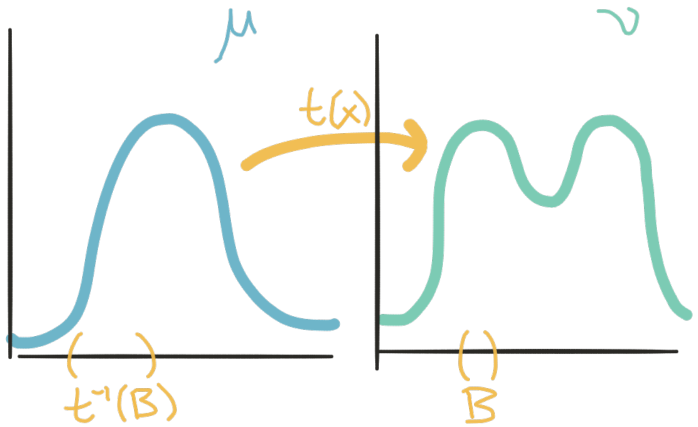
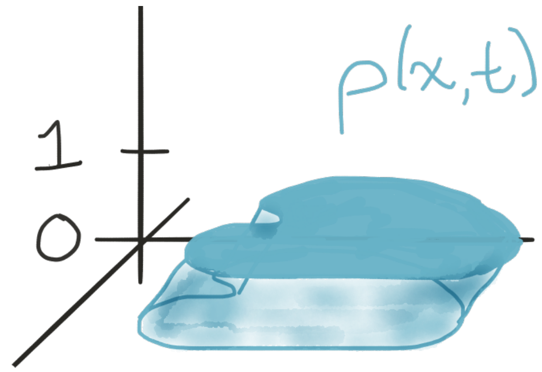
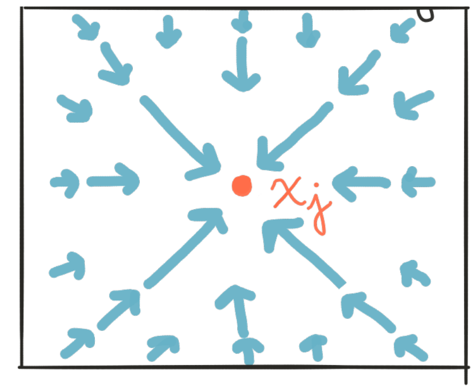
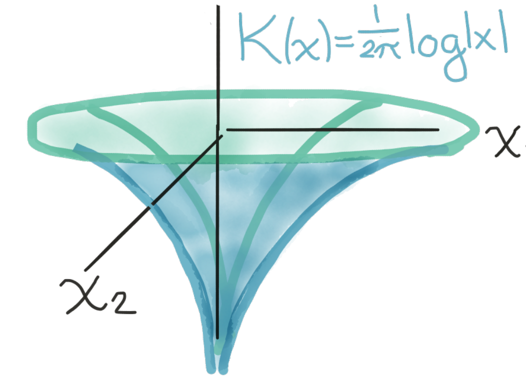








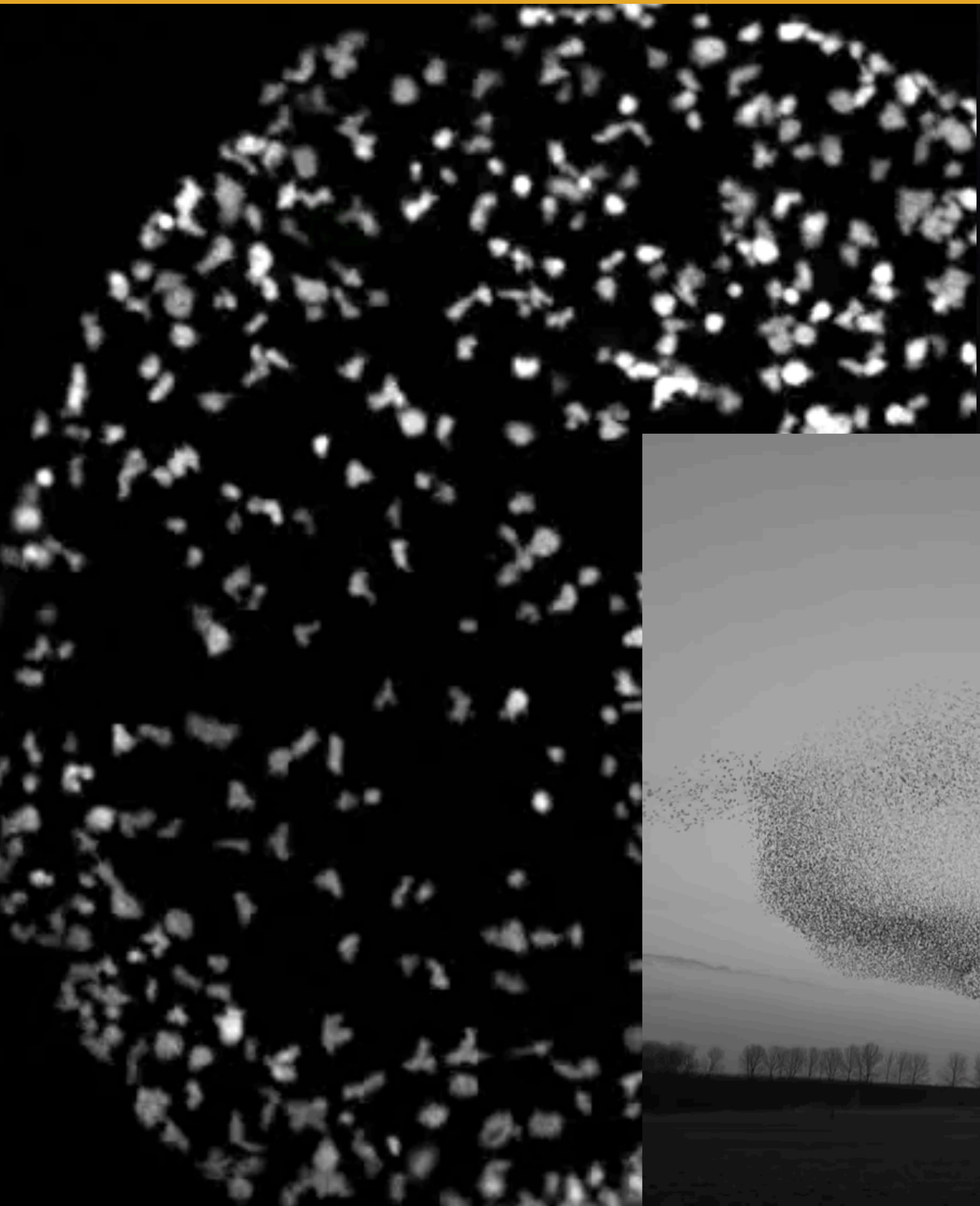
Minimizers and gradient flows in the slow diffusion limit

Katy Craig
University of California, Santa Barbara

joint with I. Topaloglu (VCU)

BIRS: Entropies, the Geometry of Nonlinear Flows, and their Applications
Banff, 4/9/18

nonlocal interactions



interaction energy / aggregation eqn

- ρ(x,t): $\mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ nonnegative density
- mass is conserved ⇒ $\int \rho(x)\,dx = M$

interaction energy:

$$\mathcal{K}(\rho) = \frac{1}{2}\int K * \rho \, d\rho$$

aggregation equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho)$$

interaction kernels, $K(x) : \mathbb{R}^d \to \mathbb{R}$

- vortex motion/chemotaxis: $K(x) = \pm|x|^{2-d}/(2-d)$ $\quad |x|^0/0 = \log(|x|)$
- granular media: $K(x) = |x|^3$
- swarming: $K(x) = |x|^a/a - |x|^b/b, \quad -d \leq b \leq a$

minimizers

interaction energy:

