Curvature effect in shear flow: slowdown of flame speeds with Markstein number

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- Jiancheng Lyu and Yifeng Yu, Mathematics, UC Irvine.
- Yu-Yu Liu, National Cheng Kung University, Taiwan.
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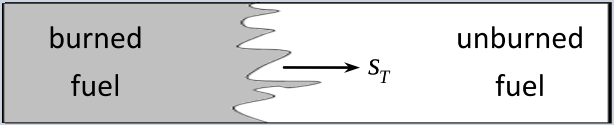
Outline

- Interface Motion by Level Set and Hamilton-Jacobi Equations.
- Curvature in Periodic Shear Flows: Homogenization, and Cell Problem as Nonlinear ODEs.
- Front Speed Analysis via Inequalities.
- Cellular Flows: Computation of Front Speeds under Curvature and Strain (Yu-Yu Liu, finite difference methods: monotone and WENO schemes).
- Conclusion and Future Work.

Origin: Premixed Turbulent Combustion

- In gasoline engine, fuel and air are well-mixed.
  Ignite the fuel and flame front propagates.



- Flame front is wrinkled and propagates at an asymptotic speed.
  (Turbulent Flame Speed "s_T")

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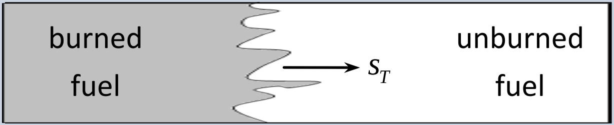

- Flame front is wrinkled and propagates at an asymptotic speed. (Turbulent Flame Speed "s_T")
- In combustion theory, understanding s_T is a fundamental issue.
  Engine Efficiency / Reducing Waste Gas Emission
- GOAL: modeling flame propagation and study s_T.

Flame Propagation Modeling

- Complete physical-chemical modeling requires:
  Navier-Stokes equations (flow)
  coupled with transport equations (chemical reaction).
- Simplified models proposed to characterize flame propagation.
  Reaction-Diffusion-Advection equation with prescribed flows.

Flame Propagation Modeling

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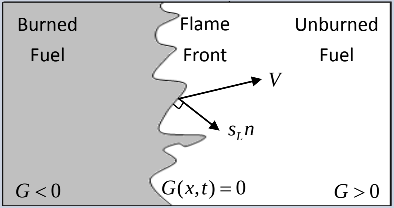
  Reaction-Diffusion-Advection equation with prescribed flows.
- Model flame front as a sharp interface.



Level set of a function: $\{(x,t) : G(x,t) = 0\}$.

Inviscid G-equation

- Motion of flame front in a velocity field (flow) driven by a laminar speed (chemical reaction):

$$\frac{dx}{dt} = V(x,t) + s_L n$$

$n = \frac{DG}{|DG|}$: unit normal (D: spatial gradient)

s_L: laminar flame speed (positive constant)

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- Level set moves in time:

$$G(x(t),t) = 0 \;\Rightarrow\; G_t + \frac{dx}{dt} \cdot DG = 0$$

Inviscid (hyperbolic) G-equation [Williams'85]:

$$G_t + V(x,t) \cdot DG + s_L |DG| = 0$$

A first order Hamilton-Jacobi (HJ) Partial Differential Equation (PDE).

Basic and Extended G-equation Models

- Inviscid G-equation:

$$G_t + V(x) \cdot DG + s_L |DG| = 0$$

- Curvature-Strain G-equation:

$$G_t + V(x) \cdot DG + \left(s_L + d_M \frac{DG \cdot DV \cdot DG}{|DG|^2} \right) |DG| = d_M s_L |DG| \operatorname{div} \left(\frac{DG}{|DG|} \right)$$

  d_M: Markstein number.

  Degenerate 2nd order nonlinear diffusion.

  Non-coercive non-convex Hamiltonian.

