Recent results on crystalline mean curvature flows

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Introduction

- Crystalline curvature motions: definition, some known results;
- "distributional" sub and supersolutions;
- Comparison;
- Construction of solutions;
- ("More") general mobilities.

Anisotropic and Crystalline mean curvature flow

Given a set $E \subset \mathbb{R}^N$ and a *surface tension* φ, which is a convex, one-homogeneous and even norm, we consider the anisotropic perimeter

$$\mathrm{Per}_\varphi(E) := \int_{\partial E} \varphi(\nu_E) d\mathcal{H}^{N-1}$$

(appropriately extended to finite perimeter sets) and its first variation

$$\kappa_\varphi = \mathrm{div}_\tau \nabla\varphi(\nu).$$

An *anisotropic* mean curvature flow is a tube of sets $E(t)$ with boundary evolving with a speed proportional to $-\kappa_\varphi$ (or $-$ a nondecreasing function of κ_φ).

Anisotropic and Crystalline mean curvature flow

Such evolutions are well defined

- if φ is smooth enough (Almgren-Taylor-Wang, 1993);
- in the level set sense: Chen-Giga-Goto (1991) show existence for

$$u_t + H(Du, D^2 u) = 0$$

in the viscosity sense, where H is "geometric" (meaning that all level sets evolve independently with same law); for anisotropic motion a typical H has the form $|Du| D^2\varphi(Du) : D^2 u$.

What if φ is not smooth?

Crystalline curvature flow

The crystalline case is the case where $\{\varphi \leq 1\}$ is a polytope. In this case, $\nabla\varphi(\nu)$ is *a priori* not well defined. We should use a selection of the subgradient $\partial\varphi(\nu)$: "$\kappa_\varphi \in \mathrm{div}_\tau \partial\varphi(\nu)$".

- loss of ellipticity / solutions are not expected to be regular in the classical sense [*not* a problem for viscosity solutions];
- infinite diffusion / the motion should be non-local.

Crystalline curvature flow

Polar function: $\varphi^\circ(\xi) := \sup\{\xi \cdot \eta : \varphi(\eta) \leq 1\}$;

$W = \{\varphi^\circ \leq 1\}$ is called the **Wulff shape**. For φ smooth it is a constant curvature compact set. It minimizes the φ-area for fixed volume

In the crystalline case it is a polytope.

Then, one knows how to define mean curvature flows

- in 2D if the initial set is a polygon with facets parallel to the faces of W (system of ODEs, cf Almgren-Taylor 95, Giga-Gurtin 96...);
- in 2D for a short time if the initial set has a "interior/exterior Wulff shape conditions" (C.-Novaga, 2012/15);
- in any dimension if the initial set is convex (Bellettini-Caselles-C-Novaga, 2006);

Crystalline curvature flow

- in 2D in the "viscosity sense" adapted to crystalline motions (Giga-Giga, 2001)
- in 3D and more, viscosity: Giga-Požar 2016 (preprint 2014) and 2018 (preprint 2017).

An important advantage of the viscosity approach is that it solves **any** equation of the form $V = -F(\nu, \kappa_\varphi)$ (with F nondecreasing wr κ).

Recent results

• With Massimiliano Morini (Parma) and Marcello Ponsiglione (Roma I): motion with "*natural*" mobility, in **any** dimension, **any** anisotropy. That is

$$V_N = -\varphi(\nu)\kappa_\varphi$$

or

$$V_\varphi = -\kappa_\varphi$$

where V_φ is the velocity along a *Cahn-Hoffmann* vector field $\partial\varphi(\nu)$ (*Cf.* [Bellettini-Paolini, 96])

• With the same and Matteo Novaga (Pisa): arbitrary *convex* mobility, forcing term (Lipschitz in time, bounded):

$$V_N = -\psi(\nu)(\kappa_\varphi + g(x,t)) \qquad (*)$$

• Crystalline limits of viscosity solutions of $(*)$ for a smoothed anisotropy φ and forcing g (preprint, 2017).

A formal equation

For ψ a norm ($\psi = \varphi$, or not...), one can define the ψ°-*signed distance function*:

$$d_E^{\psi^\circ}(x) = \operatorname{dist}^{\psi^\circ}(x, E) - \operatorname{dist}^{\psi^\circ}(x, E^c) = \min_{y \in E} \psi^\circ(x - y) - \min_{y \notin E} \psi^\circ(y - x)$$

Then one has $\psi(\nabla d_E^{\psi^\circ}) = 1$. If $E(t)$ evolves with a normal velocity V_N, one can relate the change in $d_E^{\psi^\circ}$ to V_N:

$$\frac{\partial d_{E(t)}^{\psi^\circ}}{\partial t} = -\frac{1}{\psi(\nu)} V_N$$

on $\partial E(t)$.