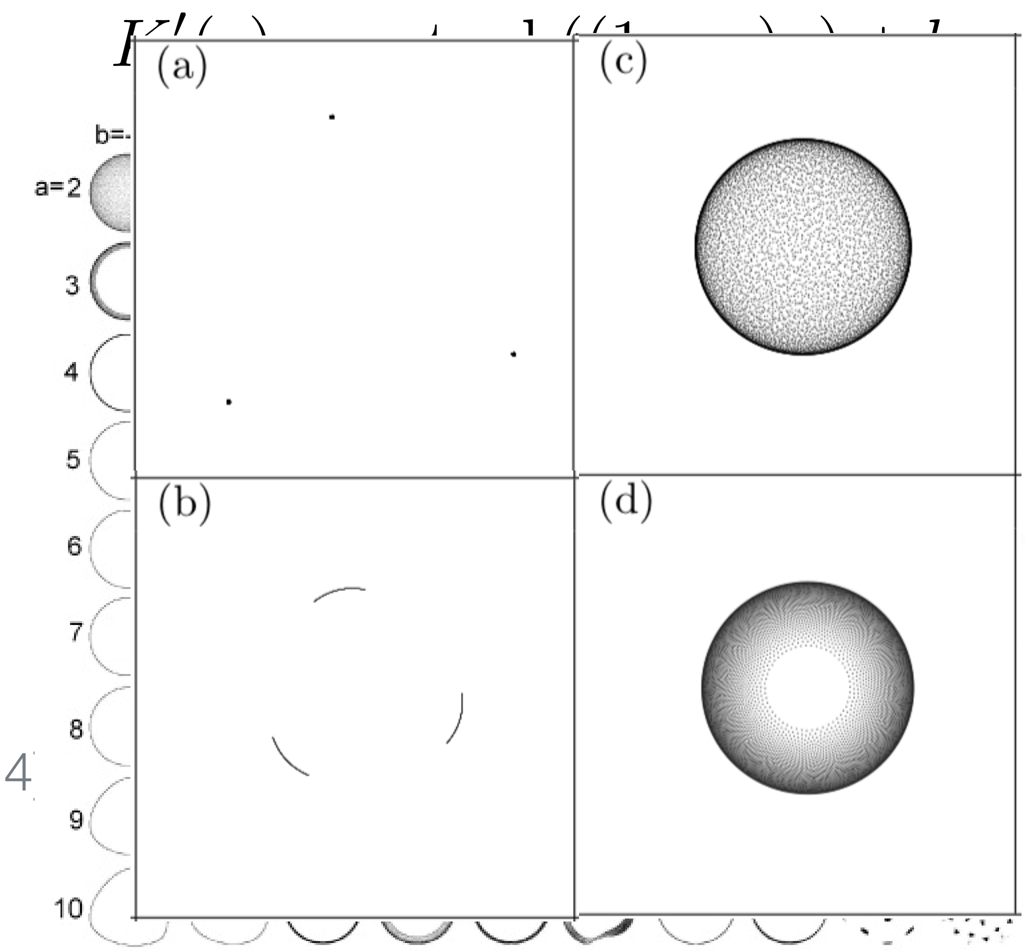
$$\mathcal{K}(\rho) = \frac{1}{2}\int K * \rho d\rho$$

aggregation equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho)$$

previous work:

- competing effects of attraction/repulsion lead to rich structure
  [Kolokolnikov, Sun, Uminsky, Bertozzi, 2011]
- more singular, repulsive K -> minimizers have higher dimensional support
  [Balague, Carrillo, Laurent, Raoul 2012]
- existence of minimizers
  [Slepçev, Simione, Topaloglu '14] [Cañizo, Carrillo, Patacchini '14

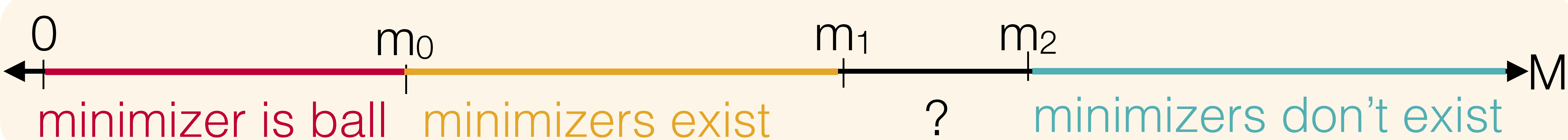


set valued minimizers

more recently: set valued minimizers

$$\min \left\{ \int_\Omega \int_\Omega K(x-y)\, dx\, dy : \Omega \subseteq \mathbb{R}^d,\ |\Omega| = M \right\}$$

Related shape optimization problems (d=3):



- Nonlocal isoperimetric problem [Knüpfer, Moratov '13]. [Lu, Otto '14], [Frank, Lieb '15], …

$$\min \left\{ \text{Perimeter}(\Omega) + \int_\Omega \int_\Omega \frac{1}{|x-y|}\, dx\, dy : \Omega \subseteq \mathbb{R}^3,\ |\Omega| = M \right\}$$

set valued minimizers

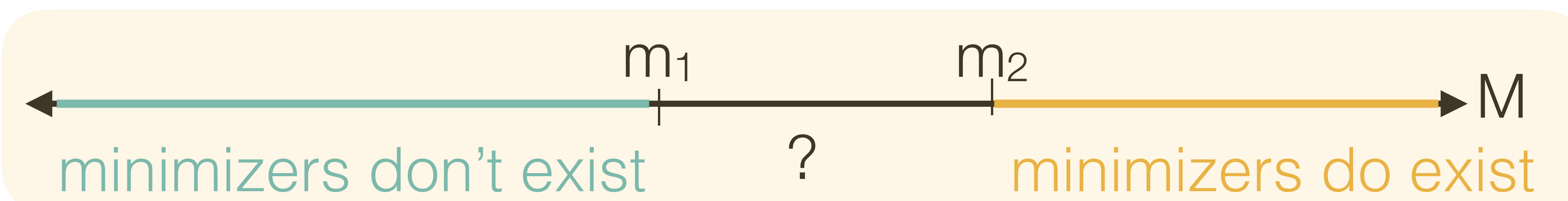
more recently: set valued minimizers $K(x) = |x|^a/a - |x|^b/b, \;\; -d < b \leq a$

$$\min \left\{ \int K * \rho d\rho : \rho = 1_\Omega \text{ for } \Omega \subseteq \mathbb{R}^d, |\Omega| = M \right\}$$

critical mass:

- a = 2, -d < b ≤ 2-d [Burchard, Choksi, Topaloglu 2016] a > 0, b = 2-d [Frank, Lieb 2017]

  there exist $m_1 \leq m_2$ s.t.



