Unstable entropy and pressure for partially hyperbolic systems

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Partially hyperbolic diffeomorphisms

Let M be a compact manifold, and $f : M \longrightarrow M$ be a diffeomorphism.

Definition

A diffeomorphism $f : M \longrightarrow M$ is said to be a partially hyperbolic diffeomorphism (PHD) if $TM = E^s \oplus E^c \oplus E^u$

and $\exists$ numbers $0 < \lambda < \lambda' \leq \mu' < \mu$ with $0 < \lambda < 1 < \mu$ s.t. for any $n \geq 0$,

$$\begin{aligned} \|d_x f^n v\| &\leq C\lambda^n \|v\| && \text{as } v \in E^s(x), \\ C^{-1}(\lambda')^n \|v\| \leq \|d_x f^n v\| &\leq C(\mu')^n \|v\| && \text{as } v \in E^c(x), \\ C^{-1}\mu^n \|v\| \leq \|d_x f^n v\| & && \text{as } v \in E^u(x) \end{aligned}$$

hold for some $C > 1$.

If $E^c = \{0\}$, then the diffeomorphism is hyperbolic.

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Observation

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Remark

All results holds if the systems have a dominate splitting for unstable subbundle and center-stable subbundle.

Quasi-stability

Theorem (Zhu-H, 2014)

A PHD $f : M \longrightarrow M$ is topologically quasi-stable, that is, $\forall$ homeomorphism $g \sim_{C^0} f$, $\exists$ a continous map $\pi : M \longrightarrow M$ s.t.

$$\pi \circ g = \tau \circ f \circ \pi,$$

where τ is a motion along the center direction.
If f has C^1 center foliation, then τ can be chosen as a motion along the center leaves

Quasi-shadowing

Theorem (Zhou, Zhu and H, 2015)

A PHD f has the quasi-shadowing property.
That is, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that any pseudo orbit $\{x_k\}_{k=-\infty}^{\infty}$, there is sequence of points $\{y_k\}_{k=-\infty}^{\infty}$ such that

$$d(x_k, y_k) < \varepsilon,$$

and y_{k+1} is obtained from $f(y_k)$ by a motion τ along the center direction.
If f has C^1 center foliation, then τ can be chosen as a motion along the center leaves.

Partitions

Let μ be an f-invariant measure.
Take $\varepsilon_0 > 0$ small.
Let $\mathcal{P}$ be set of finite partitions α of M, $\operatorname{diam}\alpha < \varepsilon_0$, $\mu(\partial\alpha) = 0$,
i.e. $\operatorname{diam} A \leq \varepsilon_0$, $\mu(\partial A) = 0 \ \forall A \in \alpha$.
For each $\beta \in \mathcal{P}$, def¡ine $\eta \geq \beta$ such that $\eta(x) = \beta(x) \cap W^u_{\mathrm{loc}}(x)$.
η is a measurable partition.
Let $\mathcal{P}^u$ denote the set of partitions η obtained this way.

A partition ξ of M is said to be subordinate to unstable manifolds of f if for μ-a.e. x, $\exists r_x > 0$ s. t. $B^u(x, r_x) \subset \xi(x) \subset W^u_{\mathrm{loc}}(x)$. It is clear that any $\eta \in \mathcal{P}^u$ is subordinate to unstable manifolds of f.

Any element in $\mathcal{P}^u$ is a uncountable partition.

Definition

Definition

The conditional entropy of f w.r.t. α given $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f, \alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta).$$

The conditional entropy of f given $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f|\eta) = \sup_{\alpha\in\mathcal{P}} h_\mu(f, \alpha|\eta),$$

and the unstable metric entropy of f is defined as

$$h_\mu^u(f) = \sup_{\eta\in\mathcal{P}^u} h_\mu(f|\eta).$$

Remarks

Recall

$$h_\mu(f,\alpha|\eta)=\limsup_{n\to\infty}\frac{1}{n}H_\mu(\alpha_0^{n-1}|\eta),$$

$$h_\mu(f|\eta)=\sup_{\alpha\in\mathcal{P}}h_\mu(f,\alpha|\eta),\qquad h_\mu^u(f)=\sup_{\eta\in\mathcal{P}^u}h_\mu(f|\eta).$$

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In the definition of $h_\mu(f,\alpha|\eta)$ we take $\limsup$ instead of $\lim$, because the sequence $\{H_\mu(\alpha_0^{n-1}|\eta)\}$ is not subadditive, since η is not invariant under f. Therefore, existence of the limit is not obvious.

