Well-posedness of fully nonlinear PDEs with Caputo time fractional derivatives

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Diffusion in heterogeneous media

- ordinary diffusion by Brownian motion

$$\langle x^2 \rangle = 2Dt, \quad D = \text{const}$$

- anomalous diffusion

$$\langle x^2 \rangle \propto t^\alpha, \quad D \propto t^{\alpha-1}, \quad (0 < \alpha < 1)$$

The behavior of the anomalous diffusion is due to an influence that heterogeneous factors of medium inhibit an movement of diffusing particles.

Reference

- Fomin, Chugunov, and Hashida (2011)
- Sun, Meerschaert, Zhang, Zhu, and Chen (2013)
- Tao, Besant, and Rezkallah (1993)

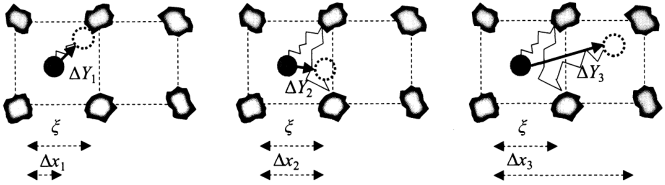
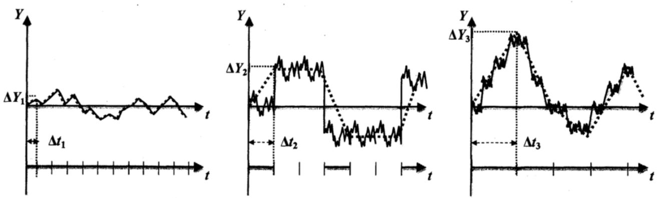


Figure: ξ is the distance between barriers and Δx_i are displacements per observation time Δt_i



Source: N. Shimamoto, RIMS Kokyuroku 1810, 59-84 (2012) (in Japanese)

Modeling by CTRW method

Two key probability density function (pdf)

$$\begin{cases} \lambda(x) & : \quad \text{pdf of the jumping length} \\ \omega(t) & : \quad \text{pdf of the waiting time} \end{cases}$$

τ : a mean waiting time in the Brownian motion.

✓ Gaussian distribution λ and Poisson distribution $\omega \sim e^{-t/\tau} \Rightarrow$ the master equation of this random walk is ordinary diffusion equation

$$\partial_t u - D\Delta u = 0$$

✓ Gaussian distribution λ and $\omega \sim (t/\tau)^{-(1+\alpha)} \Rightarrow$ the master equation is a fractional differential equation

$$\partial_t^\alpha u - D(t)\Delta u = 0,$$

where ∂_t^α is **Caputo fractional derivative**

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_t u(x,s)}{(t-s)^\alpha} ds \quad (\Gamma : \text{gamma function})$$

Reference

- Metzler and Klafter, Physics Reports '00

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Mathematical works for eqns with CTFDs

For linear eqns like $\partial_t^\alpha u - \mathrm{div}(a(x)\nabla u) = f$,

- Prüss, '93
  General theory of linear abstract Volterra eq, strong sol, mild sol
- Luchko, JMAA '09
  classical sol, generalized sol using the eigenfunction expansion
- Sakamoto-Yamamoto, JMAA '11
  distributional weak sol by Fourier method in L^2
- Zacher, Funkcial. Ekvac. '09
  distributional weak sol, weak form

For fully nonlinear eqns with CTFDs,

- Allen, arXiv '15
  viscosity solns to a eqn that appears in optimal control / regularity pb
- Giga and N., CPDE '17
  Well-posedness of (1st order) HJ eqs in $\mathbb{T}^d$
- Topp and Yangari, JDE '17
  Well-posedness of 2nd order FNL eqns in $\mathbb{R}^d$ and large-time behavior
- N., NoDEA '18
  Well-posedness of IBVPs of 2nd order FNL eqns

What I would like to do

A goal is to introduce the viscosity solution to

$$\begin{cases} \partial_t^\alpha u - \Delta u = 0 & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u|_{t=0} = u_0 & \text{on } \overline{\Omega} \end{cases}$$

and show a unique existence result. Here, for the sake of simplicity, we assume that Ω is bounded.