- for a = 2, b = 2-d, we have $m_1 = m_2 = \omega_d$

in general, it is unknown whether $m_1 = m_2$ and how m_1, m_2 depend on a, b.

set valued minimizers

[B

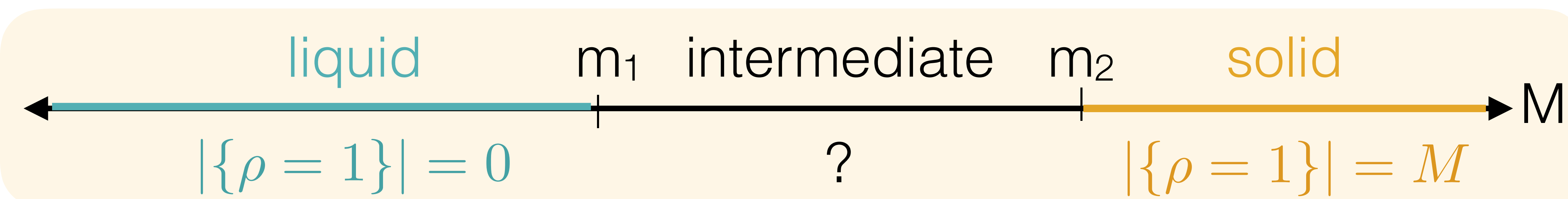
more recently: set valued minimizers $K(x) = |x|^a/a - |x|^b/b, \;\; -d < b \leq a$

$$\min\left\{\int K * \rho d\rho : \rho = 1_\Omega \text{ for } \Omega \subseteq \mathbb{R}^d, |\Omega| = M\right\}$$

relaxed problem:

$$\min\left\{\int K * \rho d\rho : 0 \leq \rho \leq 1, \; \int \rho = M\right\}$$

- minimizers exist
- $2 \leq a \leq 4$, $2-d \leq b < 0$: minimizers unique [Burchard, Choksi, Topaloglu '16] [Lopes '17]
- $a > 0$, $b = 2-d$: there exist $m_1 \leq m_2$ s.t [Frank, Lieb 2017]



constrained aggregation

constrained aggregation:

"$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$"

aggregation diffusion equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta\rho^m$$

motivation:

- previous work on congested drift equation (pedestrian crown motion)
  [Maury, Roudneff-Chupin, Santambrogio 2010], [Alexander, Kim, Yao 2014]

- competition between nonlocal attraction and repulsion from constraint.

- heuristically, height constraint is singular limit of degenerate diffusion:

  Idea: $\Delta\rho^m = \nabla \cdot (\underbrace{m\rho^{m-1}}_{D} \nabla\rho)$, so as m→+∞, $D \to \begin{cases} +\infty & \text{if } \rho > 1 \\ 0 & \text{if } \rho < 1 \end{cases}$

constrained aggregation

constrained aggregation:

"$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla (K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$"

aggregation diffusion equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

$m \to +\infty$?

results: [C. 2017, C. Kim, Yao 2017]

- the constrained aggregation eqn is well-posed as a W_2 gradient flow

$$\mathcal{E}_\infty(\rho) = \begin{cases} \int K * \rho d\rho & \text{if } \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

- $K(x) = -|x|^{2-d}/(2-d) = \Delta^{-1}$
  - solutions with "set-valued" initial data remain "set-valued"
  - characterize via Hele-Shaw type free boundary problem
  - d=2: quantify convergence to equilibrium

slow diffusion limit

goals

- prove slow diffusion limit,

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m \xrightarrow{m \to +\infty} \begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla (K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

$$K(x) = |x|^b/b, \; K(x) = |x|^a/a - |x|^b/b, \;\; 2 - d \leq b \leq a$$

- use numerical methods for aggregation diffusion equations to shed light on properties of minimizers of constrained interaction energy

gradient flow

Def: $\rho(t):\mathbb{R}\to P_2(\mathbb{R}^d)$ is the **Wasserstein gradient flow** of $E:P_2(\mathbb{R}^d)\to\mathbb{R}$ if

$$E(\rho(t)) - E(\rho(0) \leq -\frac{1}{2}\int_0^t |\partial E|(\rho(s))ds - \frac{1}{2}\int_0^t |\rho'|(s)ds$$

where

$$|\partial E|(\mu) := \limsup_{\nu\to\mu} \frac{(E(\mu)-E(\nu))^+}{W_2(\mu,\nu)} \quad \text{and} \quad |\rho'|(t) = \lim_{s\to t}\frac{W_2(\rho(s),\rho(t))}{|s-t|}$$

Analogy with **Euclidean gradient flow**:

$$\frac{d}{dt}x(t) = -\nabla E(x(t)) \iff \begin{cases} \left|\frac{d}{dt}x(t)\right| = |\nabla E(x(t))| \\ \frac{d}{dt}E(x(t)) = -|\nabla E(x(t))|\left|\frac{d}{dt}x(t)\right| \end{cases}$$

$$\iff \frac{d}{dt}E(x(t)) \leq -\frac{1}{2}|\nabla E(x(t))| - \frac{1}{2}\left|\frac{d}{dt}x(t)\right|$$

gradient flow

goal:

show solutions of aggregation diffusion equations converge to congested aggregation equation

or equivalently:

show gradient flows of $\mathcal{E}_m$ converge to the gradient flow of $\mathcal{E}_\infty$

aggregation diffusion equation:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

interaction energy + Rényi entropy

$$\mathcal{E}_m(\rho) = \int K * \rho d\rho + \frac{1}{m-1}\int \rho^m$$

constrained aggregation:

$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

constrained interaction energy

$$\mathcal{E}_\infty(\rho) = \begin{cases} \int K * \rho d\rho & \text{if } \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

Γ-convergence of gradient flows

Theorem: (Serfaty 2010)**:** Let $\rho_m(x,t)$ be grad flows of $\mathcal{E}_m$ such that

$$\rho_m(x,t) \to \rho_\infty(x,t) \text{ and } \mathcal{E}_m(\rho_m(x,0)) \to \mathcal{E}_\infty(\rho_\infty(x,0))$$

$$\uparrow$$

$$\rho_m(x,0) \to \rho_\infty(x,0) \text{ and } \mathcal{E}_m(\rho_m(x,0)) \to \mathcal{E}_\infty(\rho_\infty(x,0))$$

Recall: $\rho(t):\mathbb{R}\to P_2(\mathbb{R}^d)$ is the Wasserstein gradient flow of $E:P_2(\mathbb{R}^d)\to\mathbb{R}$ if

$$E(\rho(t)) - E(\rho(0)) \leq -\frac{1}{2}\int_0^t |\partial E|(\rho(s))ds - \frac{1}{2}\int_0^t |\rho'|(s)ds$$

Γ-convergence of gradient flows

Goal:

1. $\liminf_{m\to+\infty} E_m(\rho_m(t)) \geq E_\infty(\rho_\infty(t))$

3. $\liminf_{m\to+\infty} \int_0^t |\partial E_m|^2(\rho_m(s))ds \geq \int_0^t |\partial E_\infty|^2(\rho_\infty(s))ds$

$$\mathcal{E}_m(\rho) = \int K * \rho d\rho + \frac{1}{m-1}\int \rho^m$$

$$\mathcal{E}_\infty(\rho) = \begin{cases} \int K * \rho d\rho & \text{if } \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

- (1) follows by interpolation of L^p norms
- (3) is more difficult, due to the lack of convexity (or even ω-convexity) uniformly in m

instead, we must use specific structure of metric slope

$$|\partial \mathcal{E}_m|(\mu_m) = \left\| \nabla K * \mu_m + \frac{\nabla \mu_m^m}{\mu_m} \right\|_{L^2(\mu_m)}$$

slow diffusion limit

Theorem: (C., Topaloglu, in preparation)

Suppose $\rho_m(x,t)$ are gradient flows of $\mathcal{E}_m$ satisfying

$$\rho_m(x,0) \to \rho_\infty(x,0) \text{ and } \mathcal{E}_m(\rho_m(x,0)) \to \mathcal{E}_\infty(\rho_\infty(x,0))$$

Then $\rho_m(x,t) \to \rho_\infty(x,t)$, the gradient flow of $\mathcal{E}_\infty$.

