A DISCRETE STOCHASTIC INTERPRETATION OF THE DOMINATIVE p-LAPLACIAN

BIRS Workshop:
Advanced Developments for Surface and Interface Dynamics – Analysis and Computation

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The Dominative p-Laplacian $\mathcal{L}_p$

For $p \geq 2$, the DOMINATIVE p-LAPLACIAN, introduced by K. Brustad,is the operator

$$\mathcal{L}_p u(x) = \frac{1}{p}(\lambda_1 + \ldots + \lambda_{N-1}) + \frac{(p-1)}{p}\lambda_N,$$

where we have ordered the eigenvalues of $D^2 u(x)$ as

$$\lambda_1 \leq \lambda_2 \ldots \leq \lambda_N.$$

The operator $\mathcal{L}_p$ is sublinear (thus convex) and uniformly elliptic. Thus, the viscosity solutions of the equation $\mathcal{L}_p u(x) = 0$ are locally in the class $C^{2,\alpha}$.

We will discuss the relation between $\mathcal{L}_p$ and the regular p-Laplacian and then present a discrete stochastic approximation to the unique viscosity solution of the Dirichlet problem for the Dominative p-Laplace Equation.

The Dominative p-Laplacian $\mathcal{L}_p$,II

Recall that the ordinary p-Laplacian is the operator

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= |\nabla u|^{p-2}\Delta_p^h u \\ \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) &= |\nabla u|^{p-2}\sum_{i,j=1}^N \left\{\delta_{ij} + (p-2)\frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right\} u_{x_i x_j} \end{aligned}$$

Proposition (K. Brustad'17)

$$\Delta_p^h u \leq p\,\mathcal{L}_p u = \lambda_1 + \ldots + \lambda_{N-1} + (p-1)\lambda_N,$$

with equality for radial functions.

Theorem(Crandall-Zhang'03, Lindqvidst-M'08, K. Brustad'17)

Let $p \geq 2$ and $u_1, u_2, \ldots, u_k$ be radial p-superhamonic functions, then the $\sum_{i=1}^k u_i(x - y_i)$ is p-superharmonic.

Discretization

Fix $\epsilon > 0$ and small. Given a Lipschitz domain $\Omega \subset \mathbb{R}^N$, we build a strip around $\partial\Omega$

$$\Gamma_\epsilon = \{x \in \mathbb{H} \setminus \Omega \colon d(x, \partial\Omega) \leq \epsilon\}$$

and set $X = \Omega \cup \Gamma_\epsilon$.

Note that for $x \in \Omega$, we always have $B_\epsilon(x) \subset X$.

We are also given a Lipschitz function $F : \partial\Omega \mapsto \mathbb{R}$ that we can extend to X when needed, called the *payoff* function.

Let $\mathcal{A}$ denoted the class of functions $v \colon X \mapsto \mathbb{R}$ that are bounded Borel measurable and such that $v = F$ on Γ_ϵ. Note that $\mathcal{A} \neq \emptyset$.

Mean Values

Set

$$q = \frac{p + 4N + 6}{2N + 4}$$

and let $v \in \mathcal{A}$. Define the (sublinear) mean value operator as follows

$$\begin{aligned} MV_q(v, B_\epsilon(x)) &= \tfrac{1}{q-1} \fint_{B_\epsilon(x)} v(y)\,dy \\ &+ \left(\tfrac{q-2}{q-1}\right) \sup_\sigma \left(\tfrac{v(x+\epsilon\,\sigma(x)) + v(x-\epsilon\,\sigma(x))}{2} \right), \end{aligned}$$

where $\sigma\colon \Omega \mapsto \mathbb{S}^{N-1}$ is a **strategy**. We also define the averaging operator $T_q\colon \mathcal{A} \mapsto \mathcal{A}$ as follows:

$$\begin{cases} \text{for } x \in \Omega, & T_q v(x) = MV_q(v, B_\epsilon(x)) \\ \text{for } x \in \Gamma_\epsilon, & T_q v(x) = v(x). \end{cases}$$

ϵ-Mean Value Solution

For smooth functions we have

$$\lim_{\epsilon \to 0} \frac{MV_q(v, B_\epsilon(x)) - v(x)}{\epsilon^2} = \frac{p}{2(N+2)+p-2} \mathcal{L}_p v(x),$$

so that if $\mathcal{L}_p v(x) = 0$ we have the asymptotic mean value property

$$v(x) = MV_q(v, B_\epsilon(x)) + o(\epsilon^2).$$

We want to solve the Dirichlet problem

$$\begin{cases} \mathcal{L}_p u(x) = 0 & \text{for } x \in \Omega \\ u(x) = F(x) & \text{for } x \in \partial\Omega. \end{cases}$$

Lemma (Solution at scale ϵ, DPP)

There exist a unique function $v_\epsilon \in \mathcal{A}$ such that such that $T_q v_\epsilon(x) = v_\epsilon(x)$ for all $x \in X$.