Notation :

$$g = 0 \quad \text{on } \partial\Omega \times [0,T], \qquad = u_0 \quad \text{on } \overline{\Omega}.$$

Remark :

- $-\Delta u$ can be generalized to $F(x,t,u,\nabla u,\nabla^2 u)$
- other boundary condition

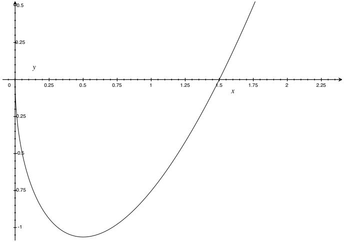
Caputo fractional derivatives

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \quad (0<\alpha<1), \quad = f'(t), \quad (\alpha = 1)$$

It has similar properties as that of the ordinary derivative

- linear operator
- $\partial_t^\alpha \text{const} = 0$
- $\partial_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$

  ex. $\partial_t^{\frac{1}{2}}(t-1)^2 = \partial_t^{\frac{1}{2}} t^2 - 2\partial_t^{\frac{1}{2}} t + \partial_t^{\frac{1}{2}} 1 = \frac{2}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} - \frac{2}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}}$



However,

- $\partial_t^\alpha (f(g)) \neq (\partial_t^\alpha f)(g) \partial_t^\alpha g$
- $\partial_t^\alpha (f \cdot g) \neq (\partial_t^\alpha f) \cdot g + f \cdot (\partial_t^\alpha g)$

<u>Textbooks</u>

- Podlubny, '99
- Kilbas, Srivastava, and Trujillo, '06
- Samko, Kilbas, and Marichev, '93

Viscosity solution (= viscosity sub- and supersolution)

Suppose that u and φ are in $\mathcal{C}$ and

$$\max_{x,t}(u-\varphi) = (u-\varphi)(\hat{x},\hat{t}), \quad (\hat{x},\hat{t}) \in Q_T.$$

Use the maximum principle by Luchko:

Lemma (Luchko, JMAA '09)

Let $f \in C^1((0,T]) \cap C([0,T])$ be s.t. $f' \in L^1(0,T)$. Assume that $\max_{[0,T]} f = f(\hat{t})$ with $\hat{t} \in (0,T]$. Then $(\partial_t^\alpha f)(\hat{t}) \geq 0$.

This implies that

$$\partial_t^\alpha (u-\varphi)(\hat{x},\hat{t}) \geq 0, \quad \nabla_x^2 (u-\varphi)(\hat{x},\hat{t}) \leq O.$$

If u satisfies the eq pointwise, then $\partial_t^\alpha \varphi(\hat{x},\hat{t}) - \Delta\varphi(\hat{x},\hat{t}) \leq 0$.

⇢ $u \in USC$ is a **viscosity subsolution** $\overset{\text{def.}}{\Leftrightarrow}$ $\partial_t^\alpha \varphi - \Delta\varphi \leq 0$ at $(\hat{x},\hat{t})$ holds whenever $u-\varphi$ attains a (local) max at $(\hat{x},\hat{t})$;

⇢ $u \in USC$ is a **viscosity subsolution of IBVP** $\overset{\text{def.}}{\Leftrightarrow}$ u is a viscosity subsolution and $u \leq g$ on $\partial_p Q_T$

$$\mathcal{C} = \{\varphi \in C^{2,1}(Q_T) \cap C(Q_{T,0}) \mid \varphi_t(x,\cdot) \in L^1, \forall x \in \Omega\}, \quad Q_{T,0} = \Omega \times [0,T]$$

Main result

Theorem (N., NoDEA '18)

Assume $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega$. Then there exists a unique sol $u \in C(Q_T \cup \partial_p Q_T)$.

Strategy (in a conventional way[1])

- Perron's method[2]
  1. Construct a subsol $u_- \in USC$ and a supersol $u_+ \in LSC$ that satisfy $u_- = u_+ = g$ on $\partial_p Q_T$ and $u_- \le u_+$ in Q_T
  2. Set $u(x,t) = \sup\{v(x,t) \mid v : \text{subsol}, u_- \le v \le u_+ \text{ in } Q_T \cup \partial_p Q_T\}$
  3. Prove that u^* and u_* are a subsol and a supersol, respectively.
  4. Prove that u is a sol by using the comparison principle
- Comparison principle

[1] cf. Crandall, Ishii, and Lions, *User's guide*
[2] cf. Ishii '87

Comparison principle

Theorem

Let u be a subsol and v be a supersol. If $u \le v$ on $\partial_p Q_T$, then $u \le v$ in Q_T.

