Stability and error analysis for a diffuse interface approach to an advection–diffusion equation on a moving surface

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Advanced Developments for Surface and Interface Dynamics
Analysis and Computation

Problem formulation

given $(\Gamma(t))_{t\in[0,T)} \subset \mathbb{R}^{n+1}$ family of evolving closed hypersurfaces

$S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$

$v : S_T \to \mathbb{R}^{n+1}$ velocity field, $v \cdot \nu = V_\Gamma$

$u_0 : \Gamma(0) \to \mathbb{R}$.

find u such that

$$\begin{aligned} \partial_t^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u &= 0 \qquad \text{on } S_T \qquad (1) \\ u(0) &= u_0 \qquad \text{on } \Gamma(0). \qquad (2) \end{aligned}$$

Existence and uniqueness

Theorem Suppose that $(\Gamma(t))_{t\in[0,T]}$ is smooth, $v \in C^1(\overline{S_T}, \mathbb{R}^{n+1})$ and $u_0 \in H^1(\Gamma(0))$. Then, (1), (2) has a unique weak solution $u \in H^1(S_T)$ such that $u(0) = u_0$ and

$$\int_{\Gamma(t)} \partial_t^\bullet u\, \varphi + \int_{\Gamma(t)} u\, \varphi\, \nabla_\Gamma \cdot v + \int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = 0$$

for all $\varphi \in H^1(\Gamma(t))$ and a.a. $t \in (0, T)$. Furthermore:

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for all $\varphi \in H^1(\Gamma(t))$ and a.a. $t \in (0, T)$. Furthermore:

$$\int_{\Gamma(t)} u(\cdot, t) = \int_{\Gamma(0)} u_0, \qquad 0 < t < T;$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} u^2 + \int_{\Gamma(t)} |\nabla_\Gamma u|^2 = -\frac{1}{2}\int_{\Gamma(t)} u^2\, \nabla_\Gamma \cdot v.$$

Triangulated surfaces:

Dziuk & Elliott (ESFEM, '07,'12,'13), Lenz, Nemadjieu & Rumpf (FV, '08)

Eulerian approach, extended PDE:

Adalsteinsson & Sethian '03, Xu & Zhao '03, Adalsteinsson, Colella, Arkin & Onsum '05, Teigen, Li, Lowengrub, Wang & Voigt '09, Dziuk & Elliott '10, Elliott, Stinner, Styles & Welford '11, Petras & Ruuth '16

Restriction of bulk FEM, CutFEM:

Olshanskii, Reusken & Xu '14, Olshanskii & Reusken '14, Hansbo, Larson & Zahedi '15, Lehrenfeld, Olshanskii & Xu '17

Preliminaries, cf. Dziuk & Elliott '10

Stationary surface $\Gamma = \{x \in \Omega \,|\, \phi(x) = 0\}, \quad \nabla\phi(x) \neq 0$:

$$\begin{aligned} \nu &= \frac{\nabla \phi}{|\nabla \phi|} \\ H &= -\nabla \cdot \nu \end{aligned}$$

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\nabla_\phi \eta &= \big(I - \nu \otimes \nu\big)\nabla\eta \\
\Delta_\phi \eta &= \nabla_\phi \cdot \nabla_\phi \eta
\end{aligned}$$

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\Delta_\phi \eta &= \nabla_\phi \cdot \nabla_\phi \eta \\
\int_\Omega \nabla_\phi f \, |\nabla\phi| &= -\int_\Omega f \, H \, \nu \, |\nabla\phi| \qquad \mathrm{supp} f \subset\subset \Omega.
\end{aligned}
$$

Extension

Moving surfaces $\quad \Gamma(t) = \{x \in \Omega \,|\, \phi(x,t) = 0\}, \; \nabla\phi(x,t) \neq 0;$

a) There exists an extension v of the velocity such that

$$\phi_t + v \cdot \nabla\phi = 0.$$

Extension

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a) There exists an extension v of the velocity such that

$$\phi_t + v \cdot \nabla\phi = 0.$$

b) Suppose that u is a solution of the surface PDE

$$\partial_t^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u = 0 \quad \text{on } S_T.$$

There exists an extension u^e of u such that $\nabla u^e \cdot \nabla\phi = 0$ and

$$\partial_t^\bullet u^e + u^e \, \nabla_\phi \cdot v - \Delta_\phi u^e = \phi \, R \quad \text{in } \Omega \times (0, T).$$

where $\partial_t^\bullet f = f_t + \nabla f \cdot v$.