We also show…

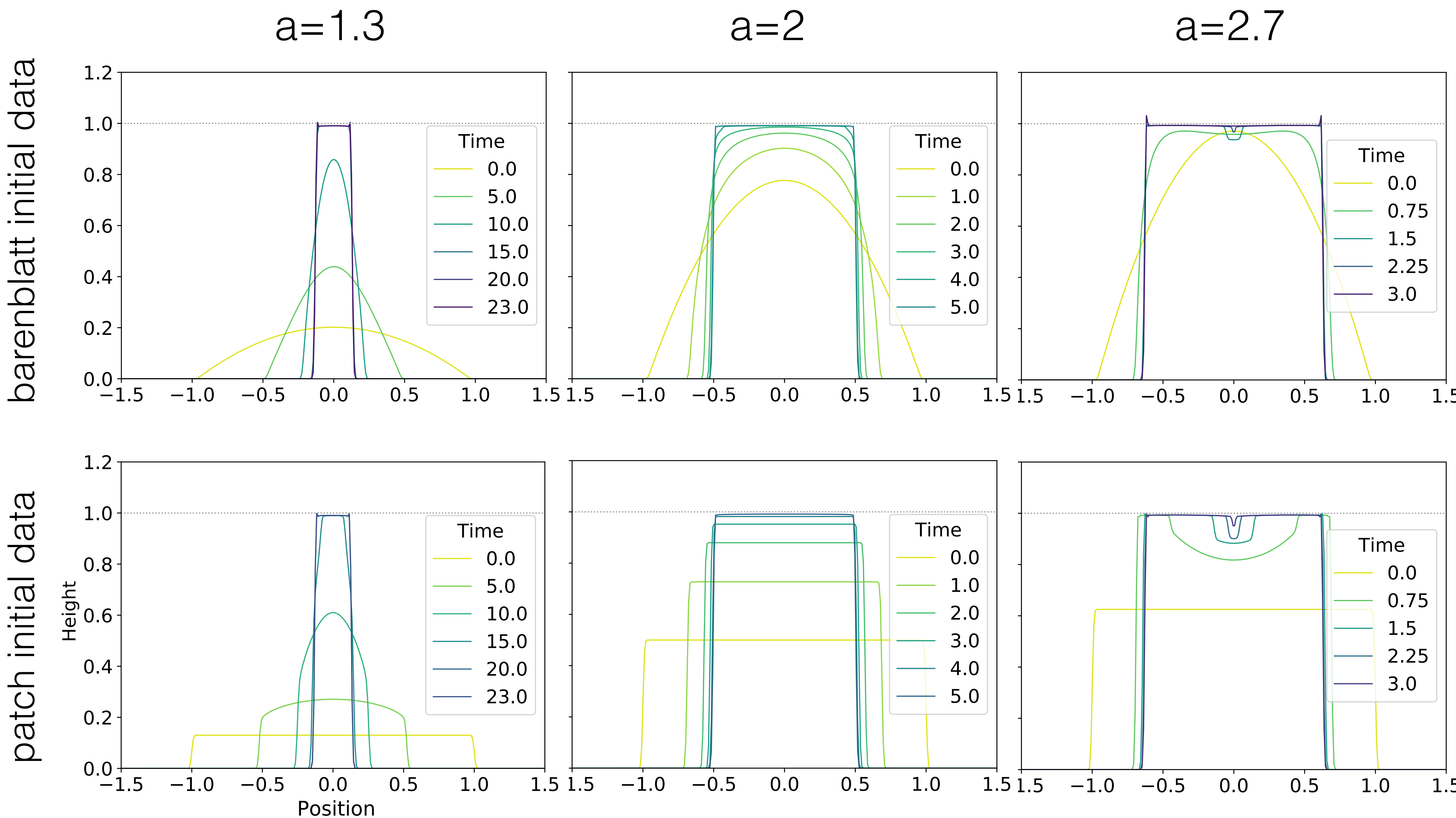
Theorem:

Suppose ρ_m are minimizers of $\mathcal{E}_m$. Then, up to a subsequence and translations, $\rho_m \to \rho_\infty$ where ρ_∞ is a minimizer of $\mathcal{E}_\infty$.

Thus, to gain numerical intuition for properties of minimizers of $\mathcal{E}_\infty$, we can simulate $\rho_m(x,t)$ for large m.

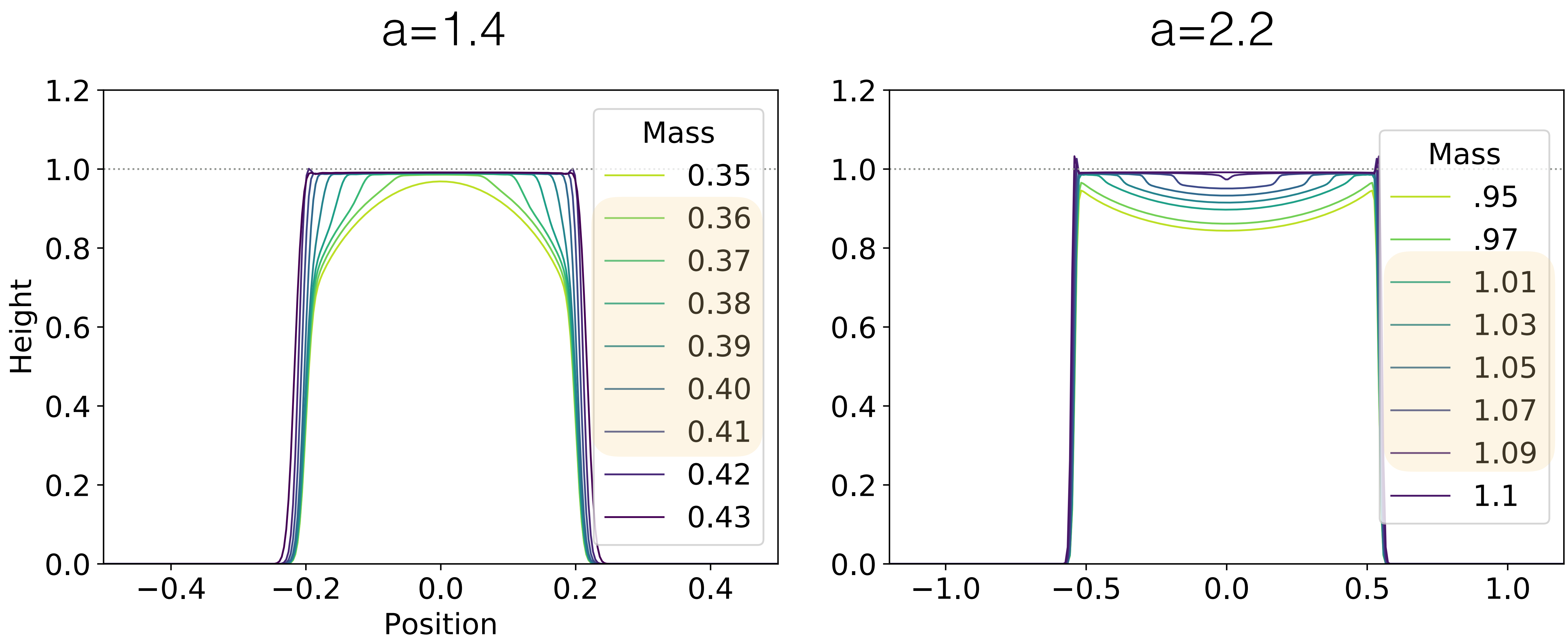
blob method for diffusion [Carrillo, Craig, Patacchini 2017]