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$h_\mu(f|\eta)$ is independent of η, as long as $\eta \in \mathcal{P}^u$. Hence, we actually have $h_\mu^u(f) = h_\mu(f|\eta)$ for any $\eta \in \mathcal{P}^u$.

Construction of incresing partitions

Let μ be ergodic with positive LE $\lambda_1 > \lambda_2 > \cdots > \lambda_{\tilde{u}} > 0$.
Let $E^{(1)} \subset E^{(2)} \subset \cdots \subset E^{(\tilde{u})}$ denote the corresponding subbundles and $W^{(1)}(x) \subset W^{(2)}(x) \subset \cdots \subset W^{(\tilde{u})}(x)$ the unstable manifolds such that $T_x W^{(i)}(x) = E_x^{(i)}$.
To construct an incresing partition, take $z \in M$, and

$$S_i(z,r) = \bigcup_{y \in W_\perp^{(i)}(z,r)} W^{(i)}(y,r)$$

where $W_\perp^{(i)}(z,r)(z,r)$ is an open ball of radius r on a surface transversal to $W^{(i)}$. Then define a partition $\hat{\xi}_{i,z}$ such that

$$\hat{\xi}_i(y) = \begin{cases} W^{(i)}(\bar{y},r) & \text{if } y \in S_i(z,r), \\ M \setminus S_i(z,r) & \text{otherwise.} \end{cases}$$

Take

$$\xi_i = \vee_{j\geq 0} f^j \hat{\xi}_i.$$

Entropies given by incresing partitions

Recall $\lambda_1 > \cdots > \lambda_{\tilde{u}} > 0$, $W^{(1)}(x) \subset \cdots \subset W^{(\tilde{u})}(x)$.
We have $\xi_1 \geq \cdots \geq \xi_{(\tilde{u})}(x)$.
ξ_i is increasing, that is, $f^{-1}\xi_i \geq \xi_i$.
Consider the condition entropy

$$h_\mu(f, \xi_i) := H_\mu(\xi_i | f\xi_i) = H_\mu(f^{-1}\xi_i | \xi_i).$$

In particular, $h_\mu(f, \xi_{\tilde{u}}) = h_\mu(f)$.
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Let $\lambda_1 > \lambda_2 > \cdots > \lambda_u > 0$ be the Lyapunov exponents in E^u, the strong unstable subbundle. (So $u \leq \tilde{u}$.)
Denote by $\mathcal{Q}^u$ the set of all ξ_u.

The equivalence

Theorem A

Suppose μ is an ergodic measure. Then for any $\alpha \in \mathcal{P}$, $\eta \in \mathcal{P}^u$ and $\xi \in \mathcal{Q}^u$,

$$h_\mu(f, \alpha|\eta) = h_\mu(f, \xi).$$

Hence,

$$h^u_\mu(f) = h_\mu(f|\eta) = h_\mu(f, \xi).$$

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Hence,

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Corollary A.1

$h_\mu^u(f) \le h_\mu(f)$, *and "$=$" holds if f is $C^{1+\alpha}$, and there is no positive Lyapunov exponent in E^c at μ-a.e. $x \in M$.*

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Corollary A.2

$$h^u_\mu(f) = h_\mu(f, \alpha|\eta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\vee_{i=0}^{n-1} f^{-i}\alpha|\eta)\ \forall \alpha \in \mathcal{P}, \eta \in \mathcal{P}^u.$$

Affineness and upper semi-continuity

Let $\mathcal{M}_f(M)$ denote the set of all f-invariant probability measures on M.

Proposition (Affineness)

The map $\mu \mapsto h^u_\mu(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is affine.

Affineness and upper semi-continuity

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Proposition (Upper semi-continuity)

The unstable entropy map $\mu \mapsto h^u_\mu(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semi-continuous at μ. i.e.

$$\limsup_{\nu \to \mu} h^u_\nu(f) \leq h^u_\mu(f).$$

A version of Shannon-McMillan-Breiman theorem

Theorem B

Suppose μ is an ergodic measure of f. Let $\eta \in \mathcal{P}^u$ be given. Then for any partition α with $H_\mu(\alpha|\eta) < \infty$, we have

$$\lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) = h_\mu(f, \alpha|\eta) \qquad \mu\text{-a.e.} x \in M.$$

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Corollary B.1

Let μ be f-ergodic and $\xi \in \mathcal{Q}^u$. Then for any partition α with $H_\mu(\alpha|\xi) < \infty$, we have

$$\lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) = h_\mu(f, \alpha|\xi) \qquad \mu\text{-a.e.} x \in M,$$

where $h_\mu(f, \alpha|\xi)$ is defined as in definition for $h_\mu(f, \alpha|\eta)$ with η replaced by ξ.

