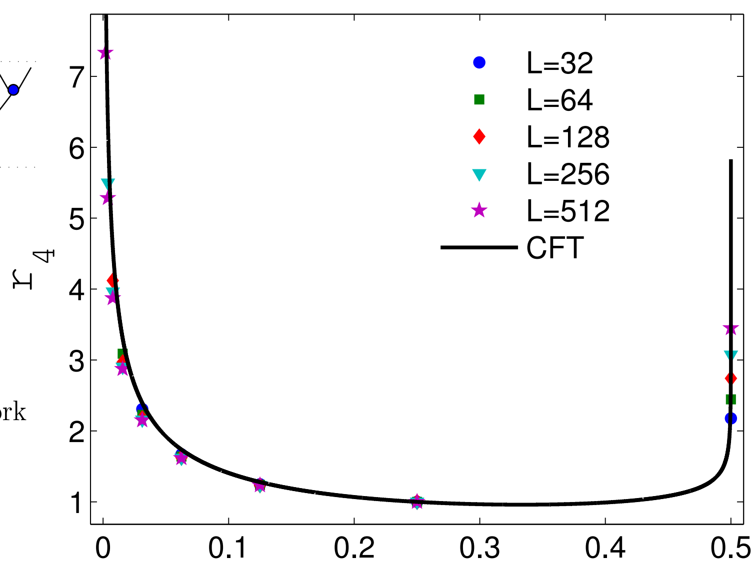
Ising model:

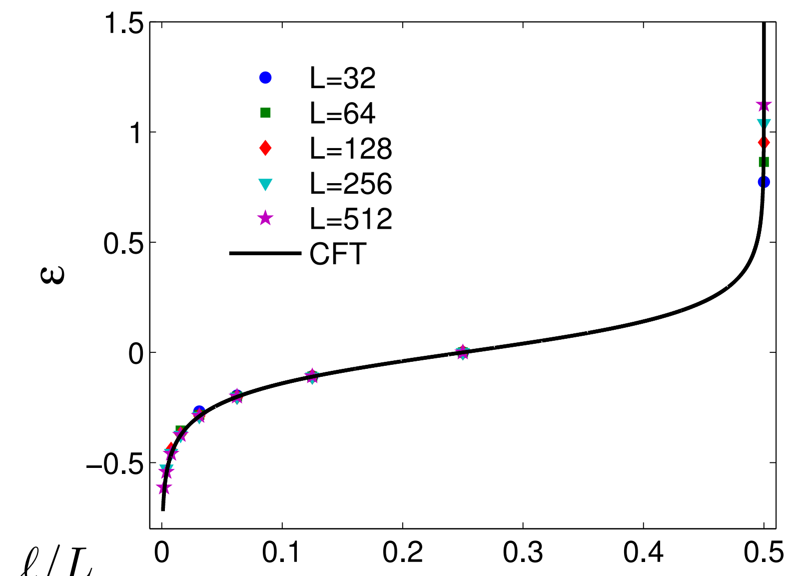
Monte-Carlo analysis [Alba, (2013)]

Tree Tensor Network [Calabrese, Tagliacozzo, E.T., (2013)]



TTN The state is encoded in the network made by smaller tensors





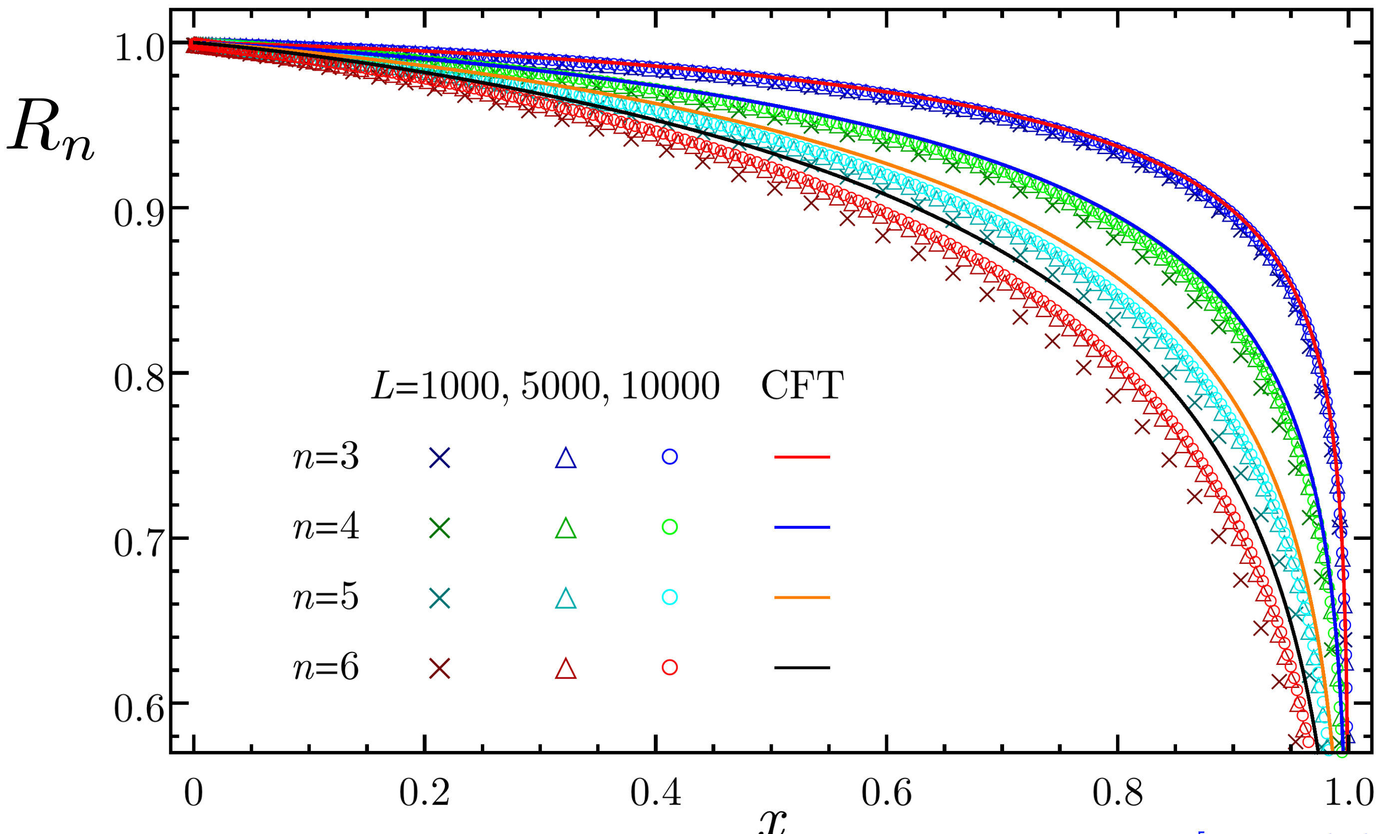
Two disjoint intervals: periodic harmonic chains

