Bounded pitch inequalities for min knapsack: approximate separation and integrality gaps

Yuri Faenza

IEOR - Columbia University

Joint work with:

Dan Bienstock (Columbia), Igor Malinovic (EPFL), Monaldo Mastrolilli (IDSIA), Ola Svensson (EPFL), Mark Zuckerberg (Maroma Optimization)

Min knapsack

$$
\begin{aligned}
\min \textstyle\sum_{i\in[n]} c_i x_i & & & \\
\text{s.t.} \quad & \textstyle\sum_{i\in[n]} w_i x_i & \geq & \quad w_0 \\
& x & \in & \quad \{0,1\}^n
\end{aligned}
$$

With $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n$, $w_i \in \mathbb{N}\ \forall i$.

	Max Knapsack	Min Knapsack
Algorithms	FPTAS (Ibarra & Kim, 75)	FPTAS (Ibarra & Kim, 75)
Polytopes	Natural LP has IG 2	Natural LP has unbounded IG
	Linear number of SA rounds keep the IG at $2-\epsilon$ (KMN)	
	t rounds of Lasserre reduce IG to $t/t-1$ (KMN)	Linear number of Lasserre rounds leave the IG unbounded (KLM)
	$\exists$ LP formulation with IG $1+\epsilon$ and $n^{f(\epsilon)}$ constraints (Bienstock, 08)	$\exists$? LP formulation with IG bounded and $poly(n)$ constraints
	No such formulation $\exists$ in the original space (F & Sanità, 15)	No such formulation $\exists$ in the original space (Dudycz & Moldenhauer, 16)

(KMN)-(Karlin, Mathieu, Nguyen, 11), (KLM)-(Kurpisz, Leppänen & Mastrolilli, 17)

Knapsack cover inequalities

How can we reduce the integrality gap?

$$\begin{array}{lll} \min \sum_{i\in[n]} c_i x_i & & \\ \text{s.t.} \quad \sum_{i\in[n]} w_i x_i & \geq & w_0 \\ x & \in & \{0,1\}^n \end{array}$$

- Pick $\mathcal{T} \subseteq [n]$ such that $w(\mathcal{T}) < w_0$.

  $\sum_{j\in[n]\setminus\mathcal{T}} w_j x_j \geq w_0 - w(\mathcal{T})$ is valid.

  $\sum_{j\in[n]\setminus\mathcal{T}} \min\{w_j, w_0 - w(\mathcal{T})\} x_j \geq w_0 - w(\mathcal{T})$ is also valid and stronger.

  - Those are the Knapsack Cover (KC) inequalities (Wolsey, 75), (Carr, Fleischer, Leung, Philipps, 00)

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Thm. (CFLP, 00) Adding all (exponentially many) KC inequalities gives an integrality gap of 2.

KC (and generalization) also used to strengthen LPs for many covering pbs.

Extended formulation for Knapsack Cover Inequalities

Thm. (Bazzi, Fiorini, Huang, Svensson, 17)
$\exists$ $(2+\epsilon)$-approximated formulation for Min Knapsack of size $(1/\epsilon)^{O(1)} n^{O(\log n)}$.

- Uses many hammers:
  - Bounds on the depth of monotone circuits computing monotone threshold functions (Beinmel & Weinreb, 06)
  - Karchmer-Wigderson games (Karchmer & Wigderson, 90)
  - Extended formulation from randomized communication protocols (F, Fiorini, Grappe, Tiwary, 15)
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- Not output-efficient.
- Made output-efficient by (Fiorini, Huynh & Weltge, 17)

Pitch of inequalities for covering problems

Consider any binary covering problem

$$\begin{array}{rccc} \min \sum_{i\in[n]} c_i x_i & & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \in & \{0,1\}^n \end{array}$$

and a valid inequality $\sum_{i\in\mathcal{S}} \alpha_i x_i \geq \alpha_0$ with $\alpha_i \in \mathbb{N}\ \forall i \in \mathcal{S}$.

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The pitch of the inequality is the minimum k such that $\forall \mathcal{T} \subseteq \mathcal{S}$, $|\mathcal{T}| = k$, we have $\alpha(\mathcal{T}) \geq \alpha_0$.

- $x_1 + x_2 + 2x_3 + x_4 + x_5 \geq 3$ is pitch-3
- $x_1 + 2x_2 + 3x_3 + 3x_4 + 3x_5 \geq 7$ is pitch-4

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Obs.

- The non-dominated pitch-1 inequalities are of the form $\sum_{i\in\mathcal{S}} x_i \geq 1$.
- $\alpha_i \in \{1,\ldots,q\}\ \forall i \in \mathcal{S} \ \Rightarrow$ pitch$\leq q$.