The Dirichlet problem for $\mathcal{L}_p$

Theorem (Brustad-Lindqvist-M'18)

$$\lim_{\epsilon \to 0} v_\epsilon = u, \ \textit{uniformly in } \Omega,$$

where u is the only solution to the Dirichlet problem for $\mathcal{L}_p$ in Ω with boundary values F.

The proof that we have uses discrete stochastic methods. We will give a stochastic interpretation to the ϵ-mean values solution v_ϵ.

Fix $x_0 \in \Omega$ and a strategy σ. We will consider a discrete process

$$x_0, x_1, x_2, \ldots, x_k \ldots$$

defined as follows:

If $x_0 \in \Gamma_\epsilon$ we set $x_1 = x_0$ and stop, otherwise $B_\epsilon(x_0) \subset X$. In this case, we move one step according to

- with probability $\frac{1}{q-1}$ select $x_1 \in B_\epsilon(x_0)$ at random,
- with probability $\frac{q-2}{2(q-1)}$ select $x_1 = x_0 + \epsilon\sigma(x_0)$, and
- with probability $\frac{q-2}{2(q-1)}$ select $x_1 = x_0 - \epsilon\sigma(x_0)$.

We continue this process so that we always have $|x_i - x_{i-1}| \leq \epsilon$, and stop when we first reach Γ_ϵ, say at x_{τ_σ}, when $k = \tau_\sigma$

$$\tau_\sigma = \inf\{k \colon x_k \notin \Omega\}$$

so that $x_{\tau_\sigma} \in \Gamma_\epsilon$.

The payoff of this run is $F(x_{\tau_\sigma})$. Averaging over all possible runs we define the *value function* for the strategy σ

$$u_\epsilon^\sigma(x_0) = \mathbb{E}_\sigma^{x_0}[F(x_{\tau_\sigma})]$$

Optimizing over all strategies we get

$$u_\epsilon(x_0) = \sup_\sigma \left(u_\epsilon^\sigma(x_0)\right) = \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[F(x_{\tau_\sigma})]\right),$$

which we call the ϵ-*stochastic solution*.

Theorem (Stochastic Solution = Mean Value Solution)

The following hold:

i) $u_\epsilon(x) = F(x)$ *for* $x \in \Gamma_\epsilon$.

ii) $u_\epsilon(x) = v_\epsilon(x)$, *where* v_ϵ *is the* ϵ*-mean value solution above. That is, we have that* u_ϵ *also satisfies the dynamic programming principle* $u_\epsilon(x) = T_q u_\epsilon(x)$.

We now study what happens when $\epsilon \to 0$.

We follow an argument from Barles-Souganidis'91. For $x \in \Omega$ define the upper-semicontinuous envelope and the lower-semicontinuous envelope

$$\overline{u}(x) = \limsup_{\substack{\epsilon \to 0 \\ y \to x}} u_\epsilon(y), \qquad \underline{u}(x) = \liminf_{\substack{\epsilon \to 0 \\ y \to x}} u_\epsilon(y)$$

Lemma

$\overline{u}$ is a viscosity subsolution of $\mathcal{L}_p$ and $\underline{u}$ is a viscosity supersolution of $\mathcal{L}_p$.

We would like to conclude that $\overline{u} \leq \underline{u}$, for which we would need the fact that $\mathcal{L}_p$ satisfies the STRONG UNIQUENESS CONDITION OF BS and that Ω is of class C^2.

But we don't know that $\mathcal{L}_p$ satisfies the STRONG UNIQUENESS CONDITION OF BS and our domain is Lipschitz, not necessarily C^2.

The condition that we use for Ω is the following

There exists $\mu > 0$ such that for all $y \in \partial\Omega$ and $\delta \in (0,1)$ we can always find a ball $\mathbb{B}(z, \mu\delta)$ such that

$$\mathbb{B}(z, \mu\,\delta) \subset \mathbb{B}(y, \delta) \setminus \Omega$$

This condition is clearly satisfied by all bounded Lipschtiz domains.