Denote by d^u the metric induced by the Riemannian structure on the unstable manifold and let $d_n^u(x,y) = \max_{0\le j\le n-1} d^u(f^j(x), f^j(y))$.
Let $W^u(x,\delta)$ be the open ball inside $W^u(x)$ centered at x of radius δ with respect to the metric d^u.
Let $N^u(f,\epsilon,n,x,\delta)$ be the maximal number of points in $\overline{W^u(x,\delta)}$ with pairwise d_n^u-distances at least ϵ. We call such set an (n,ϵ) u-separated set of $\overline{W^u(x,\delta)}$.

Definition

The unstable topological entropy of f on M is defined by

$$h^u_{\text{top}}(f) = \lim_{\delta\to 0} \sup_{x\in M} h^u_{\text{top}}(f, \overline{W^u(x,\delta)}),$$

where
$$h^u_{\text{top}}(f, \overline{W^u(x,\delta)}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N^u(f,\epsilon,n,x,\delta).$$

Using (n, ϵ) u-spanning set

A set $E \subset W^u(x)$ is called an (n, ϵ) u-spanning set of $\overline{W^u(x,\delta)}$ if $\overline{W^u(x,\delta)} \subset \bigcup_{y \in E} B_n^u(y,\epsilon)$, where $B_n^u(y,\epsilon) = \{z \in W^u(x) : d_n^u(y,z) \le \epsilon\}$ is the (n, ϵ) u-Bowen ball around y.

Let $S^u(f, \epsilon, n, x, \delta)$ be the cardinality of a minimal (n, ϵ) u-spanning set of $\overline{W^u(x,\delta)}$. Then we also have

$$h^u_{\text{top}}(f, \overline{W^u(x,\delta)}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S^u(f, \epsilon, n, x, \delta).$$

Recall

$$h^u_{\text{top}}(f) = \lim_{\delta \to 0} \sup_{x \in M} h^u_{\text{top}}(f, \overline{W^u(x,\delta)}).$$

Lemma

$h^u_{top}(f) = \sup_{x \in M} h^u_{top}(f, \overline{W^u(x,\delta)})$ for any $\delta > 0$.

Using open covers

Let $\mathcal{C}_M$ denote the set of open covers of M. Given $\mathcal{U} \in \mathcal{C}_M$, denote $\mathcal{U}_m^n := \bigvee_{i=m}^n f^{-i}\mathcal{U}$. For any $K \subset M$, set

$$N(\mathcal{U}|K) := \min\{\mathrm{Card}(\mathcal{V}) : \mathcal{V} \subset \mathcal{U}, \bigcup_{V\in\mathcal{V}} \supset K\}.$$

$$H(\mathcal{U}|K) := \log N(\mathcal{U}|K).$$

Definition

We define

$$\tilde{h}^u_{\text{top}}(f) = \lim_{\delta\to 0} \sup_{x\in M} \tilde{h}^u_{\text{top}}(f, \overline{W^u(x,\delta)}),$$

where $\tilde{h}^u_{\text{top}}(f, \overline{W^u(x,\delta)}) = \sup_{\mathcal{U}\in\mathcal{C}_M} \limsup_{n\to\infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}|\overline{W^u(x,\delta)})$.

Lemma

$\tilde{h}^u_{top}(f, \overline{W^u(x,\delta)}) = h^u_{top}(f, \overline{W^u(x,\delta)})$. So, $\tilde{h}^u_{top}(f) = h^u_{top}(f)$.

Volume growth was used by Yomdin and Newhouse for the entropy of diffeomorphisms. The unstable volume growth for partially hyperbolic systems is used by Hua-Saghin-Xia, which is defined as following:

$$\chi_u(f) = \sup_{x\in M} \chi_u(x,\delta) \tag{1}$$

where

$$\chi_u(x,\delta) = \limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{Vol}(f^n(W^u(x,\delta))). \tag{2}$$

Theorem C

$h^u_{top}(f) = \chi_u(f)$.

