The forward-backward scheme for the minimizing total variation flow in H^{-s}

Work in the collaboration with

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We consider the nonlinear, singular, $(2s+2)$-order diffusion equation

$$\frac{\partial u}{\partial t} = (-\Delta_{\mathrm{av}})^s \left[\text{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right] \qquad \text{in } \mathbb{T}^d \times (0, \infty) \tag{1}$$

with periodic boundary conditions and the initial data $u_0 \in H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$.

Here $\mathbb{T}^d := \prod_{i=1}^d \mathbb{R} \setminus \mathbb{Z}$ denotes the d-dimensional torus and s is the index in $[0,1]$.

For $s \in (0,1]$, we define by $H_{\text{av}}^{-s}(\mathbb{T}^d)$, the space dual of

$$H_{\mathrm{av}}^s(\mathbb{T}^d) := \left\{ u \in H^s(\mathbb{T}^d) \ : \ \int_{\mathbb{T}^d} u \, dx = 0 \right\},$$

where $H^s(\mathbb{T}^d)$ is the standard fractional Sobolev space.

The inner product in $H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$ is defined by

$$(u, v)_{H_{\mathrm{av}}^{-s}} := \int_{\mathbb{T}^d} (-\Delta_{\mathrm{av}})^{-s} uv \, dx \qquad \text{for all } u, v \in H_{\text{av}}^{-s}(\mathbb{T}^d).$$

The rigorous interpretation of the equation (1) is

$$\begin{cases} \dfrac{du}{dt}(t) \in -\partial_{H^{-s}_{\mathrm{av}}} \Phi(u(t)) & \text{in } H^{-s}_{\mathrm{av}}(\mathbb{T}^d) \text{ for a.e. } t \in (0,\infty)\,, \\ u(0) = u_0 & \text{in } H^{-s}_{\mathrm{av}}(\mathbb{T}^d)\,, \end{cases}$$

where the functional Φ is defined on $L^2(\mathbb{T}^d)$ by

$$\Phi(u) := \begin{cases} \displaystyle\int_{\mathbb{T}^d} |\nabla u| & \text{if } u \in BV(\mathbb{T}^d) \cap H^{-s}_{\mathrm{av}}(\mathbb{T}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

and $\partial_{H^{-s}_{\mathrm{av}}} \Phi$ denotes the subdifferential of Φ with respect to $H^{-s}_{\mathrm{av}}(\mathbb{T}^d)$-topology.

The total variation of the function u is defined by

$$\int_{\mathbb{T}^d} |\nabla u| := \sup_z \left(\int_{\mathbb{T}^d} u \,\mathrm{div} z \, dx : \; z \in C_0^1(\mathbb{T}^d, \mathbb{R}^d), \; \|z\|_\infty \leq 1 \right) ,$$

where for a vector field $z(x)$, the norm $\|\cdot\|_\infty$ is defined by $\|z\|_\infty := \sup_x |z(x)|$, and $|\cdot|$ is the standard Euclidean norm.

Characterization of subdifferential

Theorem 1

Assume that $u \in H^{-s}_{\mathrm{av}}(\mathbb{T}^d)$ *is such that* $\Phi(u) < +\infty$. *Then* $v \in \partial_{H^{-s}_{\mathrm{av}}}\Phi(u)$ *if and only if there exists* $z \in X(\mathbb{T}^d) := \{z \in L^\infty(\mathbb{T}^d, \mathbb{R}^d) \,:\, \mathit{div}\, z \in H^s_{av}(\mathbb{T}^d)\}$, *such that*

$$\begin{cases} v = -(-\Delta_{\mathrm{av}})^s \,\mathit{div}\, z\,, \\ \|z\|_\infty \le 1\,, \\ (u, -(-\Delta_{\mathrm{av}})^s \,\mathit{div}\, z)_{H^{-s}_{\mathrm{av}}(\mathbb{T}^d)} = \int_{\mathbb{T}^d} |\nabla u|\,, \end{cases}$$

where $\partial_{H^{-s}_{\mathrm{av}}}\Phi$ *is the subdifferential of* Φ *with respect to* $H^{-s}_{\mathrm{av}}(\mathbb{T}^d)$*-topology.*

- F. Andreu, C. Ballester, V. Caselles, J. M. Mazón, Minimizing total variation flow, *Diff. and Int. Eq.*, 2001.
- Y. Giga, H. Kuroda, H. Matsuoka, Fourth-order total variation flow with Dirichlet condition: characterization of evolution and extinction time estimates, *Adv. Math. Sci. App.*, 2014.