Boundary Estimate

Theorem (Key Boundary estimate)

Given $\eta > 0$ there exist $\delta = \delta(\eta, F)$, integer $k_0 = k_0(\eta, \mu, F)$, and $\epsilon_0 = \epsilon_0(\delta, \mu, k_0)$ such that

$$|u^\epsilon(p) - F(y)| \leq \frac{\eta}{2}$$

for all $y \in \partial\Omega$, $p \in B_{\delta/4^k}(y) \cap \Omega$, $k \geq k_0$ and $\epsilon \leq \epsilon_0$.

The point is that this is an estimate valid for all $\epsilon \leq \epsilon_0$. This estimate mplies

$$\limsup_{\substack{x\in\Omega, y\in\partial\Omega \\ x\to y}} \overline{u}(x) \leq F(y) \text{ and } \liminf_{\substack{x\in\Omega, y\in\partial\Omega \\ x\to y}} \underline{u}(x) \geq F(y)$$

So we we can apply the **usual** comparison principle for viscosity solutions of $\mathcal{L}^p$ to conclude $\underline{u} = \overline{u} = u$ and $u^\epsilon \to u$ locally uniformly in $\overline{\Omega}$, and thus uniformly in $\overline{\Omega}$.

Proof of the boundary estimate

This is where you get your hands dirty. The proof uses the following facts:

- Everything works for smooth C^3-functions with non-vanishing gradient (This part uses probability).
- ϵ-mean value solutions satisfy a comparison principle
- Existence of radial barriers centered at

$$U(x) = \frac{a_k}{|x - z_k|^{\frac{N-p}{p-1}}} + b_k,$$

centered at $\mathbb{B}(z_k, \mu\delta_k) \subset \mathbb{B}(y, \delta_k) \setminus \Omega$

- Iteration

Let $v \in C^3(\overline{\Omega})$ satisfying $\mathcal{L}_p v = 0$ in $\overline{\Omega}$ with non-vanishing gradient. Then, we have, uniformly in Ω that

$$v(x) = MV_q(v, B_\epsilon(x)) + O(\epsilon^3).$$

Fix a strategy σ and run the process $x_0, x_1, \dots$

Lemma

- *For an arbitrary strategy σ the sequence of random variables*

  $M_k = v(x_k) - C_1 k\epsilon^3$ *is a* SUPERMARTINGALE
- *Let* $\sigma_0(x) = \frac{\nabla v(x)}{|\nabla v(x)|}$ *by the optimal strategy, then the sequence of random variables*

  $N_k = v(x_k) + C_1 k\epsilon^3$ *is a* SUBMARTINGALE

Some Proofs

$$\begin{aligned} v^\epsilon(x_0) &= \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[v(x_{\tau_\sigma})]\right) = \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[v(x_{\tau_\sigma}) - C_1\tau_\sigma\epsilon^3 + C_1\tau_\sigma\epsilon^3]\right) \\ &\le \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[v(x_{\tau_\sigma}) - C_1\tau_\sigma\epsilon^3]\right) + \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[C_1\tau_\sigma\epsilon^3]\right) \\ &\le v(x_0) + C_1\epsilon^3 \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[\tau_\sigma]\right) \end{aligned}$$

$$\begin{aligned} v^\epsilon(x_0) &= \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[v(x_{\tau_\sigma})]\right) \ge \left(\mathbb{E}_{\sigma_0}^{x_0}[v(x_{\tau_\sigma}) + C_1\tau_\sigma\epsilon^3 - C_1\tau_\sigma\epsilon^3]\right) \\ &= \mathbb{E}_{\sigma_0}^{x_0}[v(x_{\tau_{\sigma_0}}) + C_1\tau_{\sigma_0}\epsilon^3] - \mathbb{E}_{\sigma_0}^{x_0}[C_1\tau_{\sigma_0}\epsilon^3] \\ &\ge v(x_0) - C_1\epsilon^3 \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[\tau_\sigma]\right) \end{aligned}$$

Corollary

$$|v(x) - v^\epsilon(x)| \le C_1\epsilon^3 \sup_\sigma \left(\mathbb{E}_\sigma^{x_0}[\tau_\sigma]\right) \le C\,\epsilon$$

Claim

For all strategies σ we have $\mathbb{E}_\sigma^{x_0}[\tau_\sigma] \le \frac{C}{\epsilon^2}$

Thank you very much

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