- Previous numerical results for $\mathcal{E}$: Ising (DMRG) and harmonic chains [Wichterich, Molina-Vilaplana, Bose, (2009)] [Marcovitch, Retzker, Plenio, Reznik, (2009)]

- Non compact free boson [Calabrese, Cardy, E.T., (2012)]

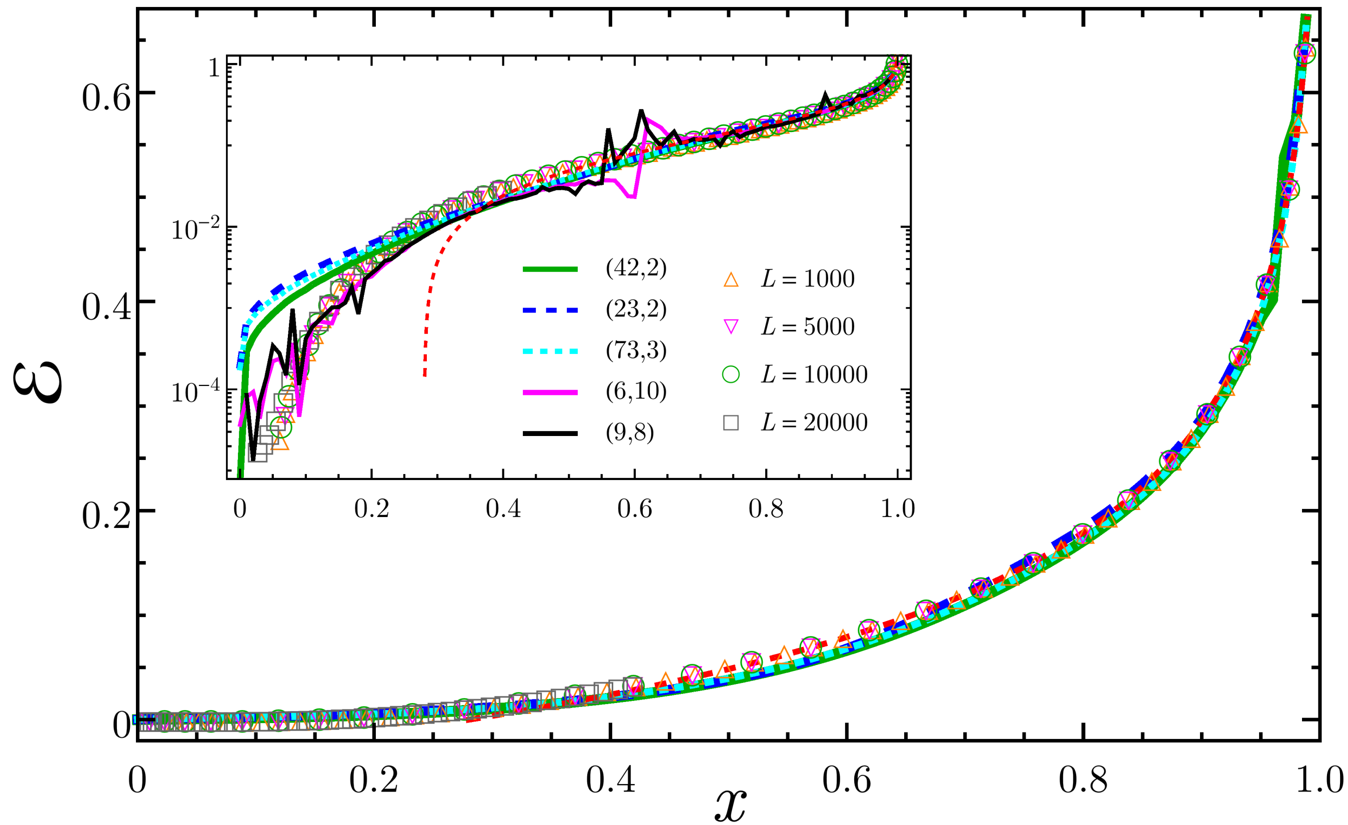
$$\boxed{R_n = \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\,\rho_A^n}}$$

$$R_n = \left[\frac{(1-x)^{\frac{2}{3}(n-\frac{1}{n})}\prod_{k=1}^{n-1} F_{\frac{k}{n}}(x)\,F_{\frac{k}{n}}(1-x)}{\prod_{k=1}^{n-1}\mathrm{Re}\big(F_{\frac{k}{n}}(\frac{x}{x-1})\,\bar{F}_{\frac{k}{n}}(\frac{1}{1-x})\big)}\right]^{\frac{1}{2}}$$



[De Nobili, Coser, E.T., (2015)]

Two disjoint intervals: periodic harmonic chains



- Analytic continuation for $x \sim 1$
  [Calabrese, Cardy, E.T., (2012)]

$$\boxed{\mathcal{E} = -\frac{1}{4}\log(1-x) + \log K(x) + \text{cnst}}$$

- Analytic continuation $n_e \to 1$ for $0 < x < 1$ not known
- $\mathcal{E}(x)$ for $x \sim 0$ vanishes faster than any power
- Numerical extrapolations (rational interpolation method) [De Nobili, Coser, E.T., (2015)]

Two disjoint intervals: Ising model

[Alba, (2013)] [Calabrese, Tagliacozzo, E.T., (2013)]