Existence and uniqueness

The existence and uniqueness of a solution to the system

$$\begin{cases} \dfrac{du}{dt}(t) \in -\partial_{H_{\mathrm{av}}^{-s}} \Phi(u(t)) & \text{in } H_{\mathrm{av}}^{-s}(\mathbb{T}^d) \text{ for a.e. } t \in (0,\infty)\,, \\ u(0) = u_0 & \text{in } H_{\mathrm{av}}^{-s}(\mathbb{T}^d)\,, \end{cases}$$

guarantees the theorem:

Theorem 2

Let $\mathcal{A}(u) := \partial_{H_{\mathrm{av}}^{-s}} \Phi(u)$ and suppose that $u_0 \in D(\mathcal{A})$. Then, there exists a unique function $u : [0,\infty) \to H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$ such that:

(1) *for all $t > 0$ we have that $u(t) \in D(\mathcal{A})$,*

(2) $\dfrac{du}{dt} \in L^\infty(0,\infty, H_{\mathrm{av}}^{-s}(\mathbb{T}^d))$ *and* $\left\| \dfrac{du}{dt} \right\|_{H_{\mathrm{av}}^{-s}(\mathbb{T}^d)} \leq \|\mathcal{A}^0(u_0)\|_{H_{\mathrm{av}}^{-s}(\mathbb{T}^d)}$,

(3) $\dfrac{du}{dt} \in -\mathcal{A}(u(t))$ *a.e. on* $(0,\infty)$,

(4) $u(0) = u_0$.

- H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, *North Holland Publishing Company*, Amsterdam, 1973.

We consider the finite set of $n+1$ equidistant points

$$\{t_i = i\tau \ : \ i = 0, \dots, n \text{ and } \tau = t/n\}$$

in the interval $[0, t]$.

We set $u_\tau(0) = u_0$. For $i = 1, \dots, n$, we define recursively by $u_\tau(t_i)$ a solution of

$$\frac{u_\tau(t_i) - u_\tau(t_{i-1})}{\tau} \in -\partial_{H^{-s}_{\mathrm{av}}} \Phi(u_\tau(t_i)) \,. \tag{2}$$

Let $\mathcal{A} := \partial_{H^{-s}_{\mathrm{av}}} \Phi$, then we can write

$$u_\tau(t_i) = (I + \tau \mathcal{A})^{-i} u_0 \,.$$

It is well known that if $\mathcal{A}$ is monotone, then the resolvent $(I + \tau\mathcal{A})^{-1}$ is non-expansive, which implies that the above implicit scheme is stable.

We observe that the equation (2) for $u_\tau(t_i)$ is the optimality condition for the minimization problem

$$\inf_{u \in H^{-s}_{\mathrm{av}}(\mathbb{T}^d)} \left\{ \frac{1}{2\tau} \|u - u_\tau(t_{i-1})\|^2_{H^{-s}_{\mathrm{av}}} + \Phi(u) \right\} .$$

- M. G. Crandall, T. M. Liggett, Generation of semi-groups of nonlinear transformations on general Banach spaces, *Amer. J. Math.*, 1971.

Dual problem

Let $f \in H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$ be a given function. To derive the dual problem to

$$\inf_{u \in H_{\mathrm{av}}^{-s}(\mathbb{T}^d)} \left\{ \frac{1}{2\tau} \|u - f\|_{H_{\mathrm{av}}^{-s}}^2 + \Phi(u) \right\}$$

we need the following two results:

Lemma 1

Let the functional Φ be convex, proper and lower-semicontinuous, then we have $v \in \partial_{H_{\mathrm{av}}^{-s}} \Phi(u)$ if and only if $u \in \partial_{H_{\mathrm{av}}^{-s}} \Phi^*(v)$.

Lemma 2

For $u \in BV(\mathbb{T}^d) \cap H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$, we have that the convex conjugate of the functional Φ in H_{av}^{-s} is given by $\Phi^*(v) = \chi_K(v)$, where K is the closure of the set

$$\{v \in \mathcal{D}'(\mathbb{T}^d) \ : \ v = -(-\Delta_{\mathrm{av}})^s \mathit{div} z, \ z \in \mathcal{D}(\mathbb{T}^d), \ \|z\|_\infty \leq 1\}$$

with respect to the $H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$-topology.