Phase field function

For $\epsilon > 0$ define

$$\rho(x,t) := \sigma\Big(\frac{\phi(x,t)}{\epsilon}\Big),$$

where $\sigma \in C^{0,1}(\mathbb{R})$ is given by

$$\sigma(r) = \begin{cases} \frac{3}{4}(1-r^2), & |r| \le 1, \\ 0, & |r| > 1. \end{cases}$$

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$$\frac{1}{\epsilon}\int_\Omega f\,\rho\,|\nabla\phi|\,dx = \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} \sigma\Big(\frac{s}{\epsilon}\Big)\int_{\{\phi=s\}} f\,d\mathcal{H}^n ds \approx \int_{\{\phi=0\}} f\,d\mathcal{H}^n.$$

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Note that

$$\nabla_\phi \rho = 0, \quad \partial_t^\bullet \rho = 0.$$

Weak formulation

Let $\eta \in H^1(\Omega)$:

$$\frac{d}{dt}\int_\Omega u^e\,\eta\,\rho\,|\nabla\phi| = \int_\Omega \partial_t^\bullet\big(u^e\,\eta\,\rho\big)|\nabla\phi| + \int_\Omega u^e\,\eta\,\nabla_\phi\cdot v\,\rho\,|\nabla\phi|$$

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$$= \int_\Omega \big(\partial_t^\bullet u^e + u^e\,\nabla_\phi\cdot v\big)\eta\,\rho\,|\nabla\phi| + \int_\Omega u^e\big(\eta\,\underbrace{\partial_t^\bullet\rho}_{=0} + \underbrace{\partial_t^\bullet\eta}_{=v\cdot\nabla\eta}\,\rho\big)|\nabla\phi|$$

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$$= \int_\Omega \big(\Delta_\phi u^e + \phi\,R\big)\eta\,\rho\,|\nabla\phi| + \int_\Omega u^e\,v\cdot\nabla\eta\,\rho\,|\nabla\phi|$$

Weak formulation

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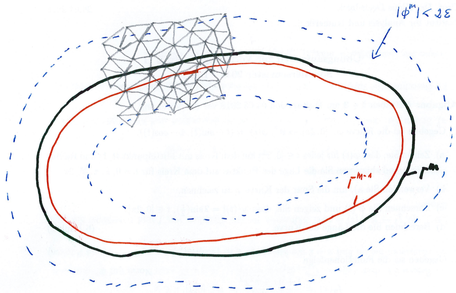
$$= \int_\Omega (\Delta_\phi u^e + \phi\,R)\eta\,\rho\,|\nabla\phi| + \int_\Omega u^e\,v\cdot\nabla\eta\,\rho\,|\nabla\phi|$$

$$\approx -\int_\Omega \underbrace{\nabla_\phi u^e\cdot\nabla_\phi\eta}_{=\nabla u^e\cdot\nabla\eta}\,\rho\,|\nabla\phi| + \int_\Omega u^e\,v\cdot\nabla\eta\,\rho\,|\nabla\phi|.$$

Discretization

- Let $\mathcal{T}_h$ be a regular triangulation of Ω, $t_m = m\tau, m = 0, 1, \ldots$
- $D_h^m = \bigcup\{T \mid |\phi^m(x)| < 2\epsilon$ for some vertex $x \in T\}$.
- $V_h^m = \{\eta_h \in C^0(D_h^m) \mid \eta_{h|T} \in P_1(T), T \subset D_h^m\}$.
- $\phi_h^m = I_h\phi^m$, $\rho_h^m = \sigma(\frac{\phi_h^m}{\epsilon})$.