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$$G_t + V(x) \cdot DG + s_L |DG| = d_M s_L |DG| \mathrm{div} \left(\frac{DG}{|DG|} \right)$$

- Viscous G-equation:

$$G_t + V(x) \cdot DG + s_L |DG| = d_M \, s_L \, \Delta G$$

- GOAL: behavior of s_T under curvature/viscosity/strain effect.

Large Space-Time Behavior

- Capture front speeds as invariants in large space-time scale, write $d_M = d$, $G^\epsilon(x,t) = \epsilon G(\frac{x}{\epsilon}, \frac{t}{\epsilon})$:

$$G_t^\epsilon + V\left(\frac{x}{\epsilon}\right) \cdot DG^\epsilon + s_L |DG^\epsilon| = \epsilon d \Delta G^\epsilon$$

$V(\cdot)$: 1-periodic. $V\left(\frac{x}{\epsilon}\right)$: small scale fluctuation

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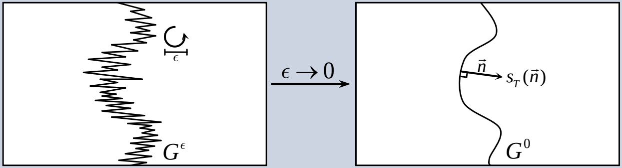

- Periodic Homogenization: as $\epsilon \to 0$, formally $G^\epsilon \to G^0$ satisfying

$$G_t^0 + \bar{H}(DG^0) = 0$$

Periodic Homogenization

- General Theory [Lions-Papanicolaou-Varadhan'86]:

$$u_t^\epsilon + H\left(\frac{x}{\epsilon}, Du^\epsilon\right) = 0$$

- Require Hamiltonian to be coercive:

$$\lim_{|p|\to+\infty} |H(x,p)| \to +\infty \text{ uniformly in } x,$$

and periodic in x. Cell problem determines $\bar{H}$:

$$H(y, P + D_y v) = \bar{H}(P), \ \forall\, y \in \mathbb{T}^n.$$

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- Second order fully nonlinear equations [Evans'89,'92]:

$$F\left(D^2 u^\epsilon, Du^\epsilon, u^\epsilon, x, \frac{x}{\epsilon}\right) = 0$$

- Introduce perturbed test function method based on viscosity solutions

Cell Problem: Formal Derivation for Viscous G-equation

- Two-scale asymptotic expansion:

$$G^{\epsilon}(x,t) = G^{0}(x,t) + \epsilon G^{1}\left(x, \frac{x}{\epsilon}, t\right) + \cdots$$

Leading order ($y = \frac{x}{\epsilon}$):

$$G_t^0 + V(y) \cdot (D_x G^0 + D_y G^1) + s_L |D_x G^0 + D_y G^1| = d \Delta_y G^1$$

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- Cell problem: given any vector $P \in \mathbb{R}^n$, find a unique number $\bar{H} = \bar{H}(P)$ such that the equation

$$-d \Delta_y u + V(y) \cdot (P + D_y u) + s_L |P + D_y u| = \bar{H}, \;\; y \in \mathbb{T}^n$$

  has a periodic solution $u = u(y)$.

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- The cell problem is solvable, G-equation has front solution:

$$G^{\epsilon}(x,t) = -\bar{H}t + P\cdot x + \epsilon u\left(\frac{x}{\epsilon}\right)$$

If $\|P\| = 1$, $\bar{H} = s_T =$ turbulent flame speed in direction P.

Curvature G-equation

- Cell problem ($s_L = 1$):

$$-d\,|p+Dw|\,\mathrm{div}\left(\frac{p+Dw}{|p+Dw|}\right)+|p+Dw|+V(y)\cdot(p+Dw)=\overline{H}_d(p) \tag{1}$$

solution is unknown in general.