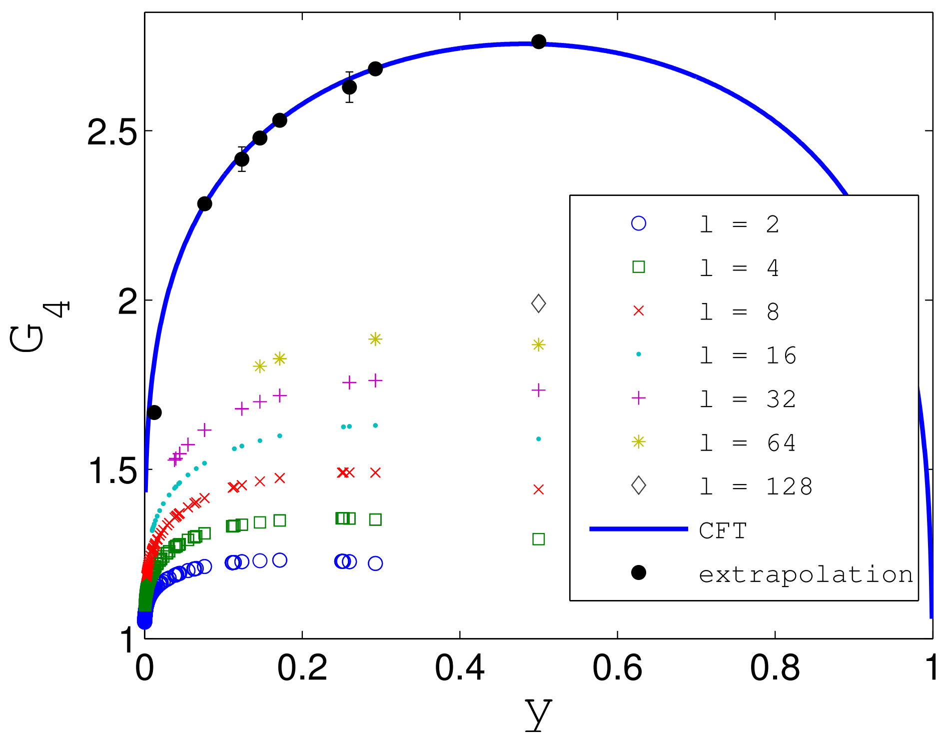
- CFT

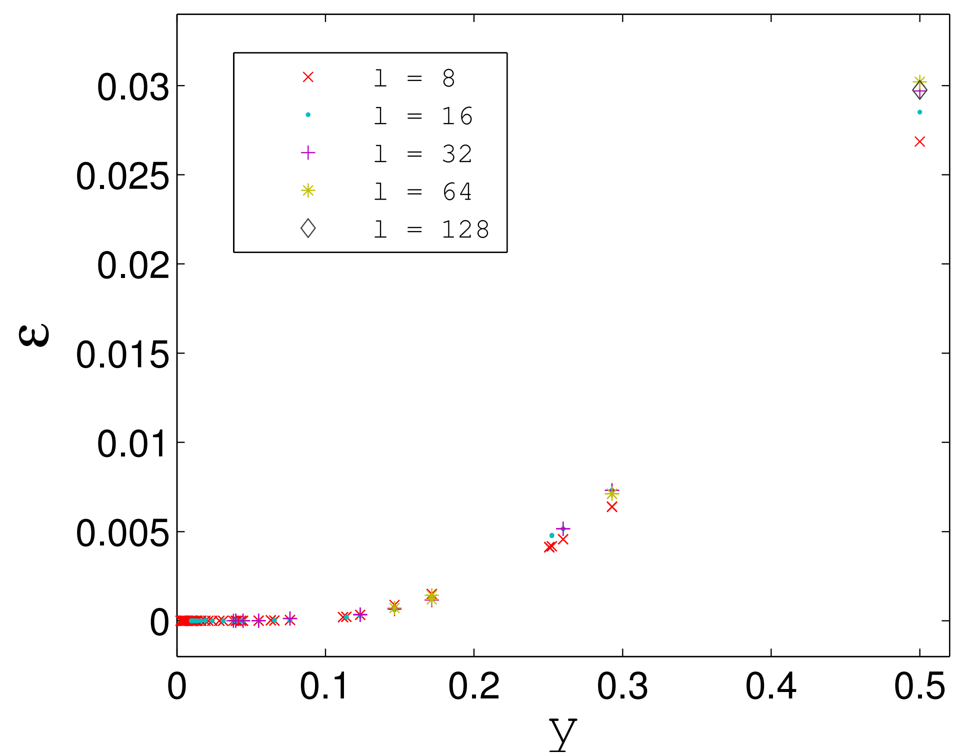
$$\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6} \, \frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](\mathbf{0}|\tau(\frac{y}{y-1}))|}{2^{n-1}\prod_{k=1}^{n-1} |F_{k/n}(\frac{y}{y-1})|^{1/2}}$$

$0 < y < 1$

- Tree tensor network:

[Calabrese, Tagliacozzo, E.T., (2013)]





XY spin chain: two disjoint blocks

- XY spin chain with periodic b.c.

$$H_{XY} = -\frac{1}{2}\sum_{j=1}^{L}\left(\frac{1+\gamma}{2}\,\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\,\sigma_j^y\sigma_{j+1}^y + h\,\sigma_j^z\right)$$

Ising model in a transverse field for $\gamma = 1$, XX spin chain for $\gamma = 0$

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- Jordan-Wigner transformation $c_j = \Big(\prod_{m<j}\sigma_m^z\Big)\frac{\sigma_j^x - \mathrm{i}\sigma_j^z}{2}$ $\qquad c_j^\dagger = \Big(\prod_{m<j}\sigma_m^z\Big)\frac{\sigma_j^x + \mathrm{i}\sigma_j^z}{2}$

Then introduce the Majorana fermions $a_j^x = c_j + c_j^\dagger$ and $a_j^y = \mathrm{i}(c_j - c_j^\dagger)$

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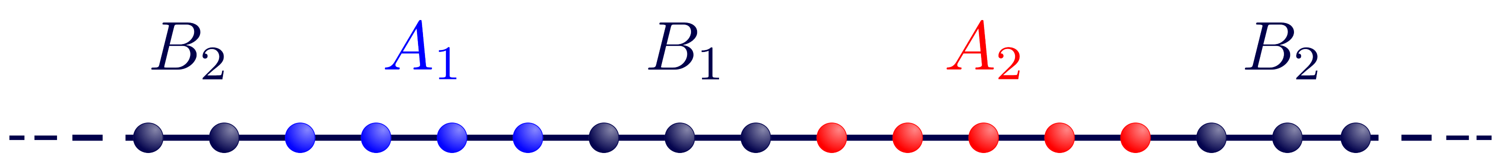
- Two disjoint blocks [Igloi, Peschel, (2010)] [Alba, Tagliacozzo, Calabrese, (2010)] [Fagotti, Calabrese, (2010)]



Crucial role played by the following strings of Majorana operators

$$P_{B_1} = \prod_{j\in B_1} \mathrm{i}\, a_j^x a_j^y \qquad\qquad P_{A_2} = \prod_{j\in A_2} \mathrm{i}\, a_j^x a_j^y$$

XY spin chain: two disjoint blocks

- The moments of ρ_A can be computed through four Gaussian operator [Fagotti, Calabrese, (2010)]

$$\rho_1 \equiv \rho_A^{\mathbf{1}} \qquad \rho_2 \equiv P_{A_2} \rho_A^{\mathbf{1}} P_{A_2} \qquad \rho_3 \equiv \langle P_{B_1} \rangle \, \rho_A^{B_1} \qquad \rho_4 \equiv \langle P_{B_1} \rangle \, P_{A_2} \rho_A^{B_1} P_{A_2}$$

where $\rho_A^{\mathbf{1}}$ is the fermionic reduced density matrix
and $\rho_A^{B_1}$ is the auxiliary density matrix

$$\boxed{\rho_A^{B_1} \equiv \frac{\mathrm{Tr}_B \left(P_{B_1} |\Psi\rangle\langle\Psi|\right)}{\langle P_{B_1} \rangle}}$$

$$\boxed{\mathrm{Tr}\rho_A^n = \frac{1}{2^n} \mathrm{Tr}\left(\rho_1 + \rho_2 + \rho_3 - \rho_4\right)^n = \frac{1}{2^{n-1}} \sum_{\boldsymbol{q}} (-1)^{\#4} \, \mathrm{Tr}\left[\prod_{k=1}^{n} \rho_{q_k}\right]}$$