Basic idea of the proof = doubling variable argument

1. Suppose $\exists \eta > 0$ s.t. $\sup_{Q_T \cup \partial_p Q_T}(u - v - \eta t^\alpha) = (u-v)(\hat{x}, \hat{t}) - \eta \hat{t}^\alpha > 0$
2. $\exists (\bar{x}, \bar{t}, \bar{y}, \bar{s}) \sim (\hat{x}, \hat{t}, \hat{x}, \hat{t})$: max pt of

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \lambda\Phi(x-y, t-s) - \eta t^\alpha, \quad \lambda > 0$$

on $(Q_T \cup \partial_p Q_T)^2$. (ex. $\Phi \sim |x-y|^2 + |t-s|^2$)

3. {viscosity ineq of u} $-$ {viscosity ineq of v} implies

$$0 \overset{??}{<} \underbrace{\lambda(\partial_t^\alpha \Phi + \partial_s^\alpha \Phi)}_{\text{might be negative}} + \underbrace{\eta\Gamma(1+\alpha)}_{0<} - \underbrace{\lambda(\Delta_x \Phi + \Delta_y \Phi)}_{\text{might be positive}} \le 0.$$

We prepare two facts:

an equivalent definition of sols and **Jensen-Ishii lemma**

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We prepare two facts:

an equivalent definition of sols and **Jensen-Ishii lemma**

Equivalent definition

Caputo derivatives are transformed using integration by parts and changing the variable of integration as follows.

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_t (u(x,t)-u(x,s))}{(t-s)^\alpha} ds$$
$$= \underbrace{\frac{u(x,t)-u(x,0)}{t^\alpha \Gamma(1-\alpha)}}_{=:J[u](x,t)} + \underbrace{\frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{u(x,t)-u(x,t-\tau)}{\tau^{\alpha+1}} ds}_{=:K_{(0,t)}[u](x,t)} .$$

Proposition

Let $u \in USC$. The following assertions are equivalent.

- u is a subsol Go back to the definition
- $\tau \mapsto [u(\hat{x},\hat{t}) - u(\hat{x},\hat{t}-\tau)]/\tau^{\alpha+1}$ is integrable on $(0,\hat{t})$ and

$$J[u] + K_{(0,\hat{t})}[u] - \Delta\varphi \le 0 \quad \text{at } (\hat{x},\hat{t})$$

whenenver $u - \varphi$ attains a local max at $(\hat{x},\hat{t}) \in Q_T$ for $\varphi \in C^{2,1} \cap C$

The problem of finding a suitable test function in time direction is eliminated.

Usability of the equivalent definition

{viscosity ineq of u} $-$ {viscosity ineq of v} implies

$$J[u](\bar{x},\bar{t}) - J[v](\bar{y},\bar{s}) + K_{(0,\bar{t})}[u](\bar{x},\bar{t}) - K_{(0,\bar{s})}[v](\bar{y},\bar{s}) - \lambda(\Delta_x\Phi + \Delta_y\Phi) \leq 0.$$

Remark $\eta\Gamma(1+\alpha)$ does not appear.

Fact

$$\liminf_{\lambda\to\infty}(J[u](\bar{x},\bar{t}) - J[v](\bar{y},\bar{s}) + K_{(0,\bar{t})}[u](\bar{x},\bar{t}) - K_{(0,\bar{s})}[v](\bar{y},\bar{s})) > 0$$

$$J\text{'s terms} \sim \frac{u(\bar{x},\bar{t}) - u(\bar{x},0)}{\bar{t}^\alpha} - \frac{v(\bar{y},\bar{s}) - v(\bar{y},0)}{\bar{s}^\alpha}$$

$$\xrightarrow{\liminf_{\lambda\to\infty}} \frac{[\overbrace{u(\hat{x},\hat{t}) - v(\hat{x},\hat{t})}^{[u(\hat{x},\hat{t})-v(\hat{x},\hat{t})-\eta\hat{t}^\alpha]+\eta\hat{t}^\alpha}] - \overbrace{[u(\hat{x},0) - v(\hat{x},0)]}^{u\leq v \text{ on } \partial_p Q_T}}{\hat{t}^\alpha} > 0.$$

What was J??