Dual problem

Theorem 4

Let $f \in H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$ *be a given function, then the problem*

$$\inf_{v \in K} \left\{ \frac{1}{2\tau} \|\tau v - f\|^2_{H_{\mathrm{av}}^{-s}} \right\}, \tag{3}$$

is dual to

$$\inf_{u \in H_{\mathrm{av}}^{-s}(\mathbb{T}^d)} \left\{ \frac{1}{2\tau} \|u - f\|^2_{H_{\mathrm{av}}^{-s}} + \Phi(u) \right\}. \tag{4}$$

Moreover, the solution u *of (4) is associated with the solution* v *of (3) by the relation*

$$u = f - \tau v\,.$$

Corollary 1

The solution of the problem (4) satisfies $u = f - \tau P_K^{H_{\mathrm{av}}^{-s}}(f/\tau)$, *where* $P_K^{H_{\mathrm{av}}^{-s}}$ *denotes the orthogonal projection on the set* K *with respect to the inner product in* H_{av}^{-s} *.*

Forward-backward splitting scheme

Let define the functional J on $H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$ by

$$J(v) := \frac{1}{2\tau}\|\tau v - f\|^2_{H_{\mathrm{av}}^{-s}}\,.$$

Then the dual problem can be written as

$$\inf_{v \in H_{\mathrm{av}}^{-s}(\mathbb{T}^d)} \left\{ J(v) + \Phi^*(v) \right\}\,.$$

To find $v^* \in K$ such that $0 \in \partial_{H_{\mathrm{av}}^{-s}}(J(v^*) + \Phi^*(v^*))$ we consider the forward-backward splitting scheme given by

$$\begin{cases} u^k \in -\partial_{H_{\mathrm{av}}^{-s}} J(v^k)\,, \\ v^{k+1} = (I + \lambda\,\partial_{H_{\mathrm{av}}^{-s}}\Phi^*)^{-1}(v^k + \lambda\,u^k)\,. \end{cases} \tag{5}$$

Remark 1

The above scheme requires that $\partial_{H_{\mathrm{av}}^{-s}}(J(v) + \Phi^*(v)) = \partial_{H_{\mathrm{av}}^{-s}} J(v) + \partial_{H_{\mathrm{av}}^{-s}}\Phi^*(v)$, *which holds since* $int(D(\Phi^*)) \cap D(J) \neq \emptyset$.

- P. L. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Num. Anal.*, 1979.
- P. L. Combettes, V. R. Wajs, Signal recovery by proximal forward-backward splitting. *SIAM: MMS*, 2005.

Convergence

Theorem 5

Let $\{u^k\}$ *and* $\{v^k\}$ *be sequences generated by the scheme*

$$\begin{cases} u^k \in -\partial_{H_{\mathrm{av}}^{-s}} J(v^k)\,, \\ v^{k+1} = (I + \lambda\, \partial_{H_{\mathrm{av}}^{-s}} \Phi^*)^{-1}(v^k + \lambda\, u^k)\,. \end{cases}$$

Moreover assume that $0 < \lambda\tau < 2$*. Then we have that* $v^k \rightharpoonup v^*$ *and* $u^k \rightharpoonup u^*$ *in* H_{av}^{-s} *as* $k \to \infty$*, where* $v^* \in K$ *is such that* $v^* \in \partial_{H_{\mathrm{av}}^{-s}} \Phi(u^*)$ *and* $u^* = f - \tau v^*$*.*

Equivalent scheme

Let $v \in H^{-s}_{\mathrm{av}}(\mathbb{T}^d)$, then by Moreau's identity

$$v = (I + \lambda \partial_{H^{-s}_{\mathrm{av}}} \Phi^*)^{-1}(v) + \lambda \left(I + 1/\lambda\, \partial_{H^{-s}_{\mathrm{av}}} \Phi \right)^{-1} (v/\lambda) \ ,$$

we obtain that the forward-backward splitting scheme

$$\begin{cases} u^k \in -\partial_{H^{-s}_{\mathrm{av}}} J(v^k)\,, \\ v^{k+1} = (I + \lambda\, \partial_{H^{-s}_{\mathrm{av}}} \Phi^*)^{-1}(v^k + \lambda\, u^k)\,, \end{cases}$$

is equivalent to

$$\begin{cases} u^k \in -\partial_{H^{-s}_{\mathrm{av}}} J(v^k)\,, \\ v^{k+1} = H_{1/\lambda}(v^k/\lambda + u^k)\,, \end{cases}$$

where $H_{1/\lambda}$ denotes the Yosida approximation of the operator $\mathcal{A} := \partial_{H^{-s}_{\mathrm{av}}} \Phi$, i.e.

$$H_{1/\lambda}(v) := \lambda \left(v - (I + 1/\lambda\, \mathcal{A})^{-1} v \right) \ .$$

It is well known that $H_{1/\lambda}$ converges as $\lambda \to \infty$ to the minimal selection $\mathcal{A}_0$ of $\mathcal{A}$.