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- Simplest examples: $n=2$ and $n=3$

$$\mathrm{Tr}\rho_A^2 \;=\; \frac{1}{2}\Big[\mathrm{Tr}(\rho_1^2)+\mathrm{Tr}(\rho_1\rho_2)+\mathrm{Tr}(\rho_3^2)-\mathrm{Tr}(\rho_3\rho_4)\Big]$$

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- We focus on the XX and Ising spin chain at criticality

Spin structures in CFT: Dirac fermion and Ising model

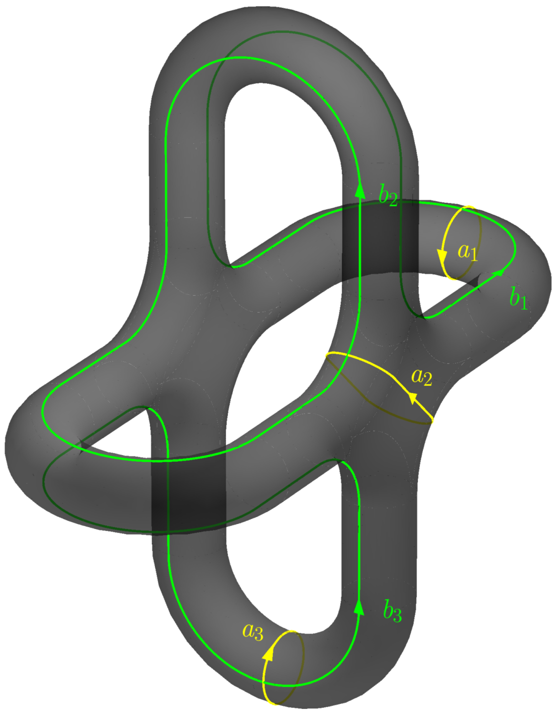
[Calabrese, Cardy, E.T., (2011)]

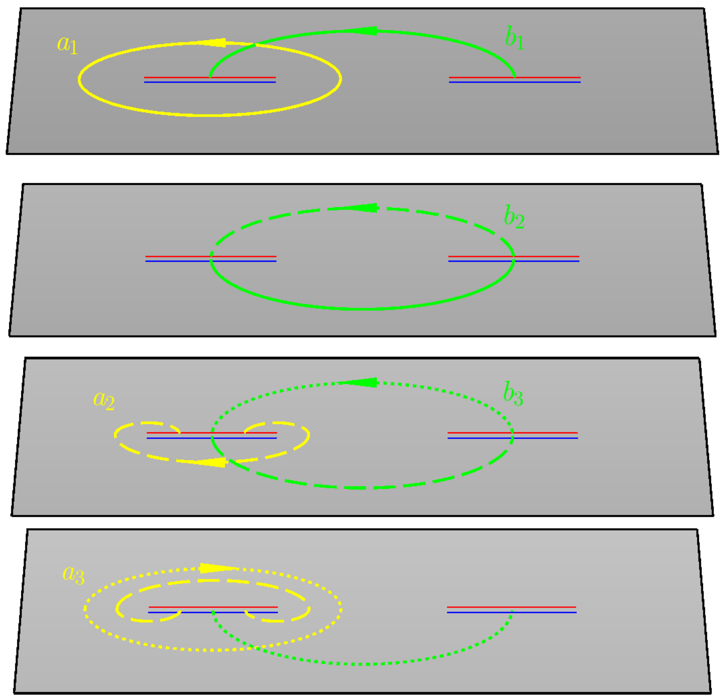
- CFT regime

$$\mathcal{F}_n^{\text{Ising}}(x) = \frac{1}{2^{n-1}} \sum_{\boldsymbol{e}} (-1)^{4\boldsymbol{\varepsilon}\cdot\boldsymbol{\delta}} \left| \frac{\Theta[\boldsymbol{e}](\mathbf{0}|\tau)}{\Theta(\mathbf{0}|\tau)} \right| \qquad \boldsymbol{e} \equiv \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \end{pmatrix}$$

A similar formula con be written for the modular invariant Dirac fermion

→ The characteristic $\boldsymbol{e}$ provides the *spin structure*
i.e. the set of boundary conditions along the canonical homology cycles



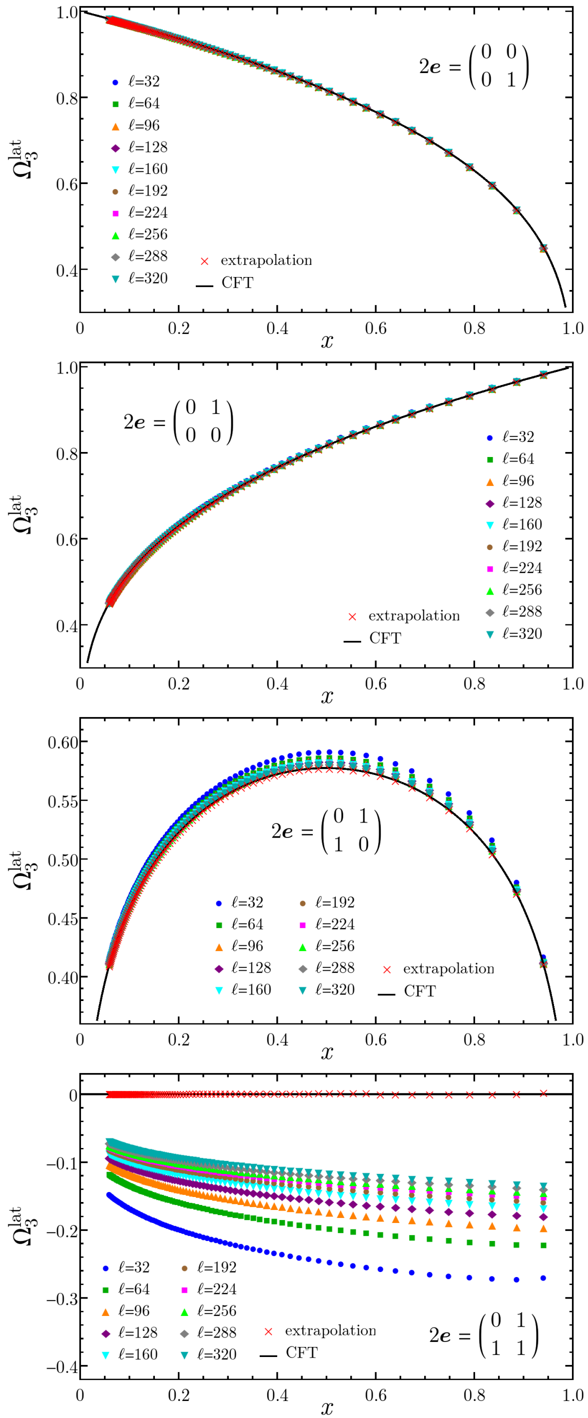


- The lattice term whose scaling limit is the term with characteristic $\boldsymbol{e}$ in $\mathcal{F}_n(x)$ can be found [Coser, E.T., Calabrese, (2015)]