numerics: convergence to equilibrium



m = 800, Nx = 500, M = critical mass

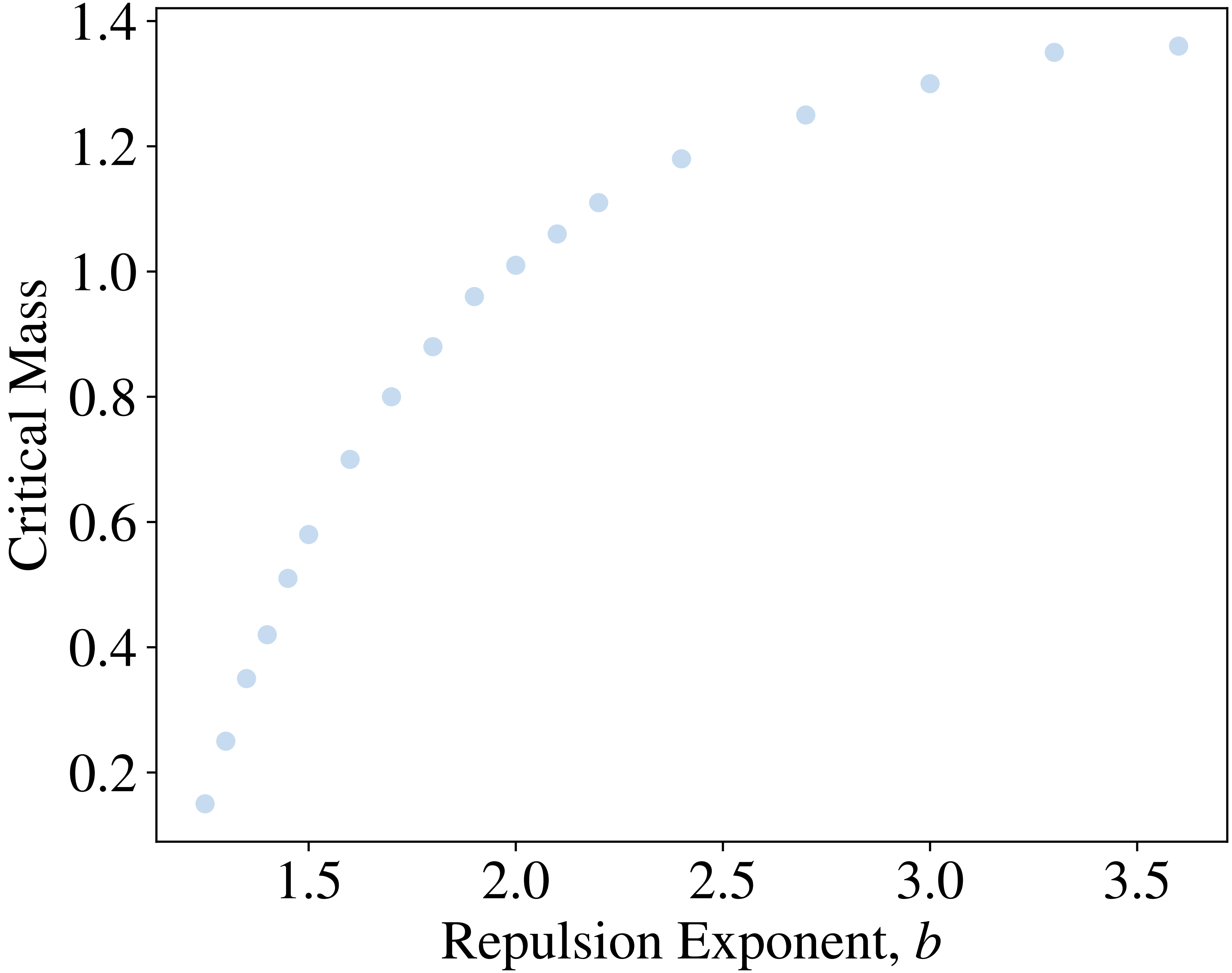
numerics: equilibria for varying mass



numerical evidence for intermediate phase

m = 800, Nx = 500

numerics: critical mass for solid state

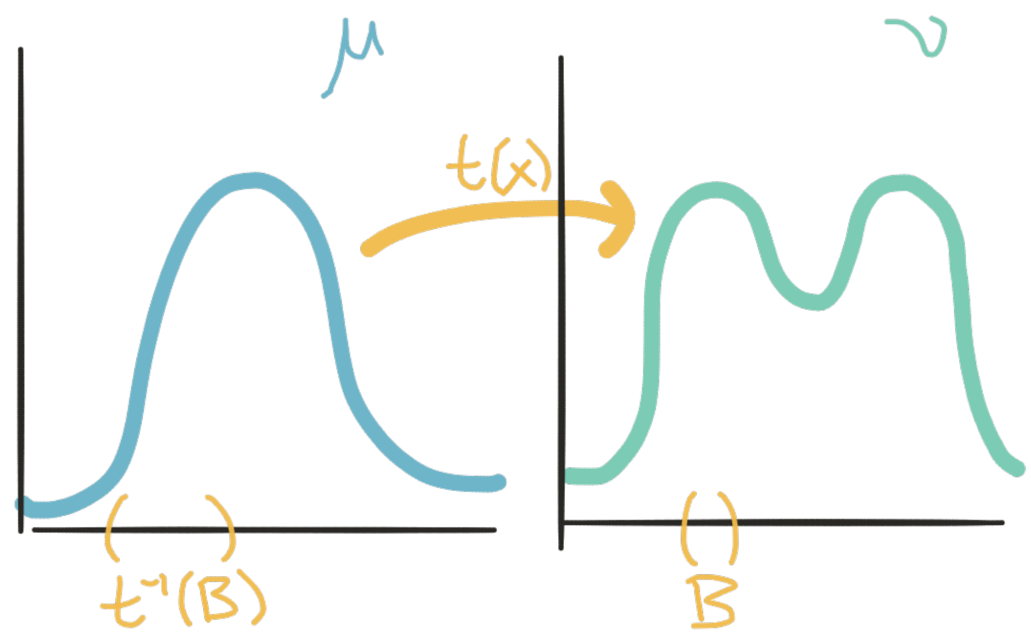


Thank you!

Backup

Wasserstein metric

- Given two probability measures μ and ν on $\mathbb{R}^d$, $\mathbf{t} : \mathbb{R}^d \to \mathbb{R}^d$ *transports* μ *onto* ν if $\nu(B) = \mu(\mathbf{t}^{-1}(B))$. Write this as $t\#\mu = \nu$.