Bounded pitch inequalities - why bother

Thm. 1 (Bienstock & Zuckerberg, 06) (informal)
In order to ϵ-approximate the t-th CG closure of a binary covering problem, it is enough to satisfy all valid inequalities of pitch $\leq f(t)$.

Thm. 2 (Bienstock & Zuckerberg, 04) (informal)
There exists a hierarchy that, given an LP for a binary covering problem that implies all pitch$\leq k$ inequalities, produces a poly-size LP that implies all pitch$\leq k+1$ inequalities.

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More hierarchies were introduced that satisfy Thm. 2 above:

- Certain Sum of Squares (Mastrolilli, 17);
- Hierarchy based on Boolean formulas (Fiorini, Huynh & Weltge, 17);
- Vector Branching (Bienstock & Zuckerberg, 18).

Bounded pitch, solved?

All those hierarchies require the original formulation to satisfy all pitch-1 inequalities. But:

Thm. (Klabjan, Nemhauser & Tovey, 98)
Optimizing over pitch-1 inequalities is NP-Hard, already for Min Knapsack.

So we do not know how to use this machinery for all covering problems (and, in particular, for min-knapsack).

This talk

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Various positive and (mostly) negative results on IG.

Notation

For a min-knapsack polytope with constraint

$$\sum_{i \in [n]} w_i x_i \geq w_0$$

with $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n$, $w_i \in \mathbb{N}\ \forall i$. We consider inequalities

$$\sum_{i \in \mathcal{S}} \alpha_i x_i \geq q$$

with $\alpha_j \in \{1, \ldots, q\}$ for all $j \in \mathcal{S}$. We let $\mathcal{S}_i := \{j : \alpha_j = i\}$.

The toy case: $q = 1$

In this case $q = 1$ iff pitch$= 1$. All non-dominated inequalities have the form:

$$\sum_{i \in \mathcal{S}} x_i \geq 1$$

- Choose, among all such valid inequalities, the one whose LHS computed in x^* is minimum.

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$$\begin{array}{rlcll} V^* := \min \sum_{i\in[n]} x_i^* z_i & & & & \leftarrow \text{minimize LHS} \\ \text{s.t.} & \sum_{i\in[n]} w_i(1 - z_i) & \leq & w_0 - 1 & \leftarrow \text{guarantee validity} \\ & z & \in & \{0,1\}^n & \end{array}$$

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$\exists$ violated pitch-1 inequality iff $V^* < 1$. Apply the FPTAS for Min Knapsack.

The almost-toy case: $q = 2$

Lemma. All pitch-2 inequalities are implied by:

- Pitch-1 inequalities;
- Valid inequalities of the form

$$\sum_{i \in \mathcal{S}_1} x_i + 2 \sum_{i \in \mathcal{S}_2} x_i \geq 2$$

where for each $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, we have $i < j$. Monotonicity property.

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Consider $\sum_j w_j x_j = 5x_1 + 6x_2 + 7x_3 + 10x_4 + 10x_5 \geq 10.$

Inequality $x_2 + x_3 + 2(x_4 + x_5 + x_1) \geq 2$ *is valid but non-monotone.*

It is dominated by

$$x_1 + x_2 + x_3 + 2(x_4 + x_5) \geq 2.$$

The almost-toy case: $q = 2$

It is enough to separate over inequalities

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where for each $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, we have $i < j$.

- Guess $k := \max\{i : i \in \mathcal{S}_1\}$. Then $\begin{cases} i \leq k, \; i \in \mathcal{S} & \Rightarrow \; i \in \mathcal{S}_1 \\ i > k, \; i \in \mathcal{S} & \Rightarrow \; i \in \mathcal{S}_2 \end{cases}$.

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$$\begin{array}{lll} V^*(k) := \min & \sum_{i \leq k} x_i^* z_i + 2 \sum_{i > k} x_i^* z_i & \leftarrow \text{minimize LHS} \\ & \sum_{i \in [n]} w_i(1 - z_i) + w_k \leq w_0 - 1 & \leftarrow \text{guarantee validity} \\ & z_k = 1 & \leftarrow \text{selection of } k \\ & z \in \{0,1\}^n & \end{array}$$

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An undominated inequality with pitch$= 2$ is violated iff $\exists \ k : V^*(k) < 2$.

Monotonicity does not hold for $q \geq 3$

Consider

$$\sum_j w_j x_j = 4x_1 + 4x_2 + 5x_3 + 6x_4 + 6x_5 \geq 13.$$

Inequality

$$x_1 + x_2 + 2x_3 + x_4 + x_5 \geq 3$$

is non-monotone and non-dominated.