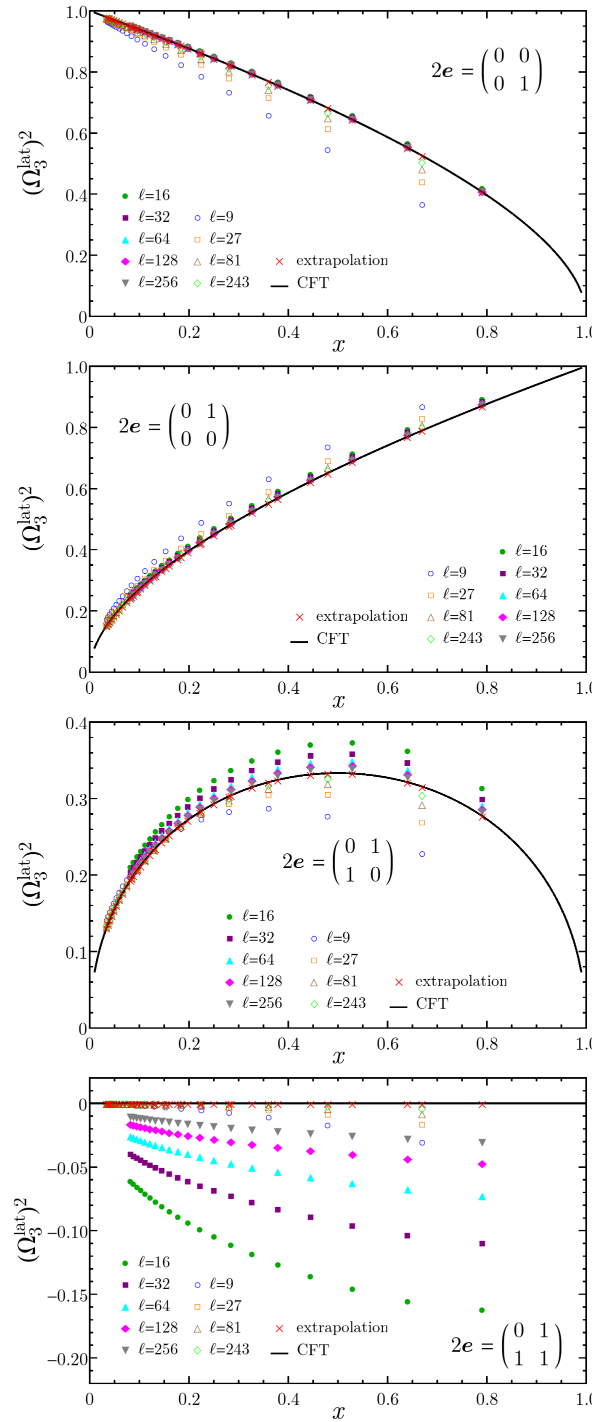
Spin structures in CFT & lattice terms: numerical analysis

[Coser, E.T., Calabrese, (2015)]

Ising spin chain



XX spin chain



- Characteristic $\boldsymbol{e}$

$$\Updownarrow$$

Lattice term $\mathrm{Tr}\left[\prod_{k=1}^{n} \rho_{q_k}\right]$

→ Odd characteristics vanish identically

- Numerics for $n = 3$

XY spin chain: partial transpose of two disjoint blocks

- Free fermion: $\rho_A^{T_2}$ is a sum of 2 fermionic Gaussian operators [Eisler, Zimboras, (2015)]

μ_2 number of Majorana operators in O_2

$$O_2^T = (-1)^{\tau(\mu_2)} O_2 \qquad \tau(\mu_2) = \begin{cases} 0 & (\mu_2 \bmod 4) \in \{0,1\} \\ 1 & (\mu_2 \bmod 4) \in \{2,3\} \end{cases}$$

- **XY spin chain** [Coser, E.T., Calabrese, (2015)]

$$\tilde{\rho}_A^{\mathbf{1}} \equiv \frac{1}{2^{\ell_1+\ell_2}} \sum \mathrm{i}^{\mu_2} \langle O_2^\dagger O_1^\dagger \rangle \, O_1 O_2 \qquad \tilde{\rho}_A^{B_1} \equiv \frac{1}{2^{\ell_1+\ell_2}} \sum \mathrm{i}^{\mu_2} \frac{\langle O_2^\dagger P_{B_1} O_1^\dagger \rangle}{\langle P_{B_1} \rangle} \, O_1 O_2$$

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- e.g.: $\mathrm{Tr}\big(\rho_A^{T_2}\big)^3 = \dfrac{1}{4}\Big[\mathrm{Tr}(\tilde{\rho}_1^3) + 3\,\mathrm{Tr}(\tilde{\rho}_1^2 \tilde{\rho}_2) + 6\,\mathrm{Tr}(\tilde{\rho}_1 \tilde{\rho}_4 \tilde{\rho}_3) - 3\,\mathrm{Tr}(\tilde{\rho}_1 \tilde{\rho}_3^2) - 3\,\mathrm{Tr}(\tilde{\rho}_2 \tilde{\rho}_3^2)\Big]$

Moments of the partial transpose: Ising chain & XX chain

- CFT regime: modular invariant Dirac fermion (scaling limit of the XX chain)

$$\mathrm{Tr}\left(\rho_A^{T_2}\right)^n = c_n^2 \left(\frac{1-x}{\ell_1 \ell_2}\right)^{2\Delta_n} \frac{1}{2^{n-1}} \sum_{\boldsymbol{e}} (-1)^{4\boldsymbol{\varepsilon}\cdot\boldsymbol{\delta}} \left|\frac{\Theta[\boldsymbol{e}](\mathbf{0}|\tilde{\tau})}{\Theta(\mathbf{0}|\tilde{\tau})}\right|^2 \qquad \boldsymbol{e} \equiv \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \end{pmatrix}$$

where $\tilde{\tau}(x) = \tau(\frac{x}{x-1})$. A similar formula can be written for the Ising model