Hence formally if $V_N = -\psi(\nu)(\kappa_\varphi + g)$ one should have

$$\frac{\partial d_{E(t)}^{\psi^\circ}}{\partial t} = \kappa_\varphi + g$$

on ∂E.

The curvature of the level sets

If $z \in \partial\varphi(\nabla d_E^{\psi^\circ})$, one can recover a φ-curvature of $\{d_E^{\psi^\circ} = 0\}$ as

$$\kappa_\varphi = \operatorname{div} z.$$

In the smooth case if $\psi = \varphi$, it is standard that $\kappa_\varphi \leq (N-1)/d_E$ where $d_E > 0$. In the crystalline case, one can build a z which satisfies such an inequality.

In general, the φ-curvature of the level sets of $d_E^{\psi^\circ}$ *decreases* as one goes further away from the set.

Super/subsolutions

Definition A (closed) "tube" $E \subseteq \mathbb{R}^N \times [0, +\infty)$ is a **supersolution** starting from the initial E^0 if

a. $E(0) \subseteq \overline{E^0}$;

b. $E(t) = \emptyset \Rightarrow E(s) = \emptyset$ if $s > t$;

c. E is (Kuratowski) left-continuous;

d. For $d = \operatorname{dist}^{\varphi^\circ}(x, E)$, there exists $z \in \partial\varphi(\nabla d)$ with, for some $M \geq 0$,

$$\partial_t d \geq \operatorname{div} z$$

in the **distributional sense** in $\mathbb{R}^N \times (0, T^*) \setminus E$ where T^* is the extinction time of E, moreover (for $t \leq T^*$)

$$(\operatorname{div} z)^+ \in L^\infty(\{d > \delta\})$$

for any $\delta > 0$

A **subsolution** is an open tube A such that A^c is a supersolution starting from $(E^0)^c$.

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d. For $d = \mathrm{dist}^{\psi^\circ}(x, E)$, there exists $z \in \partial\varphi(\nabla d)$ with, for some $M \geq 0$,

$$\partial_t d \geq \mathrm{div}\, z + g - Md$$

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Remarks

- The main equation is *linear* in (d, z) (plus constant).
- A kind of mixture of Barles-Soner-Souganidis (93), Ambrosio-Soner (96), however in the distributional sense. **Equivalent** to viscosity solutions if $\varphi, \psi, \psi^\circ$ C^2.
- d is a supersolution of the φ-total variation flow in E^c;
- if E is a supersolution and its interior a subsolution, then if $|\partial E| = 0$ one expects that $\partial_t d = \operatorname{div} z$ on ∂E, which means that $V_\varphi = -\kappa_\varphi$ and it is a *solution*;
- we can prove uniqueness (up to "fattening") of such solutions, and existence if ψ is "φ-regular".

Comparison

Theorem Let E be a supersolution with initial datum E^0 and F be a subsolution with initial datum $F^0 \supset E^0$, and assume $\Delta = \mathrm{dist}^{\psi^\circ}(E^0, (F^0)^c) > 0$. Then $\mathrm{dist}(E(t), F(t)^c) \geq \Delta e^{-Mt}$ for any $t \geq 0$.

[case $g = 0, M = 0$] The proof is by parabolic comparison. Indeed, if $d(x,t) = \mathrm{dist}(x, E(t))$ and $d'(x,t) = \mathrm{dist}(x, F^c(t))$ then between E and F

$$\partial_t d \geq \mathrm{div}\, \partial\varphi^\circ(\nabla d), \quad \partial_t d' \geq \mathrm{div}\, \partial\varphi^\circ(\nabla d'),$$

and one has $d + d' \geq \Delta$ at $t = 0$. Using a priori estimate on the speed at which d, d' decrease, one can control also $d + d'$ on a parabolic boundary of a small tube, and obtain the comparison inside this tube. As d, d' are distance function it yields global comparison.

Existence: construction

Basic idea: use Almgren-Taylor-Wang 93 / Luckhaus-Sturzenhecker 95. We pick a time step $h > 0$, and for E_0 (temporarily a compact set), we define E^{n+1} from E^n, $n \geq 0$ by solving

$$\min_E P_\varphi(E) \;+\; \frac{1}{h}\int_E \left(d^{\psi^\circ}_{E^n} + \int_{nh}^{(n+1)h} g(x,s)ds \right) dx$$

Then the Euler-Lagrange equation is

$$d^{\psi^\circ}_{E^n} = -h\left(\kappa_\varphi(E^{n+1}) - \fint_{nh}^{(n+1)h} g(x,s)ds \right)$$

$\longrightarrow$ implicit time discretization of the flow. It remains to pass to the limit...