- The *Wasserstein distance* between μ and ν ∈ $P_{2,ac}(\mathbb{R}^{d})$ is

$$W_2(\mu, \nu) := \inf \left\{ \underbrace{\left(\int |t(x) - x|^2 d\mu(x) \right)^{1/2}}_{\text{effort to rearrange } \mu \text{ to look like } \nu \text{, using } t} : \underbrace{t\#\mu = \nu}_{t \text{ sends } \mu \text{ to } \nu} \right\}$$

geodesics

Not just a metric space… a geodesic metric space: there is a constant speed geodesic $\sigma : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ connecting any μ and ν.

$$\sigma(0) = \mu,\ \ \sigma(1) = \nu,\ \ W_2(\sigma(t), \sigma(s)) = |t-s| W_2(\mu, \nu)$$

Monge Kantorovich

μ Wasserstein geodesic $\sigma(t)$ ν

μ linear interpolation $(1-t)\mu + t\nu$ ν

[Peyré, Papadakis, Oudet 2013]

convexity

Since the Wasserstein metric has geodesics, it has a notion of convexity.

Recall: in **Euclidean space**, E: $\mathbb{R}^d \to \mathbb{R}$ is…

λ-convex $$E((1-t)x+ty) \leq (1-t)E(x) + tE(y) - t(1-t)\frac{\lambda}{2}|x-y|^2$$

Likewise, in the **Wasserstein metric**, E: $P_2(\mathbb{R}^d) \to \mathbb{R}$ is…

λ-convex $$E(\sigma(t)) \leq (1-t)E(\mu) + tE(\nu) - t(1-t)\frac{\lambda}{2}W_2^2(\mu,\nu)$$

ω-convex $$E(\sigma(t)) \leq (1-t)E(\mu) + tE(\nu) - \frac{\lambda}{2}\left[(1-t)\omega\left(t^2W_2^2(\mu,\nu)\right) + t\omega\left((1-t)^2W_2^2(\mu,\nu)\right)\right]$$

$$\int_0^1 \frac{dx}{\omega(x)} = +\infty, \qquad \text{e.g. } \omega(x) = x|\log(x)|$$

[Carrillo, McCann, Villani, '06] [Ambrosio, Serfaty, '08] [Carrillo, Lisini, Mainini, '14] [C. '17] [C., Kim, Yao '17]

gradient flow

How does this relate to PDE?

- In general, given a complete metric space (X,d), a curve x(t): $\mathbb{R} \to$ X is the gradient flow of an energy E: X $\to$ $\mathbb{R}$ if

“ $$\boxed{\frac{d}{dt}x(t) = -\nabla_X E(x(t))}$$ ”

- “x(t) evolves in the direction of steepest descent of E”

Examples:

metric	energy functional	gradient flow
$(L^2(\mathbb{R}^d), \|\cdot\|_{L^2})$	$E(f) = \frac{1}{2}\int \lvert\nabla f\rvert^2$	$\frac{d}{dt}f = \Delta f$
$(\mathcal{P}_2(\mathbb{R}^d), W_2)$	$E(\rho) = \int \rho \log \rho$	$\frac{d}{dt}\rho = \Delta \rho$
	$E(\rho) = \frac{1}{m-1}\int \rho^m$	$\frac{d}{dt}\rho = \Delta \rho^m$

slow diffusion limit

Sketch of proof (iii): liminf $\int |\partial E_m|^2(\rho_m(t))\,dt \geq \int |\partial E_\infty|^2(\rho_\infty(t))$

$$E_m(\rho) = \frac{1}{2}\int K * \rho d\rho + \frac{1}{m-1}\int \rho^m$$

$$|\partial E_m|(\rho) = \left\| \nabla K * \rho + \frac{\nabla \rho^m}{\rho} \right\|_{L^2(\rho)}$$

$$\begin{tikzcd}
\rho_m \text{ grad flow} \arrow[r, "E_m"] & \|\rho_m\|_m < C \arrow[r, "\text{HLS}"] \arrow[dr] & \|\nabla K * \rho_m\|_{L^2(\rho_m)} < C & \\
{} \arrow[r, "|\partial E_m|"] & \left\| \frac{\nabla \rho_m^m}{\rho_m} \right\|_{L^2(\rho_m)} < C \arrow[r] & \|\nabla \rho_m^m\|_{2-2/(m+1)} < C \arrow[r, "S"] & \|\rho_m^m\|_2 < C
\end{tikzcd}$$

With this compactness, we get

$$\nabla K * \rho_m \to \nabla K * \rho, \qquad \frac{\nabla \rho_m^m}{\rho_m} \to \frac{\nabla \sigma}{\rho}, \qquad \liminf |\partial E_m|(\rho_m) \geq \left\| \nabla K * \rho + \frac{\nabla \sigma}{\rho} \right\|_{L^2(\rho)}$$

We conclude by showing RHS $\geq |\partial E_\infty|(\rho_\infty)$.

gradient flow

Good news: the congested aggregation equation is the Wasserstein gradient flow of the constrained interaction energy:

“
$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$
”

$$E_\infty(\rho) = \begin{cases} \frac{1}{2}\int K * \rho d\rho & \text{if } \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

Fact: If K: $\mathbb{R}^d \to \mathbb{R}$ is λ-convex, then E_∞ is λ-convex.

Bad news: $K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ C_d|x|^{2-d} & \text{otherwise} \end{cases}$ is not λ-convex.

E_∞ falls outside the scope of the existing theory.

ω-convexity

Solution: Even though we don't have

$$E_\infty(\sigma(t)) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{\lambda}{2} t(1-t) W_2^2(\mu,\nu)$$

λ-convexity

E_∞ does satisfy a similar inequality for a different modulus of convexity

$$E_\infty(\sigma(t)) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{\lambda}{2} \left[(1-t)\omega\left(t^2 W_2^2(\mu,\nu)\right) + t\omega\left((1-t)^2 W_2^2(\mu,\nu)\right)\right]$$

where $\omega(x) = x\,|\log(x)|$.

ω-convexity

[Carrillo, McCann, Villani, 2006] [Ambrosio, Serfaty, 2008]
[Carrillo, Lisini, Mainini, 2014]

Inequalities coincide for $\omega(x) = x$; ω-convexity generalizes λ-convexity.

ω-convexity: well-posedness

For merely ω-convex energies, the gradient flow is well-posed.

Theorem (C. 2016): If E is ω-convex for ω(x) = x |log(x)|, lower semicontinuous, and bounded below, solutions of its W_2 gradient flow

- exist (quantitative JKO)
- are unique
- contract (λ>0)/expand (λ≤0) double exponentially: for $W_2(\rho_1(0),\rho_2(0)) \leq 1$,

$$W_2(\rho_1(t), \rho_2(t)) \leq W_2(\rho_1(0), \rho_2(0))^{e^{2\lambda t}}$$

More generally, for ω(x) satisfying Osgood's condition, i.e.

$$\int_0^1 \frac{dx}{\omega(x)} = +\infty$$

we obtain the stability estimate

$$F_{2t}(W_2^2(\rho_1(t), \rho_2(t))) \leq W_2^2(\rho_1(0), \rho_2(0))$$
$$\frac{d}{dt} F_t(x) = \lambda\, \omega(F_t(x)), \quad F_0(x) = x$$

dynamics via free boundary problem

How does congested aggregation equation relate to free boundary problem?

$$“\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}”$$

Consider initial data: $\rho(x,0) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$

Since $\nabla K * \rho$ causes self-attraction, we expect $\rho(x,t)=1_{\Omega(t)}(x)$.