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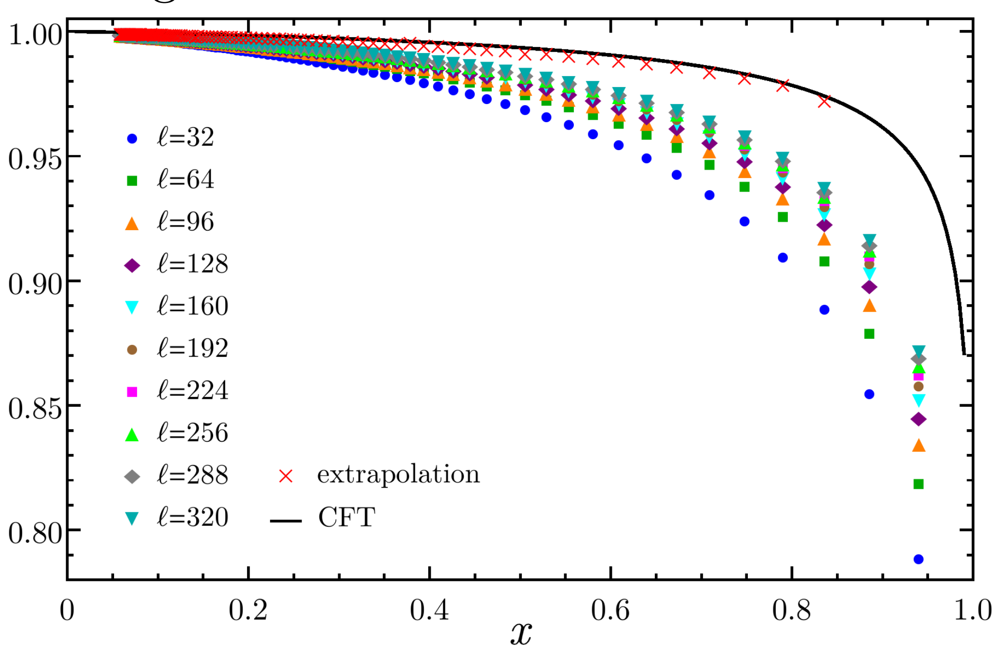
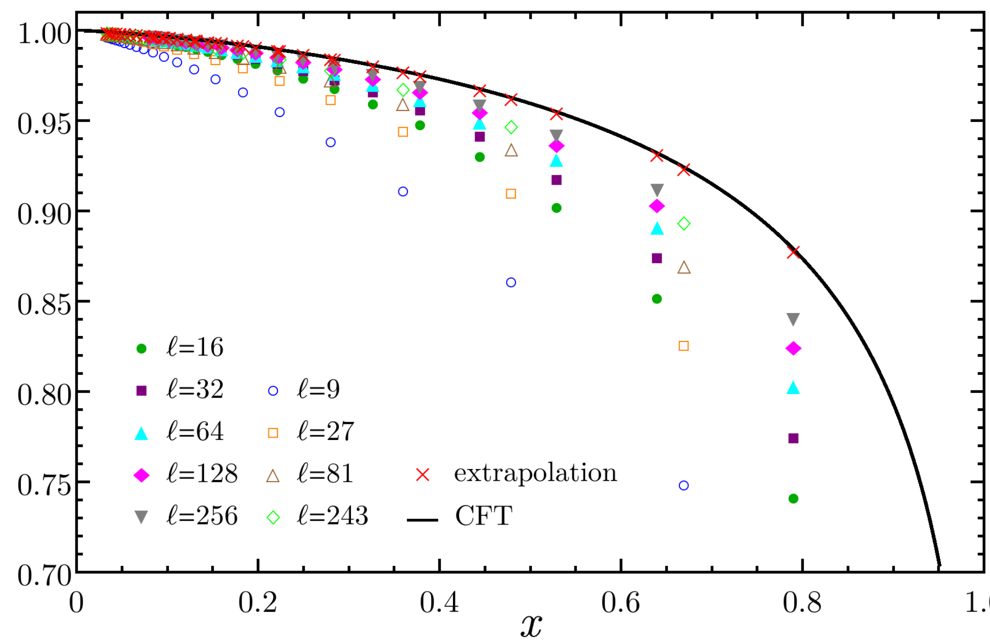
- Numerics: e.g. $\mathrm{Tr}(\rho_A^{T_2})^n / \mathrm{Tr}\rho_A^n$ for $n=4$ [Coser, E.T., Calabrese, (2015)]

Ising chain



XX chain



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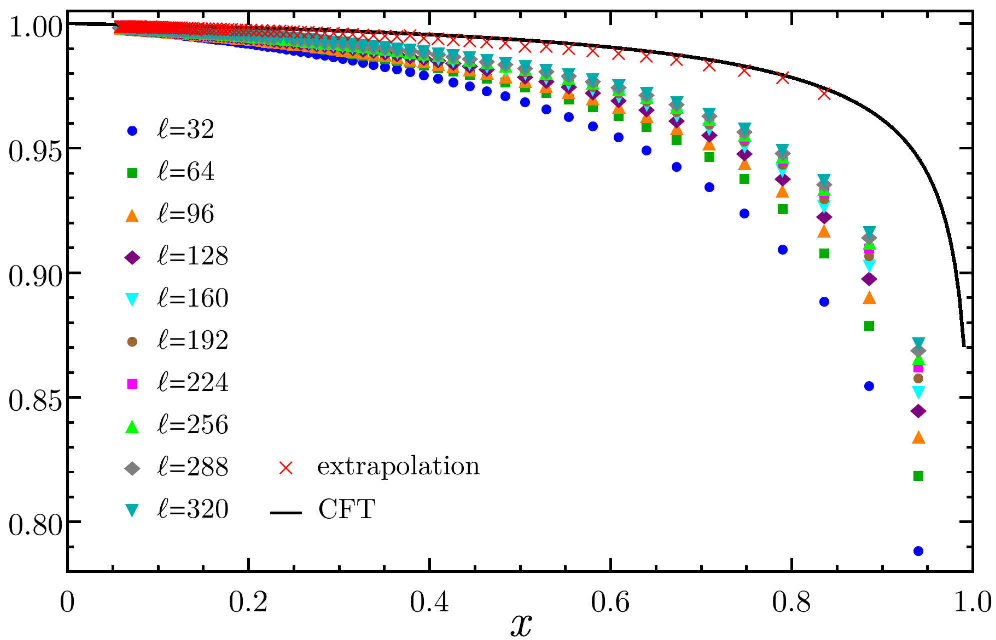
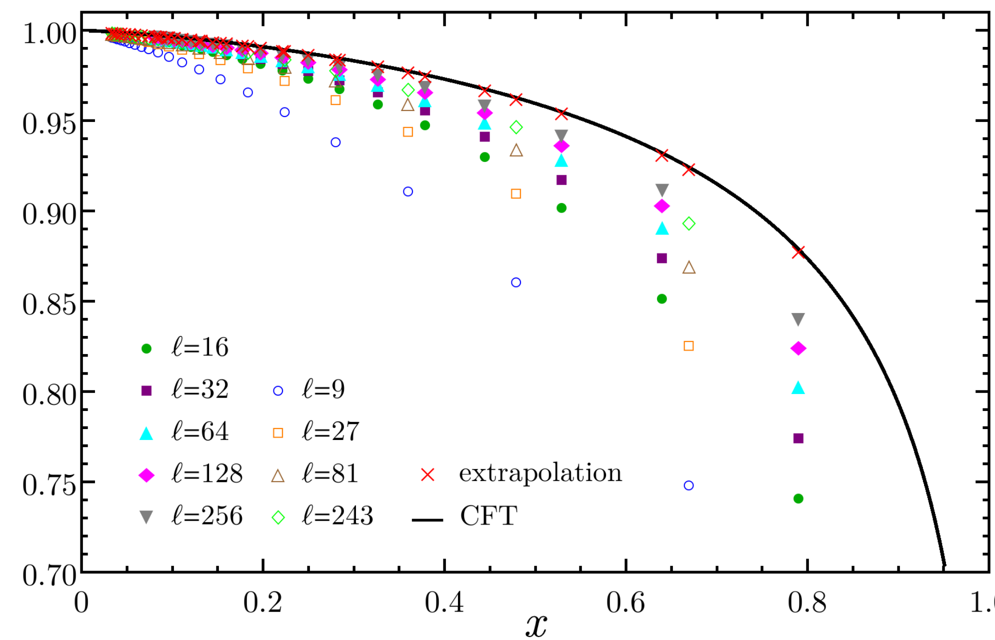