- Consider 1-periodic shear flow:

$$V(x)=(v(x_2),0), \qquad \text{for } x=(x_1,x_2)\in\mathbb{R}^2.$$

For $p=(\gamma,\mu)$, (1) becomes nonlinear ODE:

$$-\frac{d\gamma^2 w''}{\gamma^2+(\mu+w')^2}+\sqrt{\gamma^2+(\mu+w')^2}+\gamma v(y)=\overline{H}_d(p) \tag{2}$$

- $\exists$ a unique number $\overline{H}_d(p)$ such that ODE (2) has a C^2 periodic solution.

Curvature G-equation

Theorem (Lyu-X-Yu, CMP 2018)

Let 1-periodic function $v = v(y) \neq$ *constant, and* $\gamma \neq 0$. *Then*

(1)

$$\frac{\partial \overline{H}_d(p)}{\partial d} < 0.$$

Thus $\overline{H}_d$ *is* **strictly decreasing in Markstein number** d.

(2) $\lim_{d\to 0^+} \overline{H}_d = \overline{H}_0 = \overline{H}_0(p)$, *the unique number such that the inviscid cell equation below admits periodic viscosity solution*

$$\sqrt{\gamma^2 + (\mu + w_0')^2} + \gamma v(y) = \overline{H}_0(p).$$

(3) $\lim_{d\to +\infty} \overline{H}_d = |p| + \gamma \int_0^1 v(y)\, dy$, $\lim_{d\to+\infty} w = 0$ *uniformly in* $\mathbb{R}$.

- Folklore in combustion: s_T is decreasing in $d = d_M$ because curvature smoothes wrinkled flames. Rough flames move faster.

Sketch of Proof of (1)

- Let $\phi = \frac{\mu + w'}{\gamma}$, unique periodic solution to

$$-\frac{d\phi'}{1+\phi^2} + \sqrt{1+\phi^2} + v(y) = E(d) = \frac{\overline{H}_d(p)}{\gamma}$$

subject to $\int_0^1 \phi(x)\, dx = \frac{\mu}{\gamma}$. Suffices $E'(d) < 0$.

- $F(x) := \phi_d(x)$, periodic and mean zero over $[0,1]$, satisfies:

$$-d\,F' + b(x)\,F = E'(d)(1+\phi^2) + \phi',$$

where $b(x) = \frac{2d\,\phi'\phi}{1+\phi^2} + \phi\sqrt{1+\phi^2}$.

- $F(0) = F(1)$ and $\int_{[0,1]} F(x)\, dx = 0$ imply $E'(d) = -Nu/De$, Nu is:

$$e^{g(1)} \int_0^1 \phi' e^{-g(x)}\, dx \int_0^1 e^{g(x)}\, dx - (e^{g(1)} - 1) \int_0^1 e^{g(x)} \int_0^x \phi' e^{-g(y)}\, dy\, dx$$

$g(x) = \int_0^x b(y)\, dy$, and $De > 0$. Just show $Nu > 0$.

Sketch of Proof of (1)

- Let $h(x) = \int_0^x \phi\sqrt{1+\phi^2}\,dy$, $\lambda(\phi) = \arctan\phi$. Integration by parts: $Nu = A + B - C$,

$$A(\phi) = e^{h(1)} \int_0^1 \lambda(\phi) e^{-h(x)} \phi\sqrt{1+\phi^2} \int_0^x (1+\phi^2) e^{h(y)}\,dydx$$

$$B(\phi) = \int_0^1 \lambda(\phi) e^{-h(x)} \phi\sqrt{1+\phi^2} \int_x^1 (1+\phi^2) e^{h(y)}\,dydx$$

$$C(\phi) = (e^{h(1)} - 1) \int_0^1 \lambda(\phi)(1+\phi^2)\,dx.$$

- If $h(1) = 0$, $A + B - C = A + B \geq 0$ as $s\,\lambda(s) \geq 0$, "=" iff $\phi \equiv 0$.
- WLOG, $h(1) > 0$. Let $\phi_+ = \max\{\phi, 0\}$, $\phi_- = \min\{\phi, 0\}$, $h^{\pm}(x) = \int_0^x \phi_{\pm}\sqrt{1+\phi_{\pm}^2}\,dy$. Then $h(x) = h^+ + h^-$.
- Prove ("=" iff $\phi \geq 0$, i.e., $\phi_- = 0$):

$$A(\phi) + B(\phi) - C(\phi) \geq e^{h^-(1)} \left(A(\phi_+) + B(\phi_+) - C(\phi_+) \right).$$