Corollary C.1

$h^u_{top}(f) \le h_{top}(f)$, and "=" holds if there is no positive Lyapunov exponent in E^c direction at ν-a.e. w.r.t. any ergodic measure ν.

Upper bound of $h_{\text{top}}(f)$ using $h^u_{\text{top}}(f)$

Hua-Saghin-Xia proved that $\forall$ ergodic measure μ,

$$h_\mu(f) \le \chi^u(f) + \sum_{\lambda_i^c>0} \lambda_i^c m_i,$$

where $\chi^u(f)$ denotes the volume growth of the unstable foliation. Let

$$\sigma^{(i)} = \lim_{n\to\infty} \frac{1}{n} \log \left\| \bigwedge^i Df^n|_{E^c} \right\|, \quad \forall 1 \le i \le \dim E^c,$$

where $\bigwedge^i$ is the ith outer product. Then let

$$\sigma = \max\{\sigma^{(i)} : \ i = 1, \cdots, \dim E^c\}.$$

We give the topological version of the formula given by H-S-X:

Corollary C.2

$h_{top}(f) \le h^u_{top}(f) + \sigma,$ "$=$" *holds if* $\sigma^{(1)} \le 0$.

Transversal topological entropy

Let $N(f, \epsilon, n, x, \delta)$ be the maximal number of points in a (n, ϵ)-separating set in $\overline{B(x, \delta)}$.

Definition

The *transversal topological entropy* of f on M is defined by

$$h^t_{\text{top}}(f) = \lim_{\delta \to 0} \sup_{x \in M} h^t_{\text{top}}(f, \overline{B(x, \delta)}),$$

where

$$h^t_{\text{top}}(f, \overline{B(x, \delta)}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \Big[\log N(f, \epsilon, n, x, \delta) - \log N^u(f, \epsilon, n, x, \delta) \Big].$$

Corollary C.3

$h_{top}(f) \le h^u_{top}(f) + h^t_{top}(f)$.

Variational principle

Denote by $\mathcal{M}_f(M)$ the set of probability invariant measures on M and by $\mathcal{M}_f^e(M)$ the set of ergodic measures on M

Theorem D

Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism. Then

$$h^u_{top}(f) = \sup\{h^u_\mu(f) : \mu \in \mathcal{M}_f(M)\}.$$

Moreover,

$$h^u_{top}(f) = \sup\{h^u_\nu(f) : \nu \in \mathcal{M}_f^e(M)\}.$$

The theorem can be proved by the same methods for standard metric entropy and topological entropy.

Unstable topological pressure

Recall that an (n, ϵ) u-separated set of $\overline{W^u(x,\delta)}$ is a set in which the d^u_n-distances of any two points is at least ϵ. Denote by $\mathcal{S}(n,\varepsilon)$ the set of (n,ϵ) u-separated set of $\overline{W^u(x,\delta)}$.

Let

$$P^u(f,\varphi,\epsilon,n,x,\delta) = \sup\left\{ \sum_{y\in E} \exp\left((S_n\varphi)(y)\right) : \; E \in \mathcal{S}(n,\varepsilon)\right\}.$$

Definition

The unstable topological pressure of f w.r.t the potential φ is defined by

$$P^u(f,\varphi) := \lim_{\delta\to 0} \sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta)}),$$

where

$$P^u(f,\varphi,\overline{W^u(x,\delta)}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P^u(f,\varphi,\epsilon,n,x,\delta).$$

Unstable topological pressure

Two alternative ways to define unstable topological pressure are by using (n, ϵ) u-spanning sets and by using open covers.

It is clear that

$$P^u(f, 0) = h^u_{\text{top}}(f).$$

Variational principle

Theorem E (Variational principle)

Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism. Then for any $\varphi \in C(M, \mathbb{R})$,

$$P^u(f, \varphi) = \sup \left\{ h^u_\mu(f) + \int_M \varphi d\mu : \mu \in \mathcal{M}_f(M) \right\}.$$

Moreover, $P^u(f, \varphi) = \sup \left\{ h^u_\mu(f) + \int_M \varphi d\mu : \mu \in \mathcal{M}^e_f(M) \right\}.$

Corollary E.1

$P^u(f, \varphi) \leq P(f, \varphi)$, *"=" holds if f is $C^{1+\alpha}$, & has no positive Lyapunov exponent in the E^c direction at ν-a.e. $\forall \nu \in \mathcal{M}^e_f(M)$.*

u-equilibrium

Definition

Let $\varphi \in C(M,\mathbb{R})$. $\mu \in \mathcal{M}_f(M)$ is called a u-equilibrium state for φ if

$$P^u(f,\varphi) = h^u_\mu(f) + \int \varphi d\mu$$

Let $\mathcal{M}^u_\varphi(M,f)$ denote the set of all u-equilibrium states for φ.