XX chain

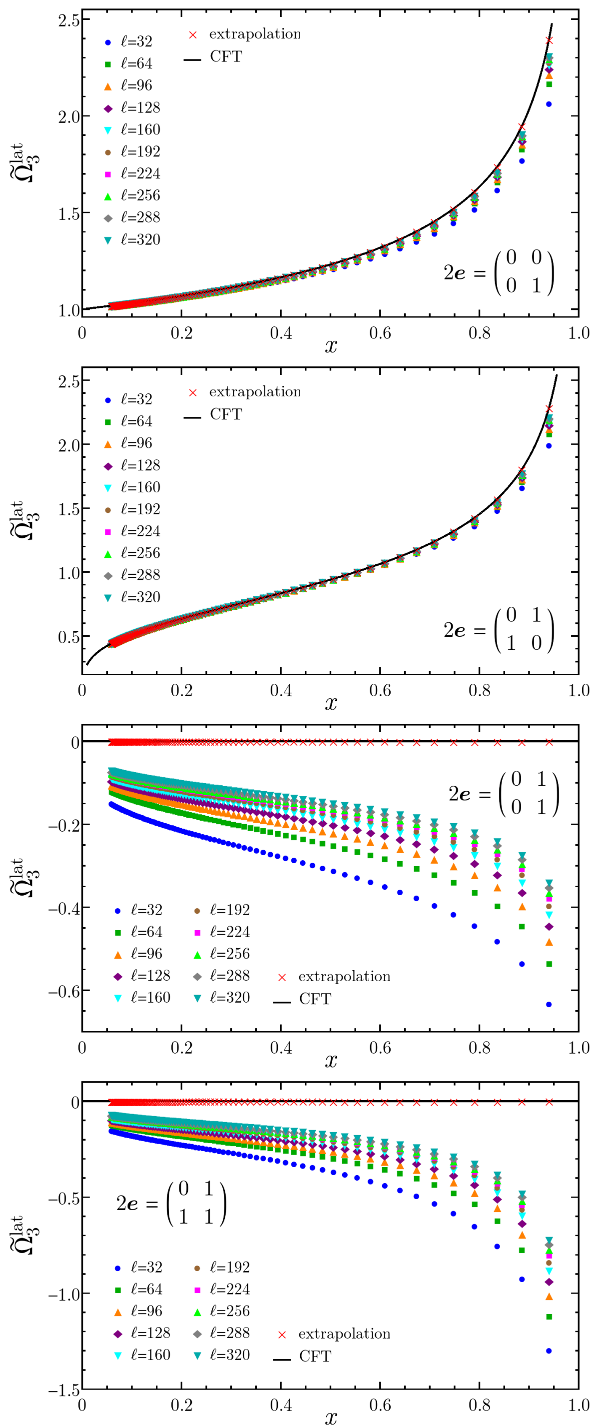


- The lattice term in $\mathrm{Tr}(\rho_A^{T_2})^n$ whose scaling limit is the term with characteristic $\boldsymbol{e}$ in the CFT formula can be found

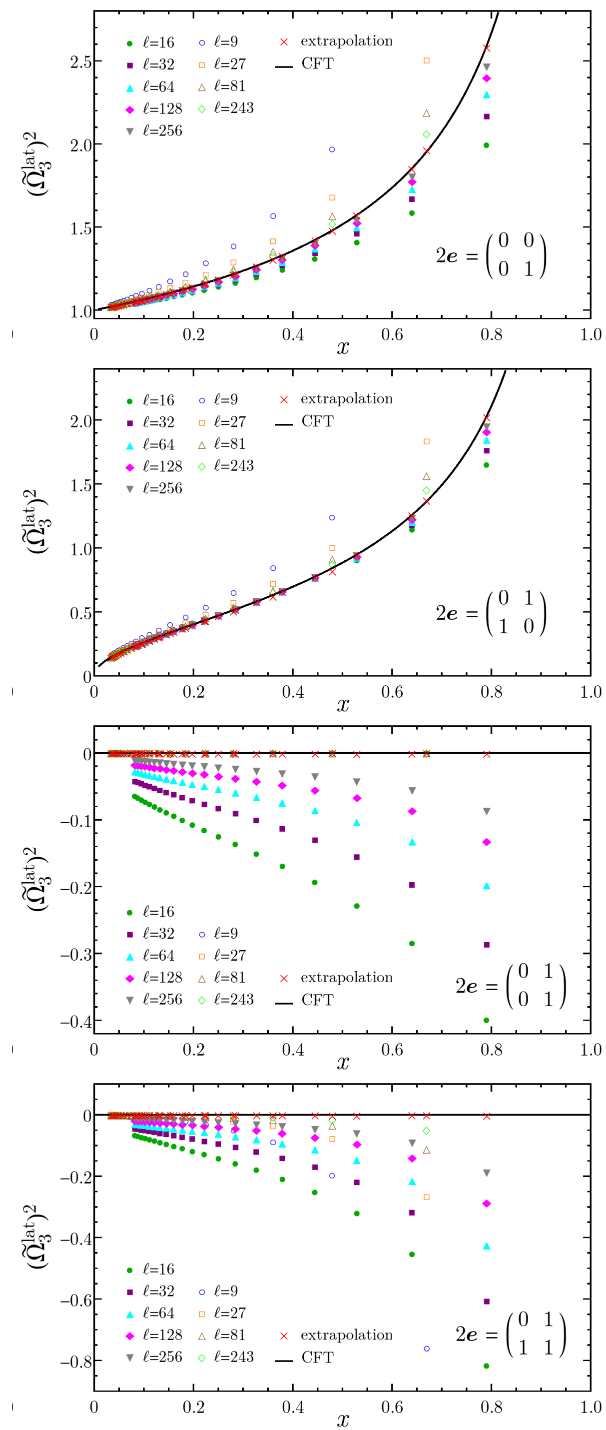
Partial transpose: spin structures in CFT & lattice terms