Sketch of Proof of (1)

- Let $\xi := h^+(x)$, strictly increasing in x; $\phi(\xi) := \phi_+(x)$, $T = h^+(1)$.

$$A(\phi_+) = A_{T,\psi} := e^T \int_0^T \lambda(\psi) e^{-x} \int_0^x \frac{\sqrt{1+\psi^2}}{\psi} e^y \, dy \, dx$$

$$B(\phi_+) = B_{T,\psi} := \int_0^T \lambda(\psi) e^{-x} \int_x^T \frac{\sqrt{1+\psi^2}}{\psi} e^y \, dy \, dx$$

$$C(\phi_+) = C_{T,\psi} := (e^T - 1) \int_0^T \lambda(\psi) \frac{\sqrt{1+\psi^2}}{\psi} \, dx.$$

- Prove

$$0 < A_{T,\psi} + B_{T,\psi} - C_{T,\psi} = e^T \int_0^T \lambda(\psi) e^{-x} \int_0^x \frac{\sqrt{1+\psi^2}}{\psi} e^y \, dy \, dx$$

$$+ \int_0^T \lambda(\psi) e^{-x} \int_x^T \frac{\sqrt{1+\psi^2}}{\psi} e^y \, dy \, dx - (e^T - 1) \int_0^T \lambda(\psi) \frac{\sqrt{1+\psi^2}}{\psi} \, dx.$$

Key Inequality

Theorem

Let $T>0$, $f\in C([0,T])$ *be positive,* $g\in C^1((0,L])$, $L:=\max_{[0,T]} f$.

(1) If $g'\le -\theta$ *for some* $\theta\ge 0$, *then*

$$e^T\int_0^T f(x)e^{-x}\int_0^x g(f(y))e^y\,dy\,dx+\int_0^T f(x)e^{-x}\int_x^T g(f(y))e^y\,dy\,dx$$

$$\ge (e^T-1)\int_0^T f(x)\,g(f(x)))\,dx+\frac{\theta}{2}\int_{[0,T]^2}|f(x)-f(y)|^2\,dx\,dy.$$

(2) If $g'\ge \theta$ *for some* $\theta\ge 0$, *then*

$$e^T\int_0^T f(x)e^{-x}\int_0^x g(f(y))e^y\,dy\,dx+\int_0^T f(x)e^{-x}\int_x^T g(f(y))e^y\,dy\,dx$$

$$\le (e^T-1)\int_0^T f(x)\,g(f(x)))\,dx-\frac{\theta}{2}\int_{[0,T]^2}|f(x)-f(y)|^2\,dx\,dy.$$

Sketch of Proof of (1)

- Let $M := \max_{[0,T]} \psi = \max_{[0,1]} \phi_+ > 0$. In key inequality (part 1), take $f(x) = \lambda(\psi) = \arctan(\psi)$, $g(y) = \frac{1}{\sin y}$, $L = \arctan(M)$ and $\theta = \frac{1}{\sqrt{1+M^2}}$, then $\frac{\sqrt{1+\psi^2}}{\psi} = g(f)$.
- It follows:

$$A_{T,\psi} + B_{T,\psi} - C_{T,\psi} \geq \frac{1}{2\sqrt{1+M^2}} \int_{[0,T]^2} |\lambda(\psi(x)) - \lambda(\psi(y))|^2 \, dxdy$$

$$= \frac{1}{2\sqrt{1+M^2}} \int_{[0,1]^2} |\lambda(\phi_+(x)) - \lambda(\phi_+(y))|^2 J(x)J(y) \, dxdy$$

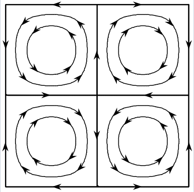
$$> 0 \qquad \text{(since } \phi_+' \neq 0\text{)}.$$

Here $J(x) = \phi_+(x)\sqrt{1 + \phi_+^2}$.