Equivalent problem

It is possible to show that this problem can be solved equivalently by solving (to simplify, $g = 0$)

$$\min_u \int \varphi(Du) + \int \frac{(u - d^{\psi^\circ}_{E^n})^2}{2h} dx$$

and letting then $E^{n+1} = \{u \leq 0\}$. If E^0 is not compact, one can consider the Euler-Lagrange equation of this problem in $\mathbb{R}^N$. It solves

$$-h \operatorname{div} z + u = d^{\psi^\circ}_{E^n},$$
$$z \in \partial\varphi(\nabla u) \quad \text{a.e.}$$

Then one lets $E^{n+1} = \{u \leq 0\}$.

The limit is a solution

We then consider $E_h(t) = E^{[t/h]}$ and consider a subsequence such that $E_{h_i} \to E$ and $(E_{h_i})^c \to A^c$ in the Kuratowski sense, with therefore $A \subset E$. One can show then:

Theorem E is a supersolution and A a subsolution. In particular if $\partial E = \partial A$, they are a solution.

Why does it work?

- The advantage of the scheme using "u" is that at each time, one has not only E (and $d_E^{\psi^\circ}$, $u \approx d_E^{\psi^\circ}$) but also a candidate field z.
- One can easily show that u is Lipschitz and $\psi(\nabla u) \leq \psi(\nabla d_{E^n}^{\psi^\circ}) = 1$ a.e.: it follows that $u \leq d_{E^{n+1}}^{\psi^\circ}$ in $\{u > 0\}$ and $u \geq d_{E^{n+1}}^{\psi^\circ}$ in $\{u < 0\}$.
- As a consequence, in $\{u > 0\}$,

$$\frac{\partial d_E^{\psi^\circ}}{\partial t} \approx \frac{d_{E^{n+1}}^{\psi^\circ} - d_{E^n}^{\psi^\circ}}{h} \geq \operatorname{div} z\,.$$

- Very easy to pass to the limit in this equation in the distributional sense. The difficult part is to show that (d and) z converge to what we expect.

"φ-regularity" of ψ

• If $\psi = \varphi$, To pass in the limit in z_h and show that the limit is calibrating for the limit of the distance we need some control. This is derived from the equation defining z_h: we can show that if $d^{\varphi^\circ}_{E_h(t)}(x) > R$ then

$$\operatorname{div} z_h(x, t+h) \leq \frac{N-1}{R}$$

(this is obtained by comparison of u, with an explicit solution).
• In case $\psi \neq \varphi$, we need ψ to be "φ-regular":

$$W^\psi = \{\psi^\circ \leq 1\} = C + \epsilon W^\varphi$$

for some convex C and $\epsilon > 0$. We can then find a similar estimate ($\operatorname{div} z_h \lessapprox C/(\epsilon R)$).

What if ψ is *not* φ-regular?

- Typical: φ crystalline, $\psi = |\cdot|$.
- Idea: we approximate ψ, for instance letting

$$W^{\psi_\epsilon} := W^{\psi} + \epsilon W^{\varphi}.$$

- Then, we can show that if $E \subset F$ are sets with boundaries positive φ°-distance Δ, evolving one with the mobility ψ and the other $\psi' \approx \psi$, then they get closer at most like $C\|\psi - \psi'\|_{L^\infty(B)}/\Delta$.
- Hence the sequence of evolutions is a Cauchy sequence and converges to a unique limit.
- The estimates are uniform in $h > 0$ small, so that we also deduce that the ATW schemes converge to a unique limit. [Up to fattening.]

Conclusion

- It is classical that such results lead to existence and uniqueness in the level-set sense, or uniqueness as long as the solution does not develop "fattening" (generic uniqueness);
- As usual, no fattening if initial set is star-shaped, or if it is a graph (hence uniqueness in these cases).
- No clear equation in the non "φ-regular" case.
- Equivalent to Giga-Giga, Giga-Požar's viscosity approach when applies;
- Also thanks to [Giga, Ohtsuka, Schätzle 2006], convergence of **Allen-Cahn** approximations.
- We have a "new" proof of existence based on the approximation with smooth anisotropies, and viscosity solution (preprint 2017).
- The distributional formulation is difficult to extend to truly nonlinear evolutions ($V_N = -F(\nu, \kappa_\varphi)$?)

Thank you for your attention