Theorem (C., Kim, Yao '18):
Suppose $\rho(x,t)$ solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
Then $\rho(x,t)=1_{\Omega(t)}(x)$, for $\Omega(t) = \{\mathbf{p(x,t)}>0\}$, where **p** a viscosity solution of

$$\begin{cases} -\Delta \mathbf{p} = 1 & \text{on} \quad \{\mathbf{p} > 0\} \\ \quad V = -\partial_\nu K * 1_{\{\mathbf{p}>0\}} - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}$$

long time behavior

Using free boundary characterization, we can describe long time behavior:

- In any dimension, the Riesz Rearrangement Inequality guarantees that the unique minimizer of E_∞ is $1_B(x)$.

- Need to show mass of $\rho(x,t)$ doesn't escape to $+\infty$. To accomplish this, we use an inequality due to Talenti, which holds in d=2.

Theorem (C., Kim, Yao 2016):
- Suppose $\rho(x,t)$ solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
- Then, in two dimensions,

$$\rho(x,t) \xrightarrow{L^p} 1_B(x) \text{ for all } 1 \le p < +\infty$$

and

$$|E_\infty(\rho(\cdot,t)) - E_\infty(1_B)| \le C_{\Omega(0)} t^{-1/6}$$

questions

K= Δ^{-1}

Congested aggregation eqn: "$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$"

- ☑ Well-posed? (nonconvex) Wasserstein gradient flow
- ☑ Dynamics/long time behavior? gradient flow + viscosity solution theory
- ☐ Slow diffusion limit? Gamma convergence

previous work

Congested drift equation: “$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot ((\nabla V)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$”

[Maury, Roudneff-Chupin, Santambrogio 2010]

- introduced as a model of crowd motion in an evacuation scenario, where V(x) = distance to exit.
- showed well-posedness as a W_2 gradient flow for V(x) convex.

[Alexander, Kim, Yao 2014]

- for $\Delta V > 0$, characterized patch dynamics via free boundary problem

$$\begin{cases} -\Delta \mathbf{p} = \Delta \Phi & \text{on } \{\mathbf{p} > 0\} \\ V = -\partial_\nu V - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}$$

gradient flow

We want to define the gradient flow as $\frac{d}{dt}\rho(t) = -\nabla_{W_2} E(\rho(t))$,

but without a Riemannian structure, we don't have a notion of **gradient**.

- Given E: $P_2(\mathbb{R}^d) \to \mathbb{R}$, its **local slope** is:

$$|\partial E|(\mu) := \limsup_{\nu \to \mu} \frac{(E(\mu) - E(\nu))^+}{W_2(\mu, \nu)}$$

- Given ρ:[0,T] $\to P_2(\mathbb{R}^d)$, its **metric derivative** is:

$$|\rho'|(t) = \lim_{s \to t} \frac{W_2(\rho(s), \rho(t))}{|s - t|}$$

DEF: $\rho(t):\mathbb{R}\to P_2(\mathbb{R}^d)$ is the **Wasserstein gradient flow** of $E:P_2(\mathbb{R}^d)\to\mathbb{R}$ if

$$\frac{d}{dt} E(\rho(t)) \leq -\frac{1}{2}\,|\partial E(\rho(t))| - \frac{1}{2}\,|\rho'|\,(t)$$

gradient flow

ρ(t): $\mathbb{R}$ → $P_2(\mathbb{R}^d)$ is the gradient flow of energy E: $P_2(\mathbb{R}^d)$ → $\mathbb{R}$ if

“ $\boxed{\dfrac{d}{dt}x(t) = -\nabla_X E(x(t))}$ ”

More precisely, ρ(t) is the gradient flow of E if…

- $\exists\ v(t) \in L^2_{\mathrm{loc}}((0,+\infty), L^2(\rho(t)))\ s.t.$ $\dfrac{d}{dt}\rho(x,t) + \nabla$
- $-v(t) \in \partial E(\rho(t))$ for a.e. t>0

The term brackets is analogous to ξ (ν - ρ)

Tangent space?

- ξ belongs to the subdifferential of E at ρ if as μ → ν,

$$E(\nu) - E(\rho) \geq \int \langle \xi, \mathbf{t}_\rho^\nu - \mathrm{id}\rangle \mathrm{d}\mu + \mathrm{o}(\mathrm{W}_2(\rho,\nu))$$

- If E and ρ are nice, $\partial E(\rho) = \left\{ \nabla \dfrac{\partial E}{\partial \rho} \right\}$
- Then solutions of the gradient flow can be characterized via a PDE.

aside: ω-convexity & Euler equations

In fact, when $\omega(x) = x\,|\log(x)|$, ω-convexity is related to well-posedness of bounded solutions of the the Euler equations.

- λ-convexity in W_2 is analogous to D^2E being bounded from below in Euclidean space, or that ∇E is one-sided Lipschitz.

- Likewise, ω-convexity in W_2 is analogous to D^2E being BMO in Euclidean space, or that ∇E is log-Lipschitz.

- Log-Lipschitz regularity of the velocity field was precisely what allowed [Yudovich 1963] to prove uniqueness of bounded solutions of the two dimensional Euler equations.

ω-convexity

Examples:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

- Chemotaxis: $K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ C_d|x|^{2-d} & \text{otherwise} \end{cases}$

ω-convex on L^∞

- Swarming: $K(x) = |x|^a/a - |x|^b/b, \quad 2 - d \leq b < a$

ω-convex on L^p, $p \geq d/(b+d-2)$

- Granular media: $K(x) = |x|^3$

ω-convex on measures with fixed center of mass; $\omega(x) = x^{3/2}$

Sufficient condition:

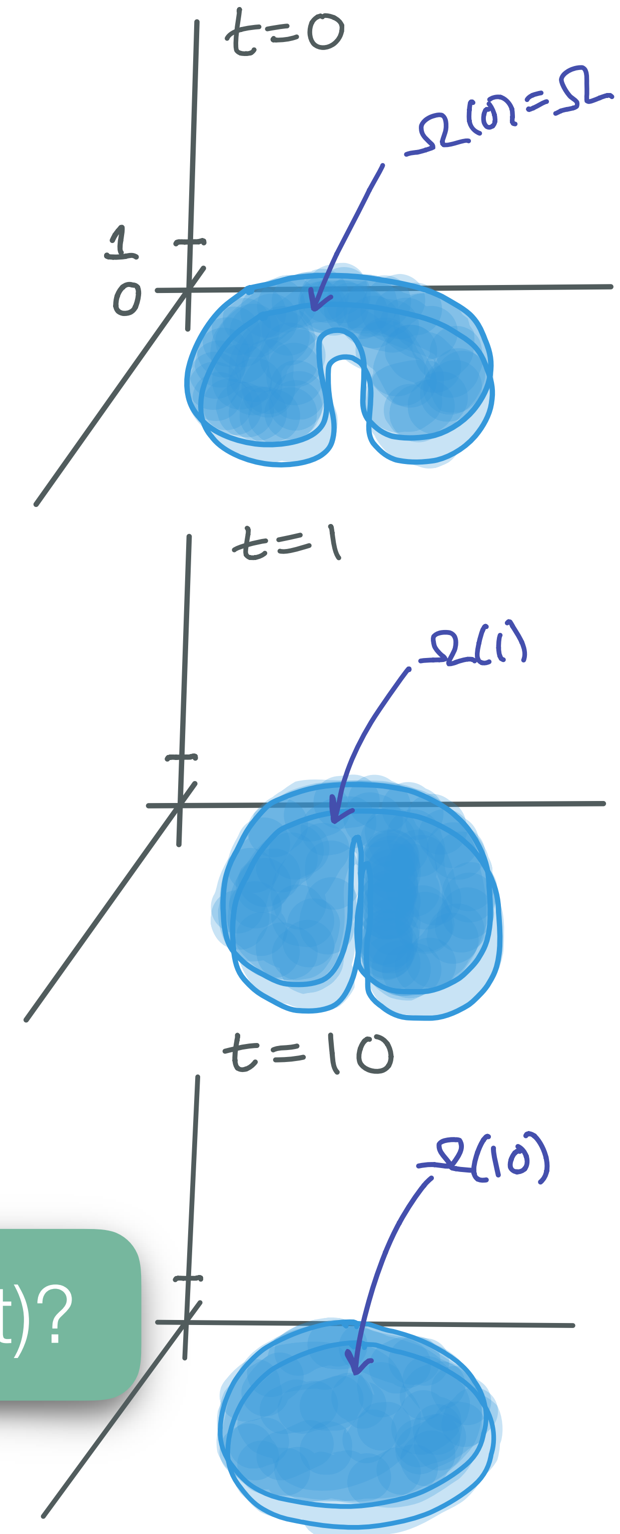
Above the tangent line inequality

$$E(\mu_1) - E(\mu_0) - \frac{d}{d\alpha}E(\mu_\alpha)|_{\alpha=0} \geq \frac{\lambda}{2}\omega(W_2^2(\mu_0, \mu_1))$$

motivation for free boundary problem

How does congested aggregation equation relate to free boundary problem?

"
$$\begin{cases} \frac{d}{dt}\rho = \nabla\cdot(\nabla(K*\rho)\rho) \text{ if } \rho < 1 \\ \rho \le 1 \text{ always} \end{cases}$$
"



- Consider patch solutions. For a domain Ω, suppose that ρ(x,t) is a solution with initial data
$$\rho(x,0) = \begin{cases} 1 & \text{if } x\in\Omega, \\ 0 & \text{otherwise.} \end{cases}$$
- Since $K=\Delta^{-1}$, $\nabla K * \rho$ causes self-attraction. Thus, we expect ρ(x,t) to remain a characteristic function.
- Let Ω(t)={ρ=1} be *congested region*, so $\rho(x,t)=1_{\Omega(t)}(x)$.

What free boundary problem describes evolution of Ω(t)?

formal derivation

- Here is a **formal** derivation of the related free boundary problem.
- Suppose ρ(x,t) solves "$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$"
- Since mass is conserved, we expect ρ(x,t) satisfies a continuity equation

$$\frac{d}{dt}\rho = \nabla \cdot (\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\rho)$$

where **∇p(x,t)** is the pressure arising from the height constraint.

Height constraint is active on the congested region {**p**>0} = Ω(t).

Height constraint is inactive outside the congested region {**p**=0}= $\Omega(t)^c$.

formal derivation

Given $\boxed{\frac{d}{dt}\rho = \nabla \cdot (\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\rho)}$ what happens on congested region?

- Because of hard height constraint, on the congested region $\Omega(t)=\{\rho=1\}$, the velocity field is incompressible, $\nabla \cdot v=0$.

- Since $K=\Delta^{-1}$, $\nabla \cdot v = \Delta K * \rho + \Delta \mathbf{p} = \rho + \Delta \mathbf{p}$, so incompressibility means

$$\boxed{-\Delta \mathbf{p} = \rho \text{ on } \Omega(t) = \{\rho = 1\}}$$

- Using that the height constraint is active on the congested region, $\Omega(t)=\{\mathbf{p}>0\}$, we obtain the following equation for the pressure:

$$\boxed{-\Delta \mathbf{p} = 1 \text{ on } \{\mathbf{p} > 0\}}$$

formal derivation

Given $\boxed{\frac{d}{dt}\rho = \nabla \cdot \big(\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\rho\big)}$ what about bdy of congested region?

outward normal velocity of ∂Ω(t)

- By conservation of mass,

$$0 = \frac{d}{dt}\int_{\Omega(t)} \rho = \int_{\Omega(t)} \frac{d}{dt}\rho + \int_{\partial\Omega(t)} V\rho$$

- Using that ρ(x,t) solves the above continuity equation, this equals

$$= \int_{\Omega(t)} \nabla \cdot ((\nabla K * \rho + \nabla \mathbf{p})\rho) + \int_{\partial\Omega(t)} V\rho = \int_{\partial\Omega(t)} (\partial_\nu K * \rho + \partial_\nu \mathbf{p} + V)\rho$$

- Since $\rho(x,t)=1_{\Omega(t)}(x)$, for $\Omega(t)=\{\mathbf{p}>0\}$, we again obtain an equation for **p**,

$$\boxed{\partial_\nu K * 1_{\{\mathbf{p}>0\}} + \partial_\nu \mathbf{p} + V = 0 \text{ on } \partial\{\mathbf{p} > 0\}}$$

free boundary problem

Remind myself the hoops we had to jump through to even define viscosity solutions

Combining the observations that…

- on the congested region,

$$\boxed{-\Delta \mathbf{p} = 1 \text{ on } \{\mathbf{p} > 0\}}$$

- and on the boundary of the congested region,

outward normal velocity of $\partial\Omega(t)$

$$\boxed{\partial_\nu K * 1_{\{\mathbf{p}>0\}} + \partial_\nu \mathbf{p} + V = 0 \text{ on } \partial\{\mathbf{p} > 0\}}$$

Theorem (C., Kim, Yao 2016):

- Suppose ρ(x,t) solves congested aggregation eqn with ρ(x,0) = $1_{\Omega(0)}$(x).
- Then ρ(x,t)=$1_{\Omega(t)}$(x), for Ω(t) = {**p(x,t)**>0}, where **p** a viscosity solution of

$$\boxed{\begin{cases} -\Delta \mathbf{p} = 1 & \text{on} \quad \{\mathbf{p} > 0\} \\ \quad V = -\partial_\nu K * 1_{\{\mathbf{p}>0\}} - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}}$$

future work

Does Keller-Segel converge to congested aggregation?

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m \xrightarrow{m \to +\infty} \begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla (K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

- [Alexander, Kim, Yao 2014] showed the analogous result for a convex drift potential V(x).
- **Obstacle:** Lack of uniform convexity as $m \to +\infty$.

Non-patch solutions?

- Recent work on $m \to +\infty$ limit in PME-type models for tumor growth with source [Kim and Pozar 2015], [Mellet, Perthame, Quiros 2015] and drift [Kim, Pozar, Woodhouse 2017].
- Can this be extended to include nonlocal interaction?
- **Obstacle:** Nucleation of new congested regions, infinite speed of propagation, neck pinching…

future work

Other characterizations of dynamics?

- Can we show $$\boxed{\frac{d}{dt}\rho = \nabla \cdot \Big(\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v} \rho \Big)}$$ in a weak sense?
- For the congested drift equation [Maury, Roudneff-Chupin, Santambrogio 2010] showed that the analogous continuity equation holds, where v is obtained by projecting ∇V onto a space of admissible velocities.
- **Obstacle:** With a nonlocal interaction term K, projection would depend nonlocally on ρ.

Height constrained aggregation with non-Newtonian kernels?

- Well-posedness theory extends to a range of interaction kernels
- **Obstacle:** Free boundary problem strongly uses Newtonian structure

Further examples of ω-convex energies? Problems with height constraint?