Theorem F

1. *$\mathcal{M}^u_\varphi(M,f)$ is nonempty and compact.*
2. *$\mathcal{M}^u_\varphi(M,f)$ is convex, and the set of extreme points is $\mathcal{M}^u_\varphi(M,f) \cap \mathcal{M}^e_f(M)$.*
3. *If $\varphi, \psi \in C(M,\mathbb{R})$, and $\exists c \in \mathbb{R}$, $h \in C(M,\mathbb{R})$ s.t. $\varphi - \psi = h \circ f - h + c$, then $\mathcal{M}^u_\varphi(M,f) = \mathcal{M}^u_\psi(M,f)$.*

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A measure of maximal unstable entropy is a u-equilibrium state for the potential 0. So it always exists by Theorem E(1).

Gibbs u-states

Take potential $\varphi^u(x) = -\log|\det Df|_{E^u(x)}|$.
A Gibbs u-state for a partially hyperbolic system is an invariant probability measures on M that has absolutely continuous conditional measures on strong unstable manifolds.

Theorem G

Let f be $C^{1+\alpha}$ and $\mu \in \mathcal{M}_f(M)$. Then μ is a Gibbs u-state of f if and only if μ is a u-equilibrium state of φ^u.

Corollary G.1

If f is $C^{1+\alpha}$, then $P^u(f, \varphi^u) = 0$.

Gibbs u-states

Corollary G.2

There always exists a Gibbs u-state for any $C^{1+\alpha}$ partially hyperbolic diffeomorphism.

Results in Corollary C.2 was obtained for partially hyperbolic attractor by Pesin-Sinai in 1982.

Jiagang Yang obtained the result for C^1 partially hyperbolic diffeomorphisms.

Topological pressure determines $\mathcal{M}_f(M)$

A finite signed measure on M is a map $\mu : \mathcal{B} \to \mathbb{R}$ which is countably additive, where $\mathcal{B}$ is the σ-algebra of Borel subsets of M. Recall that $\mu \in \mathcal{M}_f(M)$ denote the set of probability invariant measures.

Theorem

Let $T : X \to X$ be a continuous map on a compact metric space X with $h_{top}(T) < \infty$. Let μ be a finite signed measure. Then $\mu \in \mathcal{M}_f(M)$ if and only if $\int_M \varphi d\mu \leq P(T, \varphi)$ $\forall \varphi \in C(M, \mathbb{R})$.

The theorem says that when $h_{\text{top}}(T) < \infty$, the pressure of determines the set $\mathcal{M}_f(M)$.

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Theorem

Let $T : X \to X$ be a continuous map on a compact metric space X with $h_{top}(T) < \infty$. Let $\nu \in \mathcal{M}_T(M)$. Then

$$h_\nu(T) = \inf\left\{P(T,\varphi) - \int_M \varphi d\nu : \varphi \in C(M,\mathbb{R})\right\}$$

if and ony if the entropy map $\mu \to h_\mu(T)$ is upper semicontinuous.

The above two theorems can be seen in the book by Peter Walters.

Topological pressure determines $\mathcal{M}_f(M)$

Note that in our setting we have $h^u_{\text{top}}(f) < \infty$ and upper semicontinuity of the entropy map $\mu \to h^u_\mu(f)$.