The strategy for fixed $q \geq 3$

- Cover all valid inequalities $\sum_{i \in \mathcal{S}} \alpha_i x_i \geq q$ that are not dominated by any inequality with coefficients in $\{0, \ldots, q\}$ with a finite number of sets, that we call type.
- There are at most $f(q)n^{g(q)}$ types.
- Once a type is fixed, for each i, variable x_i has coefficient either 0 or t_i. Separation can again be formulated as a Min Knapsack problem, with optimum $V^*(\tau)$.
- Use the FPTAS for Min Knapsack to approximately compute each $V^*(\tau)$.

Non-monotonicity and Jealousy

Given a valid inequality

$$\sum_{i \in \mathcal{S}} \alpha_i x_i \geq q$$

we say that $i \in \mathcal{S}$ is jealous if $\exists\ j \in \mathcal{S}$ such that $w_j \leq w_i$ and $\alpha_j > \alpha_i$.

Example. Let
$\sum_j w_j x_j = 10x_1 + 10x_2 + 20x_3 + 25x_4 + 50x_5 + 80x_6 + 80x_7 + 100x_8 \geq 280$.
Consider inequality $x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 + x_6 + x_8 \geq 4$. x_5, x_6, x_8 are jealous.

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Jealousy Lemma. If $\sum_{i \in \mathcal{S}} \alpha_i x_i \geq q$ is valid and not dominated by any inequality with coefficients in $\{0, 1, \ldots, q\}$, then it has at most q^2 jealous variables.

Type of an inequality

The type associated to an inequality is a triple $\tau = (\mathbb{I}, \mathbb{M}, \mathbb{L})$.

- $\mathbb{I} = \{i : \mathcal{S}_i \neq \emptyset\}$ – Coefficients that appear in the inequality
- $\mathbb{M} = (m_i : i \in \mathbb{I})$, with $m_i \in \arg\min\{w_j : j \in \mathcal{S}_i\}$ – min weight for each $\mathcal{S}_i$
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Consider inequality $x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 + x_6 + x_8 \geq 4$. We can choose:

$$\mathbb{I} = \{1, 3, 4\}$$

$$m_1 = 1,\ m_3 = 3,\ m_4 = 4$$

$$\mathbb{L}_1 = \{6, 8, 1, 2\},\ \mathbb{L}_3 = \{5, 3\},\ \mathbb{L}_4 = \{4\}$$

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Lemma. There are at most $f(q)n^{g(q)}$ different types.

Coefficients for an inequality of a given type

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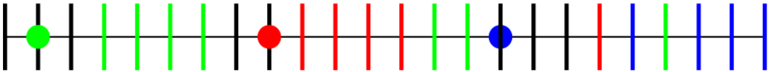

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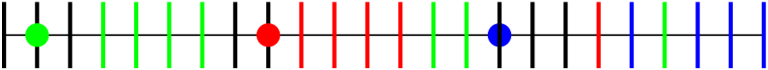

- If $j \in \mathbb{L}_i$ of $j = m_i$ for some $i \in \mathbb{I}$, then $j \in \mathcal{S}_i$.
- Else $j \in \mathcal{S}_i$, with $i \in \mathbb{I}$ s.t. $w_j \in [w_{m_i}, \min\{\min_{k \in \mathbb{L}_i} w_k, \min_{k > i} w_{m_k} - 1\}]$

We let $\mathbb{V}_i$ be the variables whose coefficient in the inequality is either 0 or i.

Guaranteeing feasibility for a given type

Consider all inequalities of type τ of the form

$$\sum_{i \in \mathcal{S}} \alpha_i x_i \geq q \tag{1}$$

$$\begin{aligned} V^*(\tau) := \min \quad & \textstyle\sum_{i \in \mathbb{I}} i(\sum_{j \in \mathbb{V}_i} x_j^* z_j) \\ & z_j = 1 \ \forall j \in \cup_i \mathbb{L}_i \cup_i \{m_i\} \\ & z_j = 0 \ \forall j \notin \cup_i \mathbb{V}_i \\ & z \in \{0,1\}^n \end{aligned}$$

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Recall how we guarantee validity for $q = 2$:

$$\underbrace{\sum_{i \in [n]} w_i(1 - z_i) +}_{w([n]\setminus\mathcal{S})} \text{ max weight of } \mathcal{S}' \overbrace{\subseteq}^{w_k} \mathcal{S} \text{ not satisfying (1)} \leq w_0 - 1$$

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Lemma. The max weight of a set $\mathcal{S}' \subseteq \mathcal{S}$ not satisfying (1) only depends on the type:$=\sigma(\tau)$. It can be computed efficiently.

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Separation algorithm for inequalities with coefficients in $\{0, 1, \dots, q\}$

- Apply the separation algorithm for pitch-1 and pitch-2. If it outputs **infeasible**, stop.
- For $k = 3, \dots, q$:
  - For each type τ:
    - Compute $\sigma(\tau)$;
    - Approximately compute $V^*(\tau)$
    - If $V^*(\tau) < q$, output **infeasible**
- Output **feasible**.

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Thank you for your attention.