Procedure

1. Divide the interval of integration by $0 < \varepsilon < \bar{t}, \bar{s}$
2. The integral on $(0, \varepsilon)$ is estimated by $k_\varepsilon \overset{\varepsilon \to 0}{\to} 0$ (if $\Phi \sim |x-y|^2 + |t-s|^2$)
3. Take $\liminf_{\lambda\to\infty}$ using Fatou lemma
4. The integral on $(\varepsilon, \hat{t})$ is nonnegative

$$K\text{'s terms} \sim \int_0^\varepsilon \frac{[u(\bar{x},\bar{t}) - v(\bar{y},\bar{s})] - [u(\bar{x},\bar{t}-\tau) - v(\bar{y},\bar{s}-\tau)]}{\tau^{\alpha+1}} d\tau$$

$$+ \int_\varepsilon^{\bar{t}} \frac{u(\bar{x},\bar{t}) - u(\bar{x},\bar{t}-\tau)}{\tau^{\alpha+1}} d\tau - \int_\varepsilon^{\bar{s}} \frac{u(\bar{y},\bar{s}) - u(\bar{y},\bar{s}-\tau)}{\tau^{\alpha+1}} d\tau$$

$$\geq k_\varepsilon + \int_\varepsilon^{\bar{t}} \frac{u(\bar{x},\bar{t}) - u(\bar{x},\bar{t}-\tau)}{\tau^{\alpha+1}} d\tau - \int_\varepsilon^{\bar{s}} \frac{u(\bar{y},\bar{s}) - u(\bar{y},\bar{s}-\tau)}{\tau^{\alpha+1}} d\tau$$

$$\overset{\liminf_{\lambda\to\infty}}{\longrightarrow} k_\varepsilon + \int_\varepsilon^{\hat{t}} \frac{[u(\hat{x},\hat{t}) - v(\hat{x},\hat{t})] - [u(\hat{x},\hat{t}-\tau) - v(\hat{x},\hat{t}-\tau)]}{\tau^{\alpha+1}} d\tau \geq k_\varepsilon \to 0$$

What was K??

Recall: $(\bar{x},\bar{t},\bar{y},\bar{s}) \in \operatorname{argmax}(u(x,t) - v(y,s) - \lambda\Phi(x-y,t-s) - \eta t^\alpha)$

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Jensen-Ishii lemma for eqns with Caputo time fractional derivatives

Let u^ε and u_ε denote the sup- and inf-convolution in space, respectively.

Lemma

Let $u \in USC$ be a subsol and $v \in LSC$ be a supersol. Assume that

$$(\bar{x}, \bar{y}, \bar{t}) \in \operatorname*{argmax}_{(x,y,t) \in \overline{\Omega}_\varepsilon \times \overline{\Omega}_\varepsilon \times (0,T]} (u^\varepsilon(x,t) - v_\varepsilon(y,t) - \varphi(x,y,t)).$$

Then there exist $X, Y \in \mathcal{S}^{d \times d}$ s.t.

$$J[u^\varepsilon](\bar{x},\bar{t}) - J[v_\varepsilon](\bar{y},\bar{t}) + K_{(0,\bar{t})}[u^\varepsilon](\bar{x},\bar{t}) - K_{(0,\bar{t})}[v_\varepsilon](\bar{y},\bar{t}) - \operatorname{tr}(X) + \operatorname{tr}(Y) \leq 0$$

and

$$-\frac{2}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \nabla^2_{(x,y)}\varphi(\bar{x},\bar{y},\bar{t})$$

If $\varphi(x,y,t) = \lambda|x-y|^2$, then

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \begin{pmatrix} 2\lambda I & O \\ O & 2\lambda I \end{pmatrix} \quad \to \quad -\operatorname{tr}(X) + \operatorname{tr}(Y) \geq 0.$$

Continuity property

Theorem

Assume $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega$. Let u_α, $\alpha \in (0,1)$, be the solution of IBVP where the order of the Caputo time fractional derivative is α. Let $\beta \in (0,1]$. Then u_α converges to a solution u_β uniformly on $Q_T \cup \partial Q_T$ as $\alpha \to \beta$.