[Coser, E.T., Calabrese, (2015)]

Ising spin chain



XX spin chain



- Characteristic $\boldsymbol{e}$

$$\Updownarrow$$

Lattice term $\mathrm{Tr}\left[\prod_{k=1}^{n} \tilde{\rho}_{\tilde{q}_k}\right]$

→ Odd characteristics vanish identically

- Numerics for $n = 3$

Free fermion: partial transpose of two disjoint intervals

[Coser, E.T., Calabrese, (2015)]

- CFT expression:

$$\mathrm{Tr}\big(\rho_A^{T_2}\big)^n = c_n^2 \left(\frac{1-x}{\ell_1 \ell_2}\right)^{2\Delta_n} \frac{1}{2^{n/2-1}} \sum_{\boldsymbol{\delta}} \cos\left[\frac{\pi}{4}\left(1+\sum_{i=1}^{n-1}(-1)^{2\sum_{j=i}^{n-1}\delta_j}\right)\right] \left|\frac{\Theta[\boldsymbol{e}](\tilde{\tau})}{\Theta(\tilde{\tau})}\right|^2$$

where $\tilde{\tau} \equiv \tau\big(x/(x-1)\big)$ and the sum is over the characteristics $\boldsymbol{e} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\delta} \end{pmatrix}$

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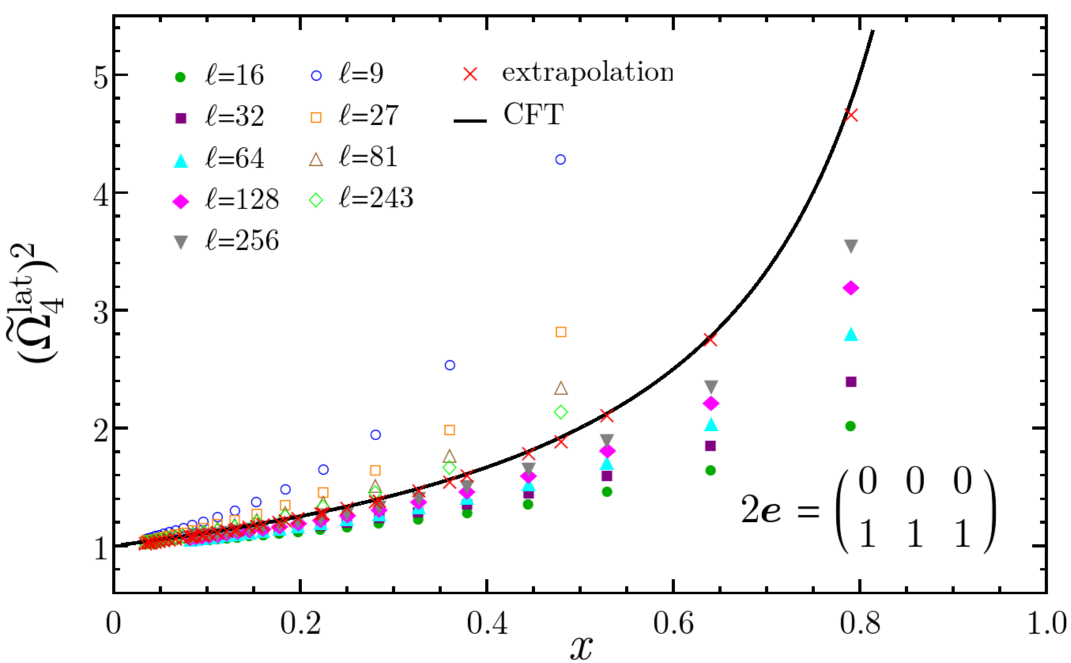
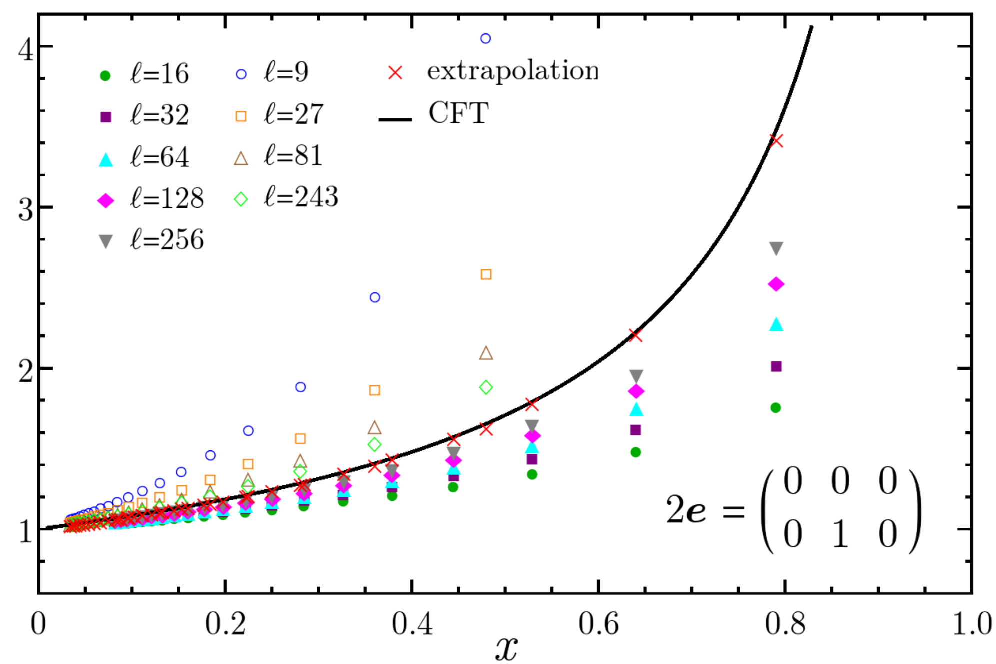
- The lattice counterpart of each term in the sum can be found





Conclusions & open issues

- Entanglement entropies for disjoint intervals for some 2D CFT

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Thank you!