Dual problem

Using the characterization of $v \in \partial_{H_{\mathrm{av}}^{-s}} \Phi(u)$, we can rewrite the dual problem to

$$\inf_{z \in Z} \left\{ \frac{1}{2\tau} \|\tau(-\Delta_{\mathrm{av}})^s \mathsf{div} z + f\|_{H_{\mathrm{av}}^{-s}}^2 \right\} ,$$

where Z is the closure of the set

$$\{ z \in \mathcal{D}(\mathbb{T}^d) \ : \ (-\Delta_{\mathrm{av}})^{-s} \mathsf{div} z \in \mathcal{D}'(\mathbb{T}^d), \ \|z\|_\infty \leq 1 \} ,$$

with respect to the $L^2(\mathbb{T}^d)$-topology.

Let define functionals F and G on $L^2(\mathbb{T}^d, \mathbb{R}^d)$ by

$$F(z) := \frac{1}{2\tau} \|\tau(-\Delta_{\mathrm{av}})^s \mathsf{div} z + f\|_{H_{\mathrm{av}}^{-s}}^2$$

and

$$G(z) := \begin{cases} 0 & \text{if } z \in Z , \\ +\infty & \text{otherwise} . \end{cases}$$

Then the dual problem can be written as

$$\inf_{z \in L^2(\mathbb{T}^d, \mathbb{R}^d)} \{ F(z) + G(z) \} ,$$

Forward-backward splitting scheme

To find $z^* \in Z$ such that $0 \in \partial_{L^2}(F(z^*) + G(z^*))$ we consider the forward-backward splitting scheme given by

$$\begin{cases} w^k \in \partial_{L^2} F(z^k)\,, \\ z^{k+1} = (I + \lambda \partial_{L^2} G)^{-1}(z^k - \lambda w^k)\,. \end{cases} \tag{6}$$

The explicit form of the scheme (6) is given by

$$\begin{cases} w^k = -\nabla(f + \tau(-\Delta_{\mathrm{av}})^s \mathsf{div} z^k)\,, \\ z^{k+1} = \dfrac{z^k - \lambda w^k}{|z^k - \lambda w^k| \vee 1}\,. \end{cases}$$

Convergence in discrete setting

We denote by X the Euclidean space $\mathbb{R}^N$.

The scalar product of two elements $u,\ v \in X$ is defined by $\langle u, v\rangle := \sum_{i=1}^N u_i v_i$ and the norm $\|u\| := \sqrt{\langle u, u\rangle}$.

Here ∇ denotes the discrete gradient operator satisfying periodic boundary conditions. Then div $:= \nabla^T$ and $\Delta :=$ div∇.

For the convenience, we also define $(u, v)_{-s} := \langle(-\Delta)^{-s}u, v\rangle$ for all $u, v \in X$ and $\|u\|_{-s} := \sqrt{(u, u)_{-s}}$.

We denote by $Z := \{z \in X \ : \ \|z\|_\infty \le 1\}$, where $\|z\|_\infty := \max_i |z_i|$.

For $f \in X$ we define functionals F and G on X by

$$F(z) := \frac{1}{2\tau}\|\tau(-\Delta)^s \mathrm{div} z + f\|_{-s}^2$$

and

$$G(z) := \begin{cases} 0 & \text{if } z \in Z\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Then the discrete version of the dual problem for z is

$$\min_{z\in X}\left\{F(z) + G(z)\right\}\,.$$

Convergence in discrete setting

Lemma 6

For $z \in X$, there exists a constant $C > 0$, such that

$$\|(-\Delta)^s \mathit{div} z\|_{-s}^2 \leq C\|z\|^2 \,.$$

Moreover, $C = \mu_{max}^{s+1}$, where μ_{max} denotes the largest eigenvalue of the discrete Laplace operator.

Theorem 6

Assume that $0 < C\lambda\tau < 2$, where the constant $C > 0$ is as in Lemma 6. Then, the sequence $\{z_k\}$ given by the scheme

$$\begin{cases} w^k = -\nabla(f + \tau(-\Delta_{\mathrm{av}})^s \mathit{div} z^k) \,, \\ z^{k+1} = \dfrac{z^k - \lambda w^k}{|z^k - \lambda w^k| \vee 1} \,, \end{cases}$$

is such that $z^k \rightharpoonup z^*$ in X as $k \to \infty$, where $z^* \in Z$ is such that

$$0 \in \partial_X (F(z^*) + G(z^*)) \,.$$

Ergodic convergence

From Theorem 5 we have that if $0 < \lambda\tau < 2$, then the sequence $\{v^k\}$ converges weakly in $H^{-s}_{\mathrm{av}}(\mathbb{T}^d)$ to $v^* \in K$, where v^* is a unique solution of the dual problem.