Cellular Flow

- Front motion in 2D cellular flow (Hamiltonian flow):

$$V(x) = (-\partial_{x_2}\mathcal{H}, \partial_{x_1}\mathcal{H})$$



Stream function:

$$\mathcal{H} = \frac{A}{2\pi}\sin(2\pi x_1)\sin(2\pi x_2)$$

Time independent, incompressible, periodic flow.

A: amplitude/ flow intensity

- QUESTION: How does turbulent flame speed s_T depend on flow ?

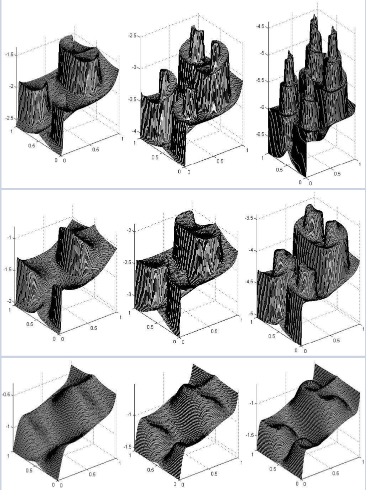
In particular at high flow intensity?

Parameterize s_T as a function of A:

$$s_T = s_T(A)$$

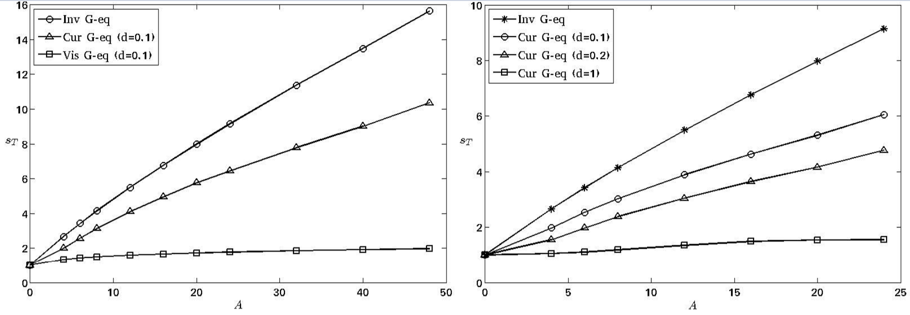
GOAL: behavior of $s_T(A)$ in $A \gg 1$ (similar to $d_M \ll 1$ at fixed A).

Compare $G(x,t)$ of Inviscid, Curvature, Viscous G-eq



Graphs of $G(x,1)$ for inviscid (1st row in $A = 4, 8, 16$), curvature, viscous G-equation with $s_L = 1$, $P = e_1$, $d = 0.1$.

Compare $s_T(A,d)$ of Inviscid, Curvature, Viscous G-eq



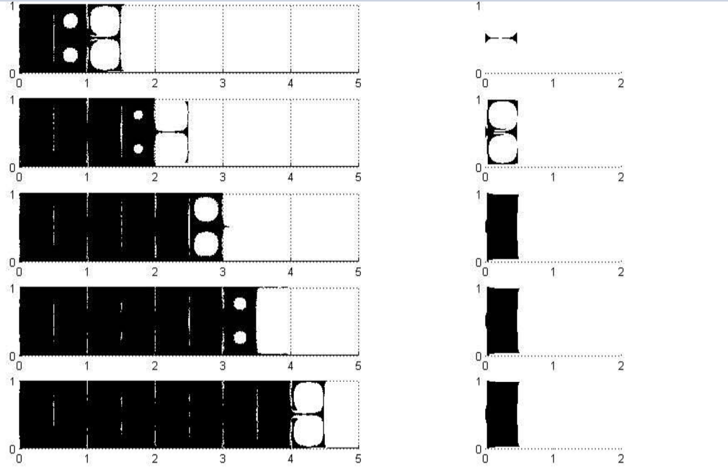
(L) Plots of $s_T = s_T(A)$ of inviscid, curvature, viscous G-equations, suggesting:

$$s_L < s_T^{vis} \leq s_T^{cur} \leq s_T^{inv}$$

(R) Plots of $s_T = s_T(A)$ for curvature G-equation with various d.

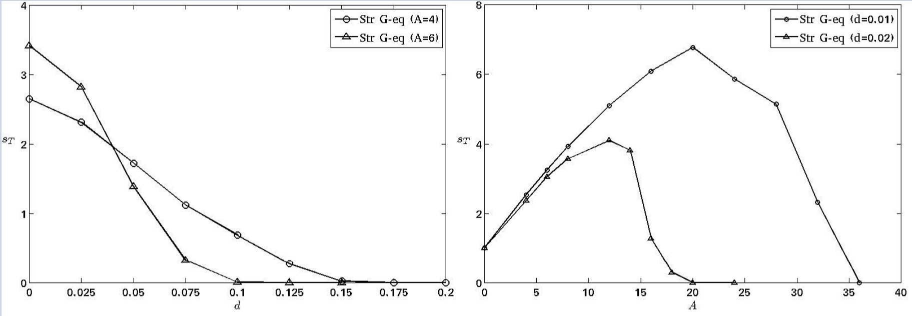
$$s_T^{inv} = O\left(\frac{A}{\log A}\right), \; s_T^{cur} = O(??), \; s_T^{vis} = O(1).$$

Propagation in Curvature-Strain G-equation



Front Propagation in curvature-strain G-eq. at $A = 32$, $t = 0.3, 0.6, 0.9, 1.2, 1.5$. (L) incomplete burning ($d = 0.01$) (R) front stops moving at a finite time ($d = 0.02$).

s_T vs. d, and s_T vs. A in Curvature-Strain G-equation



(L) Plots of $s_T = s_T(d)$ for curvature-strain G-eq at $A = 4, 6$.

In inviscid/curvature/viscous G-equation, $s_T \geq s_L$ for all $d > 0$.

(R) Plots of $s_T = s_T(A)$ in curvature-strain G-equation at $d = 0.01, 0.02$.

s_T in Cellular Flow and Strain G-equation

Theorem [X-Yu'14 (Arch Ration Mech Analysis)]

Let G be the unique viscosity solution of the Strain G-equation with cellular flow ($\mathcal{H} = A \sin x_1 \sin x_2$), and initial data $G(x, 0) = p \cdot x$, unit vector p, there exists a universal constant $d_0 \in (0, 1)$ such that when $d < d_0$ and $A > \frac{8}{d^3}$

$$|G(x,t) - p \cdot x| \leq 3\sqrt{2}\pi \quad \text{for all } t \geq 0.$$

In particular,

$$s_T(p, A) = \lim_{t \to +\infty} \frac{-G(x,t)}{t} = 0 \quad \text{locally uniformly in } \mathbb{R}^2.$$

- Stretching of the cellular flow dramatically reduces s_T.
- Proof is based on two-player differential game representation of non-convex Hamilton-Jacobi equation.

Conclusion and Future Work

- Front speed slow down in Markstein number (curvature smoothing) is proved for shear flows using structures of nonlinear ODEs.
- Curvature effects in cellular flows ?
- Main challenge: analyzing cell problem (a non-coercive, non-convex Hamilton-Jacobi PDE) under curvature smoothing.
- Any method to simplify the curvature term ?