Scheme: Find $u_h^m \in V_h^m$ such that for all $\eta_h \in V_h^m$

$$\frac{1}{\tau}\Big\{ \int_\Omega u_h^m \, \eta_h \, \rho_h^m \, |\nabla \phi_h^m| - \int_\Omega u_h^{m-1} \, \eta_h \, \rho_h^{m-1} \, |\nabla \phi_h^{m-1}| \Big\}$$

$$+ \int_\Omega \nabla u_h^m \cdot \nabla \eta_h \, \rho_h^m \, |\nabla \phi_h^m| - \int_\Omega u_h^m \, (v^m \cdot \nabla \eta_h) \, \rho_h^m \, |\nabla \phi_h^m|$$

$$+ \int_{D_h^m} (\nabla u_h^m \cdot \nu_h^m)(\nabla v_h \cdot \nu_h^m) = 0$$

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$$+ \int_\Omega \nabla u_h^m \cdot \nabla \eta_h \, \rho_h^m \, |\nabla \phi_h^m| - \int_\Omega u_h^m \, (v^m \cdot \nabla \eta_h) \, \rho_h^m \, |\nabla \phi_h^m|$$

$$+ \int_{D_h^m} (\nabla u_h^m \cdot \nu_h^m)(\nabla v_h \cdot \nu_h^m) = 0$$

Mass conservation, stability: For $m = 1, \ldots, M$

$$(a) \qquad \frac{1}{\epsilon} \int_\Omega u_h^m \rho_h^m \, |\nabla \phi_h^m| = \frac{1}{\epsilon} \int_\Omega u_h^0 \rho_h^0 \, |\nabla \phi_h^0|$$

$$(b) \qquad \frac{1}{\epsilon} \int_\Omega |u_h^m|^2 \, \rho_h^m \, |\nabla \phi_h^m| + \tau \sum_{m=1}^{M} \frac{1}{\epsilon} \int_\Omega |\nabla u_h^m|^2 \rho_h^m \, |\nabla \phi_h^m| \leq C(u_0)$$

provided that $\epsilon = ch$ and $\tau \leq \gamma h$.

Sketch of the proof of (b)

To simplify: ϕ, ρ instead of ϕ_h, ρ_h; $\phi(x,t) = d(x,t) = d_{\Gamma(t)}(x)$.

Choose $\eta_h = u_h^m$:

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Choose $\eta_h = u_h^m$:

$$\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m - \frac{1}{2}\int_\Omega (u_h^{m-1})^2\,\rho^{m-1} + \frac{1}{2}\int_\Omega (u_h^m - u_h^{m-1})^2\,\rho^{m-1}$$

$$\tau\int_\Omega |\nabla u_h^m|^2\,\rho^m + \tau\int_{D_h^m} (\nabla u_h^m \cdot \nu^m)^2$$

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Choose $\eta_h = u_h^m$:

$$
\begin{aligned}
& \frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m - \frac{1}{2}\int_\Omega (u_h^{m-1})^2\,\rho^{m-1} + \frac{1}{2}\int_\Omega (u_h^m - u_h^{m-1})^2\,\rho^{m-1} \\
& \qquad \tau\int_\Omega |\nabla u_h^m|^2\,\rho^m + \tau\int_{D_h^m} (\nabla u_h^m \cdot \nu^m)^2 \\
& = \quad \tau\int_\Omega u_h^m\,(v^m \cdot \nabla u_h^m)\,\rho^m - \frac{1}{2}\int_\Omega (u_h^m)^2(\rho^m - \rho^{m-1}) \\
& = \quad I + II.
\end{aligned}
$$

$I = \tau \int_\Omega u_h^m \, (v^m \cdot \nabla u_h^m) \, \rho^m$

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$$\begin{aligned} I_1 &= \frac{1}{2}\tau \int_\Omega (\nabla_\phi (u_h^m)^2 \cdot v^m) \, \rho^m \\ &= -\frac{1}{2}\tau \int_\Omega (u_h^m)^2 \, \nabla_\phi \cdot v^m \rho^m - \frac{1}{2}\tau \int_\Omega (u_h^m)^2 H^m \underbrace{v^m \cdot \nu^m}_{=-d_t^m} \rho^m; \end{aligned}$$