Then, Mazur's lemma implies existence of the sequence $\{\bar{v}^n\}$ given by

$$\bar{v}^n = \sum_{k=0}^{n} \alpha_k v^k \,,$$

where $\{\alpha_k\}$ is such that $\sum_{k=0}^{n} \alpha_k = 1$, which converges strongly in $H^{-s}_{\mathrm{av}}(\mathbb{T}^d)$ to v^* as $n \to \infty$.

We aim to construct a sequence $\{\alpha_k\}$ such that $\bar{v}^n \to v^*$ as $n \to \infty$, and next, to use this result in order to prove that the sequence $\{\bar{z}^n\}$ given by

$$\bar{z}^n = \sum_{k=0}^{n} \alpha_k z^k \,,$$

converges weakly in $QL^2(\mathbb{T}^d)$ to $z^* \in Z$ as $n \to \infty$, where Q is the orthogonal projection onto the space of gradient fields.

Ergodic convergence

Theorem 6

Let $\{v^k\}$ be a weakly convergent sequence generated by the scheme (5) and let $\{\beta_k\}$ be a sequence of positive real numbers such that $\{\beta_k\} \in l^2 \setminus l^1$. Then, for

$$\alpha_k = \frac{1}{\sum_{j=1}^{n} \beta_j} \beta_k \,,$$

the sequence $\{\bar{v}^n\}$ given by

$$\bar{v}^n = \sum_{k=0}^{n} \alpha_k v^k \,,$$

converges strongly in $H_{\mathrm{av}}^{-s}(\mathbb{T}^d)$ to $v^* \in K$ as $n \to \infty$. Moreover, the sequence $\{\bar{z}^n\}$ given by

$$\bar{z}^n = \sum_{k=0}^{n} \alpha_k z^k \,,$$

where $\{z^k\}$ is generated by the scheme (6), converges weakly in $QL^2(\mathbb{T}^d)$ to $z^* \in Z$ as $n \to \infty$.

For experiments, we were considering initial data f, $g : [-10, 10] \to \mathbb{R}$, given by explicit formulas

$$f(x) = \begin{cases} 20 & \text{if } |x| \leq 2 \\ 50\,|x|^{-1} - 5 & \text{otherwise} \end{cases}, \qquad g(x) = \begin{cases} 20 & \text{if } |x| \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

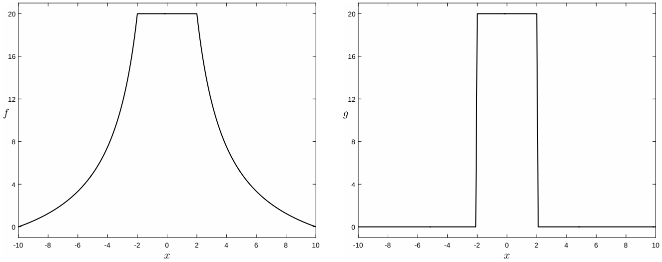

Figure: Graphs of functions f and g considered in experiments as initial data.

Evolution in 1d

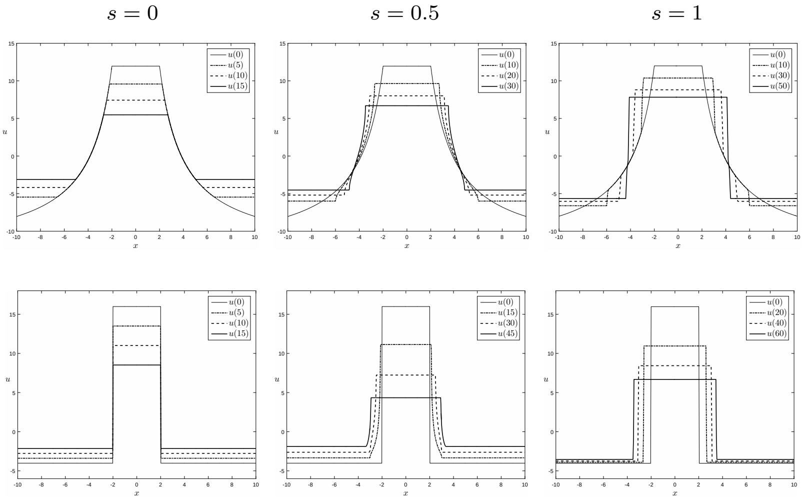

Figure: Evolution of solutions to the H^{-s} total variation flow with periodic boundary conditions and initial data f and g.