Theorem H

1. *Let μ be a finite signed measure. Then $\mu \in \mathcal{M}_f(M)$ if and only if $\int_M \varphi d\mu \leq P^u(f,\varphi)$ $\forall \varphi \in C(M,\mathbb{R})$.*
2. *Let $\nu \in \mathcal{M}_f(M)$. Then*

$$h^u_\nu(f) = \inf\left\{P^u(f,\varphi) - \int_M \varphi d\nu : \varphi \in C(M,\mathbb{R})\right\}.$$

We mention here that the first part is still true even if we replace $P(f,\varphi)$ by $P^u(f,\varphi) \leq P(f,\varphi)$.

u-tangent functional

Definition

Let $\varphi \in C(M, \mathbb{R})$. A u-tangent functional to $P^u(f, \cdot)$ at φ is a finite signed measure $\mu : \mathcal{B} \to \mathbb{R}$ such that

$$P^u(f, \varphi + \psi) - P^u(f, \varphi) \geq \int_M \psi d\mu, \quad \forall \psi \in C(M, \mathbb{R}).$$

Let $t^u_\varphi(M, f)$ be the set of u-tangent functionals to $P^u(f, \cdot)$ at φ.

Theorem I

$\mathcal{M}^u_\varphi(M, f) = t^u_\varphi(M, f)$.

In classical case for the equality $\mathcal{M}_\varphi(M, f) = t_\varphi(M, f)$ upper semicontinuity of the map $\mu \mapsto h_\mu(f)$ is required.
The assumption is always holds for $\mu \mapsto h^u_\mu(f)$.

Gateaux differentiability

Definition

The unstable topological pressure $P^u(f,\cdot) : C(M,\mathbb{R}) \to \mathbb{R}$ is said to be Gateaux differentiable at φ if

$$\lim_{t\to 0} \frac{1}{t}(P^u(f,\varphi+t\psi) - P^u(f,\varphi))$$

exists for any $\psi \in C(M,\mathbb{R})$.

Theorem J

$P^u(f,\cdot)$ is Gateaux differentiable at φ if and only if there is a unique unstable tangent functional to $P^u(f,\cdot)$ at φ, if and only if there is a unique u-equilibrium state of φ..

The last equivalent conditions follows from the first ones and Theorem I.

Fréchet differentiability

Definition

$P^u(f,\cdot): C(M,\mathbb{R}) \to \mathbb{R}$ is said to be Fréchet differentiable at φ if $\exists \gamma \in C(M,\mathbb{R})^*$ such that

$$\lim_{\psi\to 0} \frac{|P^u(f,\varphi+\psi) - P^u(f,\varphi) - \gamma(\psi)|}{\|\psi\|} = 0.$$

Fréchet differentiability of $P^u(f,\cdot)$ is stronger than Gateaux differentiability of $P^u(f,\cdot)$, either by the definitions or by Theorem J and Theorem K below.

Hence, Fréchet differentiability of $P^u(f,\cdot)$ also implies the uniqueness of u-equilibrium state.

Let $\mu_n \to \mu$ denote the convergence in weak* topology, and $\|\mu_n - \mu\| \to 0$ the convergence in norm topology on $\mathcal{M}_f(M)$.

Theorem K

The following statements are mutually equivalent.

1. $P^u(f,\cdot)$ *is Fréchet differentiable at* φ.
2. $\exists \mu_\varphi \in \mathcal{M}_f(M)$ *s.t.* $(\mu_n) \subset \mathcal{M}_f(M)$ *with* $h^u_{\mu_n}(f) + \int_M \varphi d\mu_n \to P^u(f,\varphi)$ *implies* $\|\mu_n - \mu_\varphi\| \to 0$ *as* $n \to \infty$.
3. $t^u_\varphi(M,f) = \{\mu_\varphi\}$ *and* $P^u(f,\varphi) > \sup\left\{h^u_\mu(f) + \int_M \varphi d\mu : \mu \text{ is ergodic and } \mu \neq \mu_\varphi\right\}$.
4. $t^u_\varphi(M,f) = \{\mu_\varphi\}$ *and* $\exists$ *a weak* neighborhood* $V \ni \mu_\varphi$ *s.t.* $h^u_{\mu_\varphi}(f) > \sup\{h^u_\mu(f) : \mu \in V \text{ is ergodic and } \mu \neq \mu_\varphi\}$.
5. $P^u(f,\cdot)$ *is affine on a neighborhood of* φ.
6. $t^u_\varphi(M,f) = \{\mu_\varphi\}\ \&\ \sup\{\|\mu - \mu_\varphi\| : \mu \in t^u_{\varphi+\psi}(M,f)\} \to 0$ *as* $\psi \to 0$.
7. $t^u_\varphi(M,f) = \{\mu_\varphi\}\ \&\ \inf\{\|\mu - \mu_\varphi\| : \mu \in t^u_{\varphi+\psi}(M,f)\} \to 0$ *as* $\psi \to 0$.

Thank you!