→ The definition of viscosity solution is its natural extension in the integer order case.

→ The behavior of anomalous diffusion look like ordinary diffusion when the medium is almost homogeneous.

Proof : Prove that

$$\overline{u}_\beta(x,t) = \lim_{\delta \searrow 0} \sup\{u_\alpha(y,s) \mid (y,s) \in \overline{B_\delta(x,t)} \cap (Q_T \cup \partial_p Q_T), 0 < |\alpha - \beta| < \delta\}$$

and $\underline{u}_\beta = -\overline{(-u)}_\beta$ are a sub- and supersolution, respectively. Clearly, $\overline{u}_\beta \geq \underline{u}_\beta$. Use the comparison principle to see $\overline{u}_\beta \leq \underline{u}_\beta$. Therefore u_α converges to $\overline{u}_\beta = \underline{u}_\beta$ uniformly.

Problems

1. Free boundary value problem
2. Relationship with other notions of solutions
   - What kind of solution is equivalent?
3. Extension of fractional derivatives
   - the distributed order Caputo fractional derivative

$$(\partial_t^{(\omega)} f)(t) = \int_0^1 \partial_t^\alpha f(t)\omega(\alpha)d\alpha, \quad \text{where } \omega \in C(0,1) \text{ and } \omega > 0.$$

Consider the heat balance

$$-\int_{\Sigma}\int_0^t q^{\alpha}\cdot \nu dt' ds = \int_{\Gamma(t)} \rho c u + \rho L dx,$$

where q is a "fractional heat flux" defined by

$$q^{\alpha}(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{q(x,t')}{(t-t')^{1-\alpha}}dt', \quad (q : \text{ordinary heat flux})$$

and ρ, c, K respectively represent the density, the volumetric specific heat, the latent heat. Then u should satisfy

$$\begin{cases} \rho c \partial_t^{\alpha} u = -\operatorname{div} q & \text{in } \Gamma(t),\ t > 0, \\ \rho L v^{\alpha} = q \cdot \nu & \text{on } \partial\Gamma(t),\ t > 0, \end{cases}$$

where v^{α} is a "fractional normal velocity" defined through the integral identity

$$\int_{\partial\Gamma(t)} v^{\alpha}(x,t)dx = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\int_{\partial\Gamma(t')} v(x,t')dx}{(t-t')^{\alpha}}dt' \quad (v : \text{ordinary normal velocity}).$$

How to reduce to a level set equation?

Summary

- anomalous diffusion is observed in various fields and modeled using Caputo time fractional derivatives by CTRW method
- the notion of viscosity solutions is extended to eqns with Caputo time fractional derivatives
- techniques for Perron's method and the comparison principle are extended to obtain a continuous viscosity solution
- Jensen-Ishii lemma for eqns with Caputo time fractional derivatives
- continuity property

The development of viscosity solution theory to equations with Caputo time fractional derivatives has just begun, and many interesting problems remain.

Thank you very much for your kind attention.

$$\partial_t^\alpha :\ C_\alpha \int_{-\infty}^{t} \frac{\tilde{f}(t)-\tilde{f}(s)}{(t-s)^{\alpha+1}} ds, \qquad (-\Delta)^\alpha :\ C \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2\alpha}} dy$$

Viscosity solution theory for eqns with space-fractional derivatives

- Soner '86 (first result)
- Barles-Imbert '08 (2nd order eqs with Lévy op)
- Alibaud-Imbert '08
- Caffarelli-Silvestre '09 (regularity)
- ...

Lévy op: $-\int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \frac{\nabla f(x) \cdot z}{1+|z|^2} \right) d\mu(z)$, where μ is a Lévy measure, i.e., non-negative Radon measure s.t.

$$\int_{\mathbb{R}^d} \min\{1, |z|^2\} d\mu(z) < +\infty.$$