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$$I_2 = -\tau \int_\Omega u_h^m (\nabla u_h^m \cdot \nu^m) d_t^m \rho^m.$$

$II = -\frac{1}{2} \int_\Omega (u_h^m)^2 (\rho^m - \rho^{m-1})$

$$\rho_t = \frac{1}{\epsilon}\sigma'(\frac{d}{\epsilon})d_t = \frac{1}{\epsilon}\sigma'(\frac{d}{\epsilon})\underbrace{\nabla d \cdot \nu}_{=1} d_t = \nabla \rho \cdot \nu \, d_t.$$

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$$II \;=\; -\frac{1}{2}\int_{t_{m-1}}^{t_m}\int_\Omega (u_h^m)^2 \rho_t = -\frac{1}{2}\int_{t_{m-1}}^{t_m}\int_\Omega (u_h^m)^2 \nabla\rho \cdot \nu\, d_t$$

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since $\nu \cdot \nabla d_t = \nabla d \cdot \nabla d_t = \frac{1}{2} \partial_t |\nabla d|^2 = 0.$

Recall $\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} u^2 + \int_{\Gamma(t)} |\nabla_\Gamma u|^2 = -\frac{1}{2}\int_{\Gamma(t)} u^2\, \nabla_\Gamma \cdot v$

$$\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m - \frac{1}{2}\int_\Omega (u_h^{m-1})^2\,\rho^{m-1} + \tau\int_\Omega |\nabla u_h^m|^2\,\rho^m$$

$$+\tau\int_{D_h^m} (\nabla u_h^m\cdot \nu^m)^2 \le -\frac{1}{2}\tau\int_\Omega (u_h^m)^2\,(\nabla_\phi \cdot v^m)\rho^m$$

$$+\quad \underbrace{\int_{t_{m-1}}^{t_m}\int_\Omega \big((u_h^m)^2\, r + u_h^m\,\nabla u_h^m\cdot \tilde{r}\big)}_{=S}$$

where

$$r = \frac{1}{2}\big(H^m d_t^m \rho^m - H d_t\rho\big), \quad \tilde{r} = d_t\rho\nu - d_t^m\rho^m\nu^m.$$

Recall $\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} u^2 + \int_{\Gamma(t)} |\nabla_\Gamma u|^2 = -\frac{1}{2}\int_{\Gamma(t)} u^2\,\nabla_\Gamma \cdot v$

$$\frac{1}{2}\int_\Omega (u_h^m)^2\,\rho^m - \frac{1}{2}\int_\Omega (u_h^{m-1})^2\,\rho^{m-1} + \tau\int_\Omega |\nabla u_h^m|^2\,\rho^m$$

$$+\tau\int_{D_h^m} (\nabla u_h^m \cdot \nu^m)^2 \le -\frac{1}{2}\tau\int_\Omega (u_h^m)^2\,(\nabla_\phi \cdot v^m)\rho^m$$

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where

$$r = \frac{1}{2}\big(H^m d_t^m \rho^m - H d_t \rho\big), \quad \tilde{r} = d_t \rho \nu - d_t^m \rho^m \nu^m.$$

$$|S| \le C\tau\int_\Omega (u_h^m)^2\,\rho^m + \tau\left(\frac{1}{2} + C\epsilon^2\right)\int_{D_h^m} (\nabla u_h^m \cdot \nu^m)^2.$$

Theorem (Error bound)

Let u be the solution of the surface PDE, $u_h^m \in V_h^m, m = 0, \ldots, M$ the discrete solution. Then

$$\max_m \int_{\Gamma(t_m)} |u^m - u_h^m|^2 + \tau \sum_{m=1}^{M} \int_{\Gamma(t_m)} |\nabla_\Gamma (u^m - u_h^m)|^2 \leq C h^2,$$

provided that $\epsilon = ch, \tau \leq \epsilon^2$.

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provided that $\epsilon = ch, \tau \leq \epsilon^2$.

Work in progress

- Error bound under time step restriction $\tau \leq c\epsilon$;
- Analysis of a scheme with

$$I_h[\sigma(\frac{\phi}{\epsilon})] \quad \text{instead of} \quad \sigma(\frac{I_h \phi}{